SOME REMARKS ON THE DISCRETE MORSE-SMALE CHARACTERISTIC

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Abstract. In this paper, we quickly review some basic facts from discrete Morse theory, we introduce the Morse-Smale characteristic for a finite simplicial complex and we give few examples of exact discrete Morse functions on Möbius band, Klein bottle and torus.

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1. Introduction

Let K be a finite simplicial complex. A function $f : K \to \mathbb{R}$ is a *discrete Morse function* if for every simplex $\alpha^{(p)} \in K$ we have:

(1) $\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} \leq 1$ and

(2) $\#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} \leq 1$.

Example 1.1. One considers two simplicial complexes. We indicate functions by writing next to each simplex the value of the function on that simplex. The function (i) from the figure from below is not a discrete Morse function as edge $f^{-1}(0)$ violates rule (2), since it has two lower dimensional neighbors on which f takes on higher values. Moreover, the vertex $f^{-1}(5)$ violates rule (1), since it has two higher dimensional neighbors on which f takes on lower values. The function (ii) from the Figure 1 is a discrete Morse function. Note that a discrete Morse function is not a continuous function on K since we have not considered any topology on K. Rather, it is an assignment of a single number to each simplex.

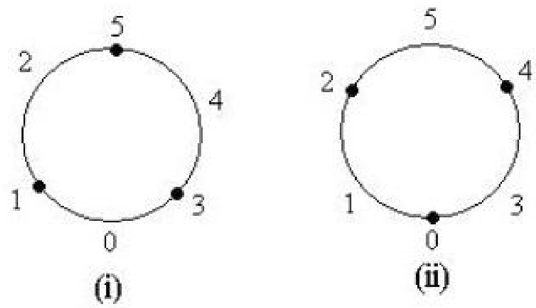

Figure 1:

The other main ingredient in discrete Morse theory is the notion of a critical point. A p-dimensional simplex $\alpha^{(p)}$ is *critical* if the following relations hold:

(1) $\#\{\beta^{(p+1)} > \alpha^{(p)} \mid f(\beta) \leq f(\alpha)\} = 0$ and

(2) $\#\{\gamma^{(p+1)} < \alpha^{(p)} \mid f(\gamma) \geq f(\alpha)\} = 0$.

For example, in Figure 1(ii), the vertex $f^{-1}(0)$ and the edge $f^{-1}(5)$ are critical and there are no other critical simplices.

If K is a m-dimensional simplicial complex with a discrete Morse function, then let μ_j denote the number of critical simplices of dimension j. For any field F, let $\beta_j = \dim H_j(K, F)$ be the j-th Betti number with respect to F, $j = 0, 1, \ldots, m$.

Then the following relations also hold in this context:

(1) The weak discrete Morse's inequalities.

(i) For each $j = 0, 1, \ldots, m$ (where m is the dimension of K), $\mu_j \geq \beta_j$;

(ii) $\mu_0 - \mu_1 + \mu_2 - \cdots + (-1)^m \mu_m = \beta_0 - \beta_1 + \beta_2 - \cdots + (-1)^m \beta_m = \chi(K)$ (Euler's relation).

(2) Also, the strong discrete Morse's inequalities are valid:

For each $j = 0, 1, \ldots, m$,

$$\mu_j - \mu_{j-1} + \cdots + (-1)^j \mu_0 \geq \beta_j - \beta_{j-i} + \cdots + (-1)^j \beta_0.$$

Let K be a simplicial complex containing exactly c_j simplices of dimension j, for each $j = 0, 1, \ldots, m$, where $m = \dim K$. Let $C_j(K, \mathbb{Z})$ denote the space $\mathbb{Z}^{c_j}$. More precisely, $C_j(K, \mathbb{Z})$ denotes the free abelian group generated by the

j-simplices of K, each endowed with an orientation. Then for each j, there are boundary maps $\partial_j : C_j(K,\mathbb{Z}) \to C_{j-1}(K,\mathbb{Z})$, such that $\partial_{j-1} \circ \partial_j = 0$.

