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# CERTAIN APPLICATION OF DIFFERENTIAL SUBORDINATION ASSOCIATED WITH GENERALIZED DERIVATIVE OPERATOR

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Abstract. The purpose of the present paper is to introduce several new subclasses of analytic function defined in the unit disk $D = \{ z \in \mathbb{C} : |z| < 1 \}$, using derivative operator for analytic function, introduced in [1]. We also investigate various inclusion properties of these subclasses. In addition we determine inclusion relationships between these new subclasses and other known classes.

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*Key words*: Analytic function; Hadamard product (or convolution); Univalent function; Convex function; Starlike function; Subordination; Derivative operator.

## 1. Introduction and Definitions

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad a_k \quad \text{is complex number} \tag{1}$$

which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Let $S$, $S^*(\alpha)$, $K(\alpha)\,(0 \le \alpha < 1)$ denote the subclasses of $A$ consisting of functions that are univalent, starlike of order $\alpha$ and convex of order $\alpha$ in $U$, respectively. In particular, the classes $S^*(0) = S^*$ and $K(0) = K$ are the familiar classes of starlike and convex functions in $U$, respectively.

Let be given two functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$ analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$. Then the Hadamard product (or convolution) $f * g$ of two functions $f$, $g$ is defined by

$$f(z) * g(z) = (f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \ .$$

Next, we give simple knowledge in subordination. If $f$ and $g$ are analytic in $U$, then the function $f$ is said to be subordinate to $g$, and can be written as

$$f \prec g \quad \text{and} \quad f(z) \prec g(z) \quad (z \in U),$$

if and only if there exists the Schwarz function $w$, analytic in $U$, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))$ $(z \in U)$.
If $g$ is univalent in $U$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(U) \subset g(U)$. [9, $p$.36].
Now, $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by

$$(x)_k = \begin{cases} 1 & \text{for } k = 0,\ x \in \mathbb{C}\backslash\{0\}, \\ x(x+1)(x+2)...(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, ...\} \text{and } x \in \mathbb{C}. \end{cases}$$

Let

$$k_a(z) = \frac{z}{(1-z)^a}$$

where $a$ is any real number. It is easy to verify that $k_a(z) = z + \sum\limits_{k=2}^{\infty} \frac{(a)_{k-1}}{(1)_{k-1}} z^k$. Thus $k_a * f$, denotes the Hadamard product of $k_a$ with $f$ that is

$$(k_a * f)(z) = z + \sum_{k=2}^{\infty} \frac{(a)_{k-1}}{(1)_{k-1}} a_k z^k.$$

Let $N$ denotes the class of functions which are analytic, convex, univalent in $U$, with normalization $h(0) = 1$ and $\operatorname{Re}(h(z)) > 0$ $(z \in U)$ Al_Shaqsi and Darus [1] defined the following generalized derivative operator.

**Definition 1 ([1]).** *For $f \in A$ the operator $\kappa_\lambda^n$ is defined by $\kappa_\lambda^n : A \to A$*

$$\kappa_\lambda^n f(z) = (1-\lambda) R^n f(z) + \lambda z (R^n f(z))', \qquad (z \in U), \tag{2}$$

*where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $\lambda \geq 0$ and $R^n f(z)$ denote for Ruscheweyh derivative operator [11].*
If $f$ is given by (1), then we easily find from the equality (2) that

$$\kappa_\lambda^n f(z) = z + \sum_{k=2}^{\infty} (1 + \lambda(k-1))\, c(n,k) a_k z^k, \qquad (z \in U),$$

where $n \in \mathbb{N}_0 = \{0, 1, 2...\}$, $\lambda \geq 0$ and $c(n,k) = \binom{n+k-1}{n} = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.
Let $\phi_\lambda^n(z) = z + \sum\limits_{k=2}^{\infty} (1 + \lambda(k-1))\, c(n,k) z^k$, where $n \in \mathbb{N}_0$, $\lambda \geq 0$ and $(z \in U)$, the operator $\kappa_\lambda^n$ written as Hadamard product of $\phi_\lambda^n(z)$ with $f(z)$, that is

$$\kappa_\lambda^n f(z) = \phi_\lambda^n(z) * f(z) = (\phi_\lambda^n * f)(z).$$

Note that for $\lambda$=0, $\kappa_0^n f(z) = R^n f(z)$ which Ruscheweyh derivative operator [11]. Now, let remind the well known Carlson-Shaffer operator $L(a,c)$ [3] associated with the incomplete beta function $\phi(a,c;z)$, defined by

$$L(a,c): A \to A$$
$$L(a,c) := \phi(a,c;z) * f(z) \qquad (z \in U), \text{ where } \phi(a,c;z) = z + \sum_{k=2}^{\infty} \tfrac{(a)_{k-1}}{(c)_{k-1}} z^k.$$