The resulting differential complex

$$0 \longrightarrow C_m(K,\mathbb{Z}) \xrightarrow{\partial_m} C_{m-1}(K,\mathbb{Z}) \xrightarrow{\partial_{m-1}} \dots \xrightarrow{\partial_1} C_0(K,\mathbb{Z}) \longrightarrow 0$$

calculates the homology of K. That is, if we define the quotient space

$$H_j(C,\partial) = \text{Ker } (\partial_j)/\text{Im } (\partial_{j+1}),$$

then for each j we have the isomorphism

$$H_j(C,\partial) \cong H_j(K,\mathbb{Z}),$$

where $H_j(K,\mathbb{Z})$ denotes the singular homology of K.

The discrete Morse theory is the main tool in studying the curvature properties of a finite simplicial complex (see [3] and [4]).

2. The discrete Morse-Smale characteristic

Consider K^m a m-dimensional finite simplicial complex.

The discrete Morse-Smale characteristic of K was considered in paper [7] and it is a natural extension of the well-known Morse-Smale characteristic of a manifold (see [2]).

Let $\Omega(K)$ be the set of all discrete Morse functions defined on K. It is clear that $\Omega(K) \neq 0$. Consider, for instance, the trivial example $f(\sigma) = \dim \sigma$, $\sigma \in K$.

For $f \in \Omega(K)$, let $\mu_j(f)$ be the number of j-dimensional critical simplices of K, $j = 0, 1, \dots, m$.

Let $\mu(f)$ be the number defined as follows:

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f),$$

i.e. $\mu(f)$ is the total number of critical simplices of K. The number

$$\gamma(K) = \min\{\mu(f) : \ f \in \Omega(K)\}$$

is called *the discrete Morse Smale characteristic* of K.

So, the discrete Morse-Smale characteristic represents the minimal number of critical simplices for all discrete Morse functions defined on K.

In analogous way, one can define the numbers $\gamma_j(K)$, for all $j = 0, 1, \ldots, m$, by

$$\gamma_j(K) = \min\{\mu_j(f) : \ f \in \Omega(K)\}$$

which represent the minimal numbers of critical simplices of j-th dimension for all discrete Morse functions defined on K.

An extremely complicated problem in combinatorial topology represents the effective computation of these numbers associated to a finite simplicial complex. A finite algorithm for the determination of these numbers for any simplicial complex is not yet known.

3. Exact discrete Morse functions and F-perfect Morse functions on some 2-dimensional complexes

Consider K^m a m-dimensional finite simplicial complex.

Let $H_j(K, F)$, $j = 0, 1, \ldots, m$ be *the singular homology groups* with the coefficients in the field F and let $\beta_j(K, F) = \text{rank } H_j(K, F) = \dim_F H_j(K, F)$, $j = 0, 1, \ldots, m$ be the Betti numbers with respect to F.

For any $f \in \Omega(K)$, the following relations hold:

$$\mu_j(f) \geq \beta_j(K, F), \quad j = 0, 1, \ldots, m$$

(the discrete weak Morse inequalities).

Definition 3.1. *The discrete Morse function $f \in \Omega(K)$ is called exact (or minimal) if $\mu_j(f) = \gamma_j(K)$, for all $j = 0, 1, \ldots, m$.*

So, an exact discrete Morse function has a minimal number of critical simplices for each dimension.

Definition 3.2. *The discrete Morse function $f \in \Omega(K)$ is called F-perfect if*

$$\mu_j(f) = \beta_j(K, F), \quad j = 0, 1, \ldots, m.$$

Using the discrete weak Morse inequalities and the definition of the discrete Morse-Smale characteristic, we obtain the inequalities:

$$\mu_j(f) \geq \min\{\mu_j(f) : \ f \in \Omega(K)\} = \gamma_j(K) \geq \beta_j(K, F).$$

Theorem 3.3. *The simplicial complex K has F-perfect discrete Morse functions if and only if $\gamma(K) = \beta(K, F)$, where*

$$\beta(K, F) = \sum_{j=0}^{m} \beta_j(K, F)$$

is the total Betti number of K with respect to the field F.

Proof. The direct implication follows in this way.

Let $f \in \Omega(K)$ be a fixed F-perfect discrete Morse function. Using the weak Morse inequalities, it follows:

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f) \geq \sum_{j=0}^{m} \beta_j(K, F) = \beta(K, F).$$

So, $\mu(f) \geq \beta(K, F)$.

Using the definition of the Morse-Smale characteristic of K, we get:

$$\gamma(K) = \min\{\mu(f) : \ f \in \Omega(K)\} \geq \beta(K, F).$$

Because f is a discrete F-perfect Morse function on K, we have: $\mu(f) = \beta(K, F)$.

On the other hand, we have the inequality:

$$\gamma(K) = \min\{\mu(f) : \ f \in (K)\} \leq \beta(K, F).$$

So, we get $\gamma(K) \leq \beta(K, F)$ hence the desired relation follows.

For the converse implication, let $f \in \Omega(K)$ be a discrete Morse function. We have that:

$$\mu(f) = \sum_{j=0}^{m} \mu_j(f) \text{ and } \beta(K, F) = \sum_{j=0}^{m} \beta_j(K, F).$$

Using the relation $\gamma(K) = \beta(K, F)$, it follows

$$\sum_{j=0}^{m} [\mu_j(f) - \beta_j(K, F)] = 0.$$

From the discrete weak Morse inequalities, we get:

$$\mu_j(f) - \beta_j(K, F) \geq 0, \quad j = 0, 1, \ldots, m.$$

All in all, the following relation holds:

$$\mu_j(f) = \beta_j(K, F), \quad j = 0, 1, \ldots, m.$$

So, f is a discrete F-perfect Morse function. □

If K is a simplicial complex of dimension m, one knows that $C_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, is a finitely generated free abelian group on as many generators as there are j-simplices in K. Since subgroups and quotient groups of finitely generated groups are again finitely generated, it follows that $H_j(K, \mathbb{Z})$ is finitely generated. Therefore, by the fundamental theorem about such groups, we can write

$$H_j(K, \mathbb{Z}) \simeq A_j \oplus B_j,$$

where A_j is a free group and B_j is the torsion subgroup of $H_j(K, \mathbb{Z})$.

So, the singular homology groups $H_j(K, \mathbb{Z})$, $j = 0, 1, \ldots, m$, are finitely generated. For any $j = 0, 1, \ldots, m$ one obtains

$$H_j(K, \mathbb{Z}) \simeq (\mathbb{Z} \oplus \cdots \oplus) \oplus (\mathbb{Z}_{n_{j_1}} \oplus \cdots \oplus \mathbb{Z}_{n_{j_{\beta(j)}}})$$

where $\mathbb{Z}$ is taken β_j times in the free group and $j = 0, 1, \ldots, m$, represent the Betti numbers of K with respect to the group $(\mathbb{Z}, +)$, i.e. $\beta_j(K, \mathbb{Z}) = \text{rank } H_j(K, \mathbb{Z})$, for $j = 0, 1, \ldots, m$.

Example 3.4. One considers the Mobius band M. The singular homology of M over $\mathbb{Z}$ is:

$$H_j(M, \mathbb{Z}) = \mathbb{Z}, \text{ for } 0, 1 \text{ and } H_2(M, \mathbb{Z}) = 0.$$

From the universal coefficients formula for homology (see [2] p. 118) it follows

$$H_k(M, \mathbb{Z}) \simeq (H_k(M, \mathbb{Z}) \otimes \mathbb{Z}_p) \oplus Tor(\mathbb{Z}_p, H_{k-1}(M, \mathbb{Z})), \quad k \in \mathbb{Z},$$

where $Tor(\mathbb{Z}_p, H_{k-1}(M, \mathbb{Z}))$ is the torsion product of the groups $(\mathbb{Z}_p, +)$ and $H_{k-1}(M, \mathbb{Z})$.

Then we have:

$$H_0(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(M, \mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_2(M, \mathbb{Z}_2) \simeq \{0\}$$

and it follows

$$\beta_0(M, \mathbb{Z}_2) = 1, \quad \beta_1(M, \mathbb{Z}_2) = 1, \quad \beta_2(M, \mathbb{Z}_2) = 0.$$

Then the total Betti number is

$$\beta(M,\mathbb{Z}_2)=\sum_{j=0}^{2}\beta_j(M,\mathbb{Z}_2)=1+1+0=2.$$

According to our Theorem 3.3, we have the relation: $\gamma(M)=\beta(M,\mathbb{Z}_2)=2$. This means that one can build on the Mobius band M a discrete Morse function with exactly two critical simplices. A such function is, according to the definition, $\mathbb{Z}_2$-exact and it is defined in Figure 2. One has encircled the critical simplices.