It is easily seen that $\kappa_0^0 f(z) = L(0,0)f(z) = f(z)$ and $\kappa_0^1 f(z) = L(2,1)f(z) = zf'$ and also if $\lambda = 0$, $n = a-1$, we see $\kappa_0^{a-1} f(z) = L(a,1)f(z)$, where $a = 1,2,3,\ldots$ Therefore, we write the following equality which can be verified easily for our result.

$$(1-\beta)\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))' = \beta(1+n)\kappa_\lambda^{n+1} f(z) - (\beta(1+n)-1)\kappa_\lambda^n f(z) \qquad (3)$$

By using the generalized derivative operator $\kappa_\lambda^n$ we define new subclasses of $A$:
For some $\beta(0 \le \beta \le 1)$, some $h \in N$ and for all $z \in U$.

$$P_\lambda^n(h,\beta) = \left\{ f \in A : \frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'} \prec h(z) \right\}.$$

For some $\alpha(\alpha \ge 0)$, some $h \in N$ and for all $z \in U$.

$$T_\lambda^n(h,\alpha) = \left\{ f \in A : (1-\alpha)\frac{\kappa_\lambda^n f(z)}{z} + \alpha(\kappa_\lambda^n f(z))' \prec h(z) \right\}$$

and finally $R_\lambda^n(h,\alpha) = \{ f \in A : (\kappa_\lambda^n f(z))' + \alpha z(\kappa_\lambda^n f(z))'' \prec h(z) \}$.

We note that the class $P_0^{a-1}(h,0) = S_a(h)$ was studied by Padmanabhan Parvatham in [8], $P_0^{a-1}(h,1) = k_a(h)$, $T_0^{a-1}(h,0) = R_a(h)$ and $T_0^{a-1}(h,1) = p_a(h)$ were studied by Padmanabhan and Manjini in [7] and the classes $P_0^{a-1}(h,\beta) = P_a(h,\beta)$, $T_0^{a-1}(h,\beta) = T_a(h,\beta)$ and $R_0^{a-1}(h,\beta) = R_a(h,\beta)$ were studied by Ozkan and Altintas [6]. Also note that the class $P_0^0(\frac{1+(1-2\alpha)z}{1-z},\beta)$ was studied by Altintas [2]. Obviously, for the special choices function $h$ and variables $\alpha, \beta, \lambda, n$ we have the following relationships:

$$P_0^0(\frac{1+z}{1-z},0) = S^*, \; P_0^0(\frac{1+z}{1-z},1) = K, \; P_0^1(\frac{1+z}{1-z},0) = K$$

and $P_0^0(\frac{1+(1-2\alpha)z}{1-z},0) = S^*(\alpha)$, $P_0^0(\frac{1+(1-2\alpha)z}{1-z},1) = K(\alpha) \quad (0 \le \alpha < 1)$.

## 2. The main inclusion relationships

In proving our main results, we need the following lemmas.

**Lemma 1** (Ruscheweyh and Sheil-Small [12,p.54]). *If $f \in K$, $g \in S^*$, then for each analytic function $h$,*

$$\frac{(f * hg)(U)}{(f * g)(U)} \subset \overline{co}h(U),$$

*where $\overline{co}h(U)$ denotes the closed convex hull of $h(U)$.*

**Lemma 2** (Ruscheweyh [10]). *Let $0 < \alpha \leq \beta$,* if $\beta \geq 2$ or $\alpha + \beta \geq 3$, *then the function*

$$\phi(\alpha, \beta, z) = z + \sum_{k=2}^{\infty} \frac{(\alpha)_{k-1}}{(\beta)_{k-1}} z^k \quad (z \in U)$$

*belongs to the class $K$ of convex functions.*

**Lemma 3** ([5]). *Let $h$ be analytic, univalent, convex in $U$, with $h(0) = 1$ and*

$$\text{Re}(\beta h(z) + \gamma) > o \quad (\beta, \gamma \in \mathbb{C};\, z \in U).$$

*If $p(z)$ is analytic in $U$, with $p(0) = h(0)$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z).$$

**Lemma 4** ([5]). *Let $h$ be analytic, univalent, convex in $U$, with $h(0) = 1$. Also let $p(z)$ be analytic in $U$, with $p(0) = h(0)$. If $p(z) + \frac{zp'(z)}{\gamma} \prec h(z)$ then $p(z) \prec q(z) \prec h(z)$, where $q(z) = \frac{\gamma}{z^\gamma} \int_0^z t^{\gamma-1} h(t) dt \quad (z \in U;\, \text{Re}(\gamma) \geq 0;\, \gamma \neq 0)$.*