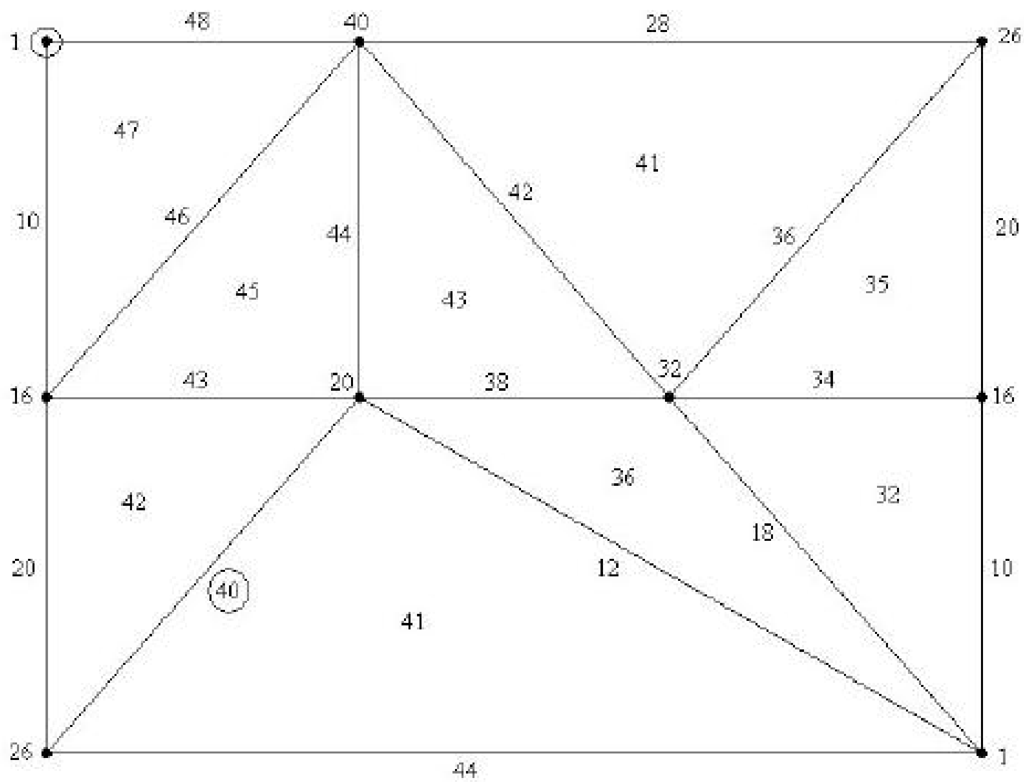

Figure 2: A Morse function with two critical simplices on Mobius band M

Example 3.5. One considers the Klein bottle K with the triangulation given in Figure 3.