**Lemma 5** ([4, p.248]). *If $\psi \in K$* and *$g \in S^*$, and $F$ is an analytic function with* $\text{Re}F(z) > 0$ for $z \in U$, *then we have*

$$\text{Re}\frac{(\psi * Fg)(z)}{(\psi * g)(z)} > 0 \quad (z \in U).$$

**Lemma 6** ([13]). *If $0 < a \leq c$* then $\text{Re}(\frac{\phi(a,c;z)}{z}) > \frac{1}{2}$ for all $z \in U$.

**Theorem 1.** *$f(z) \in P_\lambda^n(h,\beta)$ if and only if $g(z) = \beta z f'(z) + (1-\beta) f(z) \in P_\lambda^n(h,0)$.*

*Proof.* ($\Rightarrow$) Let $f \in P_\lambda^n(h,\beta)$, we want to show $\frac{z(\kappa_\lambda^n g(z))'}{\kappa_\lambda^n g(z)} \prec h(z)$. Using the well-known property of convolution $z(f*g)'(z) = (f*zg')(z)$ we obtain

$$\frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'} = \frac{z(\phi_\lambda^n(z) * f(z))' + \beta z^2(\phi_\lambda^n(z) * f(z))''}{(1-\beta)\left(\phi_\lambda^n(z) * f(z)\right) + \beta z(\phi_\lambda^n(z) * f(z))'} =$$

$$= \frac{\phi_\lambda^n(z) * z\left[f'(z) + \beta z f''(z)\right]}{\phi_\lambda^n(z) * \left[(1-\beta) f(z) + \beta z f'(z)\right]} = \frac{\phi_\lambda^n(z) * zg'(z)}{\phi_\lambda^n(z) * g(z)} \prec h(z).$$

Therefore $g(z) \in P_\lambda^n(h,0)$.
($\Leftarrow$) Obvious. Let $g(z) \in P_\lambda^n(h,0)$, by using same property of convolution and arguments, in the last proof, we obtain

$$\frac{z(\kappa_\lambda^n g(z))'}{\kappa_\lambda^n g(z)} = \frac{\phi_\lambda^n(z) * zg'(z)}{\phi_\lambda^n(z) * g(z)} = \frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'} \prec h(z).$$

Therefore $f(z) \in P_\lambda^n(h,\beta)$.

**Remark 1.** If $\beta$=1 in Theorem 1, then we deduce Theorem 3(i) of Padmanabhan and Manjini [7].

**Theorem 2.** *Let $h \in N$, $0 \le \beta \le 1$, $0 \le n_1 \le n_2$* and $n_1, n_2 \in \mathbb{N}_0$, if $n_2 \ge 1$ or $n_1 + n_2 \ge 1$, *then $P_\lambda^{n_2}(h,\beta) \subset P_\lambda^{n_1}(h,\beta)$.*

*Proof.* We suppose that $f \in P_\lambda^{n_2}(h,\beta)$. Then by the definition of the class $P_\lambda^{n_2}(h,\beta)$ we have

$$\frac{z(\kappa_\lambda^{n_2} f(z))' + \beta z^2(\kappa_\lambda^{n_2} f(z))''}{(1-\beta)\kappa_\lambda^{n_2} f(z) + \beta z(\kappa_\lambda^{n_2} f(z))'} = h(w(z)),$$

where $h$ is convex univalent in $U$ with $\mathrm{Re}(h(z)) > 0 \ \ (z \in U)$, and $|w(z)| < 1$ in $U$ with $w(0) = h(0) - 1$. By using the fact $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Setting $k_\lambda^{n_1} f(z) = k_\lambda^{n_2} f(z) * \varphi_{n_2}^{n_1}(z)$ where $\varphi_{n_2}^{n_1}(z) = z + \sum_{k=2}^{\infty} \frac{(n_1+1)_{k-1}}{(n_2+1)_{k-1}} z^k$ we get

$$\frac{z(\kappa_\lambda^{n_1} f(z))' + \beta z^2(\kappa_\lambda^{n_1} f(z))''}{(1-\beta)\kappa_\lambda^{n_1} f(z) + \beta z(\kappa_\lambda^{n_1} f(z))'} = \frac{z(\kappa_\lambda^{n_2} f(z) * \varphi_{n_2}^{n_1}(z))' + \beta z^2(\kappa_\lambda^{n_2} f(z) * \varphi_{n_2}^{n_1}(z))''}{(1-\beta)\left(\kappa_\lambda^{n_2} f(z) * \varphi_{n_2}^{n_1}(z)\right) + \beta z(\kappa_\lambda^{n_2} f(z) * \varphi_{n_2}^{n_1}(z))'}$$

$$= \frac{\varphi_{n_2}^{n_1}(z) * \left[z(\kappa_\lambda^{n_2} f(z))' + \beta z^2(\kappa_\lambda^{n_2} f(z))''\right]}{\varphi_{n_2}^{n_1}(z) * \left[(1-\beta)\left(\kappa_\lambda^{n_2} f(z)\right) + \beta z(\kappa_\lambda^{n_2} f(z))'\right]} = \frac{\varphi_{n_2}^{n_1}(z) * h(w(z)) p(z)}{\varphi_{n_2}^{n_1}(z) * p(z)}, \qquad (4)$$

where $p(z) = (1-\beta)\left(\kappa_\lambda^{n_2} f(z)\right) + \beta z(\kappa_\lambda^{n_2} f(z))'$. It follows from Lemma 2 that $\varphi_{n_2}^{n_1}(z) \in K$ and it follows from the Theorem 1 and from the definition of $P_\lambda^n(h,\beta)$ that $p(z) \in S^*$. Therefor applying Lemma 1 we get

$$\frac{\left\{\varphi_{n_2}^{n_1}(z) * h(w(z))p\right\}(U)}{\left\{\varphi_{n_2}^{n_1}(z) * p\right\}(U)} \subset \overline{co}h(w(U)) \subset h(U).$$

Since $h$ is convex univalent, thus (4) is subordinate to $h$ in $U$ and consequently $f(z) \in P_\lambda^{n_1}(h,\beta)$. This completes the proof of the Theorem 2.

**Remark 2.** Özkan and Altintas in [6] obtained the result: for $a \geq 1$, $P_{a+1}(h,\lambda) \subset P_a(h,\lambda)$. If we take $\lambda$=0, $n_1 = a_1 - 1$ and $n_2 = a_2 - 1$ in Theorem 2 we obtain following result improving the above mentiond.

**Corollary 1.** *Let* $0 < a_1 \leq a_2$, if $a_2 \geq 2$ or $a_1 + a_2 \geq 3$, then $P_{a_2}(h,\lambda) \subset P_{a_1}(h,\lambda)$.

**Theorem 3.** *For* $\lambda \geq 0$, $n \in \mathbb{N}_0$ *and* $0 \leq \beta \leq 1$, *then* $P_\lambda^{n+1}(h,\beta) \subset P_\lambda^n(h,\beta)$.

*Proof.* We suppose that $f \in P_\lambda^{n+1}(h,\beta)$. Then by the definition of the class $P_\lambda^{n+1}(h,\beta)$ we have

$$\frac{z(\kappa_\lambda^{n+1} f(z))' + \beta z^2(\kappa_\lambda^{n+1} f(z))''}{(1-\beta)\kappa_\lambda^{n+1} f(z) + \beta z(\kappa_\lambda^{n+1} f(z))'} = h(w(z)),$$

where $h$ is convex univalent in $U$ with $\mathrm{Re}(h(z)) > 0 \quad (z \in U)$, and $|w(z)| < 1$ in $U$ with $w(0) = h(0) - 1$. By using the fact $c(n,k) = \frac{(n+1)_{k-1}}{(1)_{k-1}}$.

Setting $\kappa_\lambda^n f(z) = \kappa_\lambda^{n+1} f(z) * \varphi_n(z)$, where $\varphi_n(z) = z + \sum_{k=2}^{\infty} \frac{(n+1)_{k-1}}{(n+2)_{k-1}} z^k$, we get

$$\frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'} = \frac{z(\kappa_\lambda^{n+1} f(z) * \varphi_n(z))' + \beta z^2(\kappa_\lambda^{n+1} f(z) * \varphi_n(z))''}{(1-\beta)\left(\kappa_\lambda^{n+1} f(z) * \varphi_n(z)\right) + \beta z(\kappa_\lambda^{n+1} f(z) * \varphi_n(z))'}$$

$$= \frac{\varphi_n(z) * \left[z(\kappa_\lambda^{n+1} f(z))' + \beta z^2(\kappa_\lambda^{n+1} f(z))''\right]}{\varphi_n(z) * \left[(1-\beta)\left(\kappa_\lambda^{n+1} f(z)\right) + \beta z(\kappa_\lambda^{n+1} f(z))'\right]} = \frac{\varphi_n(z) * h(w(z))p(z)}{\varphi_n(z) * p(z)}. \qquad (5)$$

Here $p(z) = (1-\beta)\left(\kappa_\lambda^{n+1} f(z)\right) + \beta z(\kappa_\lambda^{n+1} f(z))'$. It follows from Lemma 2 that $\varphi_n(z) \in K$ and it follows form the Theorem 1 and from the definition of $P_\lambda^n(h,\beta)$ that $p(z) \in S^*$. Therefore applying Lemma 1 we get

$$\frac{\left\{\varphi_n(z) * h(w(z))p\right\}(U)}{\left\{\varphi_n(z) * p\right\}(U)} \subset \overline{co}h(w(U)) \subset h(U).$$