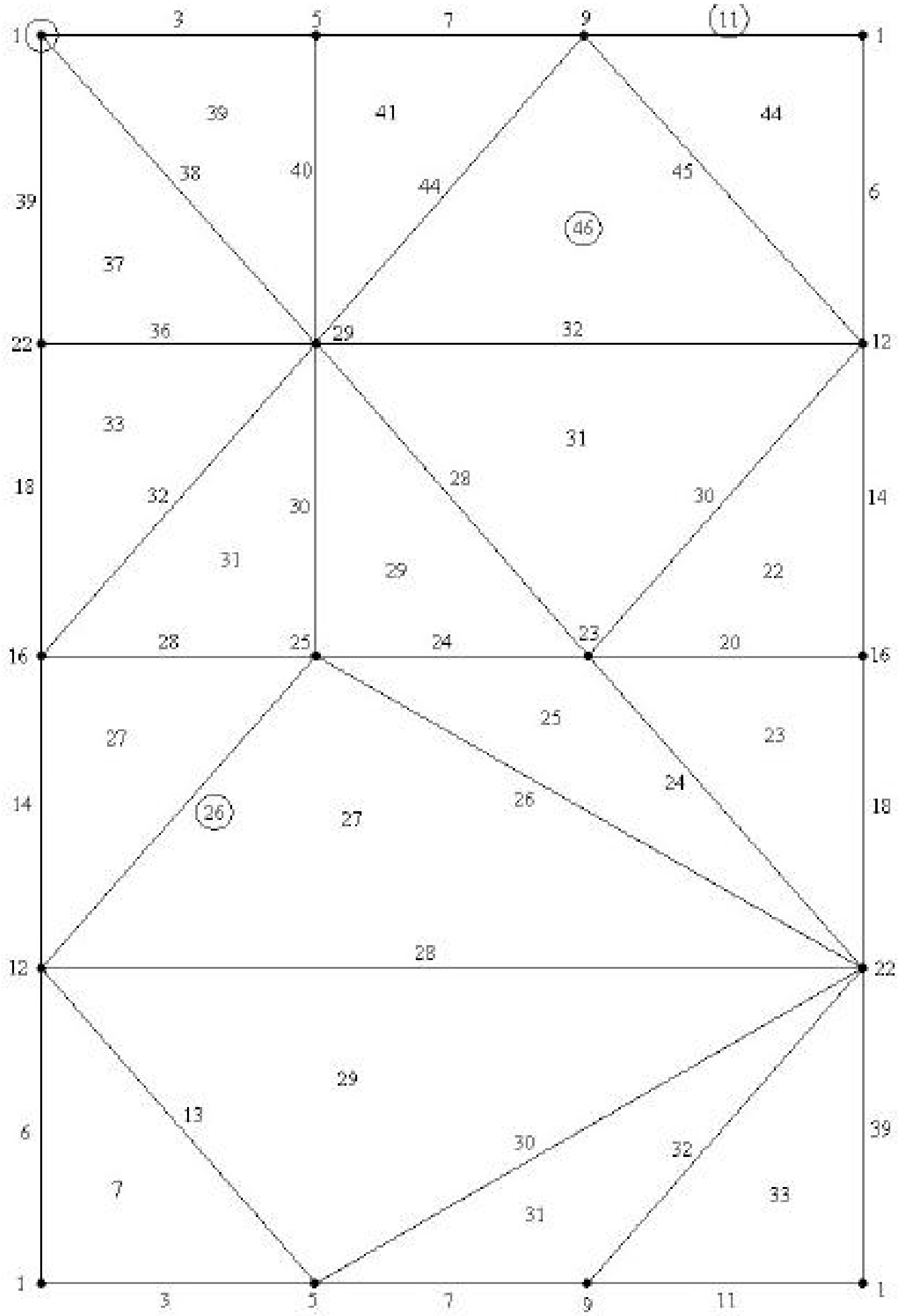

Figure 3: A Morse function with four critical simplices on Klein bottle K

The singular homology of K over $\mathbb{Z}$ is:

$$H_0(K,\mathbb{Z}) = \mathbb{Z}, \quad H_1(K,\mathbb{Z}) = \mathbb{Z}_2 \oplus \mathbb{Z} \text{ and } H_2(K,\mathbb{Z}) = 0.$$

Then we have:

$$H_0(K,\mathbb{Z}_2) \simeq \mathbb{Z}_2, \quad H_1(K,\mathbb{Z}_2) \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad H_2(K,\mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

It follows

$$\beta_0(K,\mathbb{Z}_2) = 1, \quad \beta_1(K,\mathbb{Z}_2) = 2, \quad \beta_2(M,\mathbb{Z}_2) = 1.$$

Then the total Betti number is

$$\beta(K,\mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(K,\mathbb{Z}_2) = 1 + 2 + 1 = 4.$$

According to our Theorem 3.3, we have the relation:

$$\gamma(K) = \beta(K,\mathbb{Z}_2) = 4.$$

This means that one can build on the Klein bottle K a discrete Morse function with exactly four critical simplices. A such function is, according to the definition, $\mathbb{Z}_2$-exact and it is defined in Figure 3. Again one has encircled the critical simplices.

Example 3.6. One considers the torus $S^1 \times S^1$ with the triangulation given in Figure 4.