Since $h$ is convex univalent, thus (5) is subordinate to $h$ in $U$ and consequently $f(z) \in P^n_\lambda(h,\beta)$. This completes the proof of Theorem 3.
**Theorem 4.** *Let $h \in N$, $0 \le \beta \le 1$ and $0 \le \lambda_1 \le \lambda_2$ then $P^n_{\lambda_2}(h,\beta) \subset P^n_{\lambda_1}(h,\beta)$.*

*Proof.* Let $f \in P^n_{\lambda_2}(h,\beta)$. Applying the definition of the class $P^n_{\lambda_2}(h,\beta)$. And using the same arguments as in the proof of Theorem 2. We get

$$\frac{z\left(\kappa^n_{\lambda_1} f(z)\right)' + \beta z^2\left(\kappa^n_{\lambda_1} f(z)\right)''}{(1-\beta)\kappa^n_{\lambda_1} f(z) + \beta z\left(\kappa^n_{\lambda_1} f(z)\right)'} = \frac{z\left(\kappa^n_{\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)\right)' + \beta z^2\left(\kappa^n_{\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)\right)''}{(1-\beta)(\kappa^n_{\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)) + \beta z\left(\kappa^n_{\lambda_2} f(z) * \psi^{\lambda_1}_{\lambda_2}(z)\right)'}$$
$$= \frac{\psi^{\lambda_1}_{\lambda_2}(z) * h(w(z))q(z)}{\psi^{\lambda_1}_{\lambda_2}(z) * q(z)}$$

where $|w(z)| < 1$ in $U$ with $w(0) = 0$, $q(z) = (1-\beta)\kappa^n_{\lambda_2} f(z) + \beta z\left(\kappa^n_{\lambda_2} f(z)\right)'$ and $\psi^{\lambda_1}_{\lambda_2}(z) = z + \sum\limits_{k=2}^{\infty} \frac{1+\lambda_1(k-1)}{1+\lambda_2(k-1)} z^k$. It follows from the Theorem 1 and the definition of $P^n_\lambda(h,\beta)$ that $q(z) \in S^*$. And by classical results in the class of convex, the coefficients problem for convex: $|a_n| \le 1$ we find $\psi^{\lambda_1}_{\lambda_2}(z) \in K$. Hence it follows from Lemma 1 that

$$\frac{\left\{\psi^{\lambda_1}_{\lambda_2}(z) * h(w(z))q\right\}(U)}{\left\{\psi^{\lambda_1}_{\lambda_2}(z) * q\right\}(U)} \subset \overline{co}h(w(U)) \subset h(U)$$

because $h$ is convex univalent, and consequently $f \in P^n_{\lambda_1}(h,\beta)$.

**Theorem 5.** *If $f(z) \in P^n_\lambda(h,\beta)$ for $n \in \mathbb{N}_0$ then $F_\mu(f) \in P^n_\lambda(h,\beta)$ where $F_\mu$ is the integral operator defind by*

$$F_\mu(f) = F_\mu(f)(z) := \frac{\mu+1}{z^\mu}\int_0^z t^{\mu-1} f(t)dt \qquad (\mu \ge 0). \tag{6}$$

*Proof.* Let $f(z) \in P^n_\lambda(h,\beta)$ and

$$p(z) = \frac{z(\kappa^n_\lambda F_\mu(f))'(z) + \beta z^2(\kappa^n_\lambda F_\mu(f))''(z)}{(1-\beta)(\kappa^n_\lambda F_\mu(f))(z) + \beta z(\kappa^n_\lambda F_\mu(f))'(z)}$$

from (6), we have $z(F_\mu(f))'(z) + \mu F_\mu(f)(z) = (\mu+1)f(z)$ and so

$$(\phi^n_\lambda * z(F_\mu(f))')(z) + \mu\left(\phi^n_\lambda * F_\mu(f)\right)(z) = (\mu+1)\left(\phi^n_\lambda * f\right)(z).$$

Using the fact $z(\phi_\lambda^n * F_\mu(f))'(z) = (\phi_\lambda^n * zF'_\mu(f))(z)$ we obtain

$$z(\kappa_\lambda^n F_\mu(f))'(z) + \mu(\kappa_\lambda^n F_\mu(f))(z) = (\mu+1)\kappa_\lambda^n f(z). \tag{7}$$

Differentiating (7), we have

$$p(z) + \mu = (\mu+1)\left[\frac{(1-\beta)(\kappa_\lambda^n f(z)) + \beta z(\kappa_\lambda^n f(z))'}{(1-\beta)(\kappa_\lambda^n F_\mu(f))(z) + \beta z(\kappa_\lambda^n F_\mu(f))'(z)}\right]. \tag{8}$$

Making use of the logarithmic differentiation on both sides of (8) and multiplying the resulting equation by $z$, we have

$$p(z) + \frac{zp'(z)}{p(z)+\mu} = \frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)(\kappa_\lambda^n f(z)) + \beta z(\kappa_\lambda^n f(z))'}\ . \tag{9}$$

By applying Lemma 3 to (9), it follows that $p \prec h$ in $U$, that is $F_\mu(f) \in P_\lambda^n(h,\beta)$.

**Remark 3.** Special cases of Theorems 3 and 5 with $\beta = 0,\ \lambda = 0,\ n = a-1$ and $\beta = 1,\ \lambda = 0,\ n = a-1$ were given earlier in [8,7], respectively.

**Remark 4.** By putting $\beta = 0,\ \lambda = 0,\ n = 0$ and $h(z) = \frac{1+z}{1-z}$ $(z \in U)$ in Theorem 3, we obtain $K \subset S^*$.

**Theorem 6.** *If* $f \in p_\lambda^n(h,\beta)$ *then* $\psi = \beta f + (1-\beta)\int_0^z \frac{f(t)}{t}\,dt \in p_\lambda^n(h,1)$.

*Proof.* Let $f \in P_\lambda^n(h,\beta)$. Applying Theorem 1 at $\beta$=1 we get

$$f \in P_\lambda^n(h,1) \ \Leftrightarrow\ zf' \in P_\lambda^n(h,0) \tag{10}$$

now $z\psi' = \beta zf' + (1-\beta)f$ that is $z\psi' \in P_\lambda^n(h,0)$, by (10) we see $\psi \in P_\lambda^n(h,1)$.

**Theorem 7.** *Let* $h \in N$, $\alpha \geq 0$ *and* $0 \leq n_1 \leq n_2$ *if* $n_2 \geq 1$ *or* $n_1 + n_2 \geq 1$ *then* $T_\lambda^{n_2}(h,\alpha) \subset T_\lambda^{n_1}(h,\alpha)$.

*Proof.* Let $f \in T_\lambda^{n_2}(h,\alpha)$. We obtain that

$$(1-\alpha)\frac{\kappa_\lambda^{n_2} f(z)}{z} + \alpha\left(\kappa_\lambda^{n_2} f(z)\right)' \prec h(z), \tag{11}$$

where $h \in N$.
Setting, $\kappa_\lambda^{n_1} f(z) = \kappa_\lambda^{n_2} f(z) * \psi_{n_2}^{n_1}(z)$, where $\psi_{n_2}^{n_1}(z) = z + \sum_{k=2}^{\infty} \frac{c(n_1,k)}{c(n_2,k)} z^k$. Applying (11) and the properties of convolution we find that

$$(1-\alpha)\frac{\kappa_\lambda^{n_1} f(z)}{z} + \alpha\left(\kappa_\lambda^{n_1} f(z)\right)' = \frac{\psi_{n_2}^{n_1}(z)}{z} * \left[(1-\alpha)\frac{\kappa_\lambda^{n_2} f(z)}{z} + \alpha\left(\kappa_\lambda^{n_2} f(z)\right)'\right]. \tag{12}$$

Under the hypothesis $0 \leq n_1 \leq n_2$, it follows from Lemma 6 that the function $z \to \frac{\psi_{n_2}^{n_1}(z)}{z}$ has its real part greater than or equal to $\frac{1}{2}$ in $U$. From the Herglotz Theorem we thus obtain $\frac{\psi_{n_2}^{n_1}(z)}{z} = \int\limits_{|x|=1} \frac{d\mu(x)}{1-xz} \quad (z \in U)$, when $\mu(x)$ is a probability measure on the unit circle $|x| = 1$, that is, $\int\limits_{|x|=1} d\mu(x) = 1$. It follows from (12) that

$$(1-\alpha)\frac{\kappa_\lambda^{n_1} f(z)}{z} + \alpha \left(\kappa_\lambda^{n_1} f(z)\right)' = \int\limits_{|x|=1} h(xz)d\mu(x) \prec h(z)$$

because $h$ is convex univalent in $U$. This proves Theorem 7.

**Remark 5.** By putting $\lambda = 0$, $n_2 + 1 = a_2$, and $n_1 + 1 = a_1$ in the Theorem 7 we deduce the following result which improves Theorem 5 of [6].

**Corollary 2.** *If* $0 < a_1 \leq a_2$ *then* $T_{a_2}(h, \alpha) \subset T_{a_1}(h, \alpha)$.

**Theorem 8.** *For* $n \in \mathbb{N}_0$ *and* $\lambda \geq 0$ *then* $T_\lambda^{n+1}(h, \alpha) \subset T_\lambda^n(h, \alpha)$.