The singular homology of $S^1 \times S^1$ over $\mathbb{Z}$ is:

$$H_0(S^1 \times S^1,\mathbb{Z}) = \mathbb{Z}, \quad H_1(S^1 \times S^1,\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \text{ and } H_2(S^1 \times S^1,\mathbb{Z}) = \mathbb{Z}.$$

Then we have:

$$H_0(S^1 \times S^1,\mathbb{Z}_2) \simeq \mathbb{Z}_2,\ H_1(S^1 \times S^1,\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2,\ H_2(S^1 \times S^1,\mathbb{Z}_2) \simeq \mathbb{Z}_2.$$

It follows

$$\beta_0(S^1 \times S^1,\mathbb{Z}_2) = 1, \quad \beta_1(S^1 \times S^1,\mathbb{Z}_2) = 2, \quad \beta_2(S^1 \times S^1,\mathbb{Z}_2) = 1.$$

Then the total Betti number is

$$\beta(S^1 \times S^1,\mathbb{Z}_2) = \sum_{j=0}^{2} \beta_j(S^1 \times S^1,\mathbb{Z}_2) = 1 + 2 + 1 = 4.$$

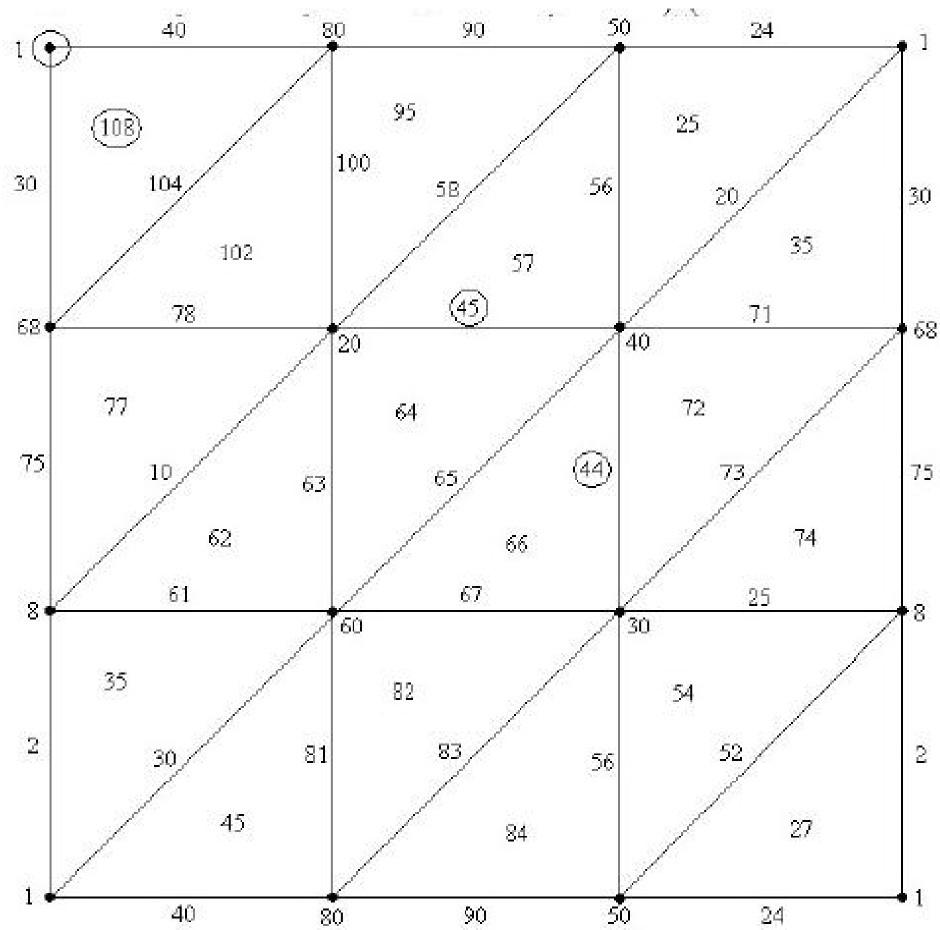

Figure 4: A Morse function with four critical simplices on the torus $S^1 \times S^1$

According to our Theorem 3.3, we have the relation:

$$\gamma(S^1 \times S^1) = \beta(S^1 \times S^1, \mathbb{Z}_2) = 4.$$

This means that one can build on the torus $S^1 \times S^1$ a discrete Morse function with exactly four critical simplices. A such function is, according to the definition, $\mathbb{Z}_2$-exact and it is defined in Figure 4. One has again encircled the critical simplices.

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