*Proof.* Let $f \in T_\lambda^{n+1}(h, \alpha)$ and $p(z) = (1-\alpha)\frac{\kappa_\lambda^n f(z)}{z} + \alpha \left(\kappa_\lambda^n f(z)\right)'$.
Taking $\beta$=1 in (3), we obtain the following equality:

$$z\left(\kappa_\lambda^n f(z)\right)' = (n+1)\,\kappa_\lambda^{n+1} f(z) - n\kappa_\lambda^n f(z). \tag{13}$$

Using (13) and the differentiation of (13), we have

$$p(z) + \frac{zp'(z)}{n+1} = (1-\alpha)\frac{\kappa_\lambda^{n+1} f(z)}{z} + \alpha \left(\kappa_\lambda^{n+1} f(z)\right)' \prec h(z). \tag{14}$$

By applying Lemma 4 to (14), we can write $p \prec h(z)$ in $U$. Thus $f \in T_\lambda^n(h, \alpha)$.

**Theorem 9.** *If* $f \in T_\lambda^n(h, \alpha)$ *then* $F_\mu(f) \in T_\lambda^n(h, \alpha)$.

*Proof.* We assume that if $f \in T_\lambda^n(h, \alpha)$ and $p(z) = (1-\alpha)\frac{\left(\kappa_\lambda^n F_\mu(f)\right)(z)}{z} + \alpha \left(\kappa_\lambda^n F_\mu(f)\right)'(z)$.
Differentiating (7), we have

$$p(z) + \frac{zp'(z)}{\mu+1} = (1-\alpha)\frac{\left(\kappa_\lambda^n f(z)\right)}{z} + \alpha \left(\kappa_\lambda^n f(z)\right)'$$

from Lemma 4, we write $p(z) \prec h(z)$ in $U$ and hence $F_\mu(f) \in T_\lambda^n(h, \alpha)$.

**Theorem 10.** *$f \in R_\lambda^n(h,\alpha)$ if and only if $zf' \in T_\lambda^n(h,\alpha)$.*

*Proof.* Using the equality $z(\phi_\lambda^n(z) * f)' = (\phi_\lambda^n * zf')(z)$ we see that.

$$(1-\alpha)\frac{\kappa_\lambda^n(zf')(z)}{z} + \alpha\left(\kappa_\lambda^n(zf')\right)'(z) = (1-\alpha)\frac{(\phi_\lambda^n * (zf'))(z)}{z} + \alpha\left(\phi_\lambda^n * (zf')\right)'(z)$$

$$= (1-\alpha)(\phi_\lambda^n * f)'(z) + \alpha\left(z(\phi_\lambda^n * f)'(z)\right)' = (\kappa_\lambda^n f(z))' + \alpha z(\kappa_\lambda^n f(z))''.$$

**Theorem 11.** *Let $h \in N$, $\alpha \geqslant 0$ and $n_1, n_2 \in \mathbb{N}_0$. If $0 \leqslant n_1 \leq n_2$ then*

$$R_\lambda^{n_2}(h,\alpha) \subset R_\lambda^{n_1}(h,\alpha).$$

*Proof.* Applying Theorem 10 we immediately find that

$$f \in R_\lambda^{n_2}(h,\alpha) \Leftrightarrow zf' \in T_\lambda^{n_2}(h,\alpha) \Rightarrow zf' \in T_\lambda^{n_1}(h,\alpha) \Leftrightarrow f \in R_\lambda^{n_1}(h,\alpha).$$

This completes the proof of Theorem 11.

**Theorem 12.** $R_\lambda^{n+1}(h,\alpha) \subset R_\lambda^n(h,\alpha)$. *Proof.* Let $f \in R_\lambda^{n+1}(h,\alpha)$ and $p(z) = (\kappa_\lambda^n f(z))' + \alpha z(\kappa_\lambda^n f(z))''$.
Differentiating (13), we have

$$p(z) + \frac{zp'(z)}{\alpha} = \left(\kappa_\lambda^{n+1} f(z)\right)' + \alpha z\left(\kappa_\lambda^{n+1} f(z)\right)''.$$

From Lemma 4, we have $p \prec h$ in $U$. Thus $f \in R_\lambda^n(h,\alpha)$.

**Theorem 13.** *If $f \in R_\lambda^n(h,\alpha)$ then $F_\mu(f) \in R_\lambda^n(h,\alpha)$.*

*Proof.* We assume that $f \in R_\lambda^n(h,\alpha)$ and $p(z) = (\kappa_\lambda^n F_\mu(f))' + \alpha z(\kappa_\lambda^n F_\mu(f))''$.
Differentiating (7), we have

$$p(z) + \frac{zp'(z)}{\mu+1} = (\kappa_\lambda^n f(z))' + \alpha z(\kappa_\lambda^n f(z))'' \prec h(z).$$

From Lemma 4, we write $p \prec h$ in $U$. Thus $F_\mu(f) \in R_\lambda^n(h,\alpha)$.

**Theorem 14.** $R_\lambda^n(h,\alpha) \subset T_\lambda^n(h,\alpha)$.

*Proof.* Let $f \in R_\lambda^n(h,\alpha)$ and $p(z) = (1-\alpha)\frac{\kappa_\lambda^n f(z)}{z} + \alpha(\kappa_\lambda^n f(z))'$.
Thus, we obtain

$$p(z) + zp'(z) = (\kappa_\lambda^n f(z))' + \alpha z(\kappa_\lambda^n f(z))'' \prec h(z).$$

Hence, from Lemma 4, we have $f \in T_{\lambda}^{n}(h, \alpha)$.

**Theorem 15.**
(i) *If* $f \in T_{\lambda}^{n}(h, \alpha)$ *then* $f \in T_{\lambda}^{n}(h, 0)$.
(ii) *For* $\alpha > \beta \geqslant 0$, $T_{\lambda}^{n}(h, \alpha) \subset T_{\lambda}^{n}(h, \beta)$.

*Proof.*
(i) Let $f \in T_{\lambda}^{n}(h, \alpha)$ and $p(z) = \frac{\kappa_{\lambda}^{n} f(z)}{z}$. Then, we find that

$$p(z) + \alpha z p'(z) = (1-\alpha) \frac{\kappa_{\lambda}^{n} f(z)}{z} + \alpha \left(\kappa_{\lambda}^{n} f(z)\right)'.$$

From Lemma 4, we have $p \prec h$ in $U$. Thus $f(z) \in T_{\lambda}^{n}(h, 0)$.
(ii) If $\beta$=0, then the statement reduces to (i). Hence we suppose that $\beta \neq 0$ and let $f \in T_{\lambda}^{n}(h, \alpha)$. Let $z_1$ be arbitrary point in $U$. Then
$(1-\alpha) \frac{\kappa_{\lambda}^{n} f(z_1)}{z_1} + \alpha \left(\kappa_{\lambda}^{n} f(z_1)\right)' \in h(U)$. From (i), since $\frac{\kappa_{\lambda}^{n} f(z)}{z} \in h(U)$, we write the following equality:

$$(1-\beta) \frac{\kappa_{\lambda}^{n} f(z)}{z} + \beta \left(\kappa_{\lambda}^{n} f(z)\right)' = \left(1 - \frac{\beta}{\alpha}\right) \frac{\kappa_{\lambda}^{n} f(z)}{z} + \frac{\beta}{\alpha} \left[(1-\alpha) \frac{\kappa_{\lambda}^{n} f(z)}{z} + \alpha \left(\kappa_{\lambda}^{n} f(z)\right)'\right].$$

Since ${}^{\beta}/_{\alpha} < 1$ and $h(U)$ is convex,

$$(1-\beta) \frac{\kappa_{\lambda}^{n} f(z)}{z} + \beta \left(\kappa_{\lambda}^{n} f(z)\right)' \in h(U).$$

Thus $f \in T_{\lambda}^{n}(h, \beta)$.

**Theorem 16.**
(i) *If* $f \in R_{\lambda}^{n}(h, \alpha)$ *then* $f \in R_{\lambda}^{n}(h, 0)$.

(ii)*For* $\alpha > \beta \geqslant 0$, $R_{\lambda}^{n}(h, \alpha) \subset R_{\lambda}^{n}(h, \beta)$.

*Proof.* (i) Let $f \in R_{\lambda}^{n}(h, \alpha)$ and $p(z) = \left(\kappa_{\lambda}^{n} f(z)\right)'$ then we have

$$p(z) + \alpha z p'(z) = \left(\kappa_{\lambda}^{n} f(z)\right)' + \alpha z \left(\kappa_{\lambda}^{n} f(z)\right)''.$$

Hence from Lemma 4, we have $p \prec h$ in $U$. Thus $f(z) \in R_{\lambda}^{n}(h, 0)$.
(ii) If $\beta$=0, then the statement reduces to (i). Hence we suppose that $\beta \neq 0$ and let $f \in R_{\lambda}^{n}(h, \alpha)$. Let $z_1$ be arbitrary point in $U$. Then

$$\left(\kappa_{\lambda}^{n} f(z_1)\right)' + \alpha z_1 \left(\kappa_{\lambda}^{n} f(z_1)\right)'' \in h(U).$$

From (i) we write the following equality:

$$(\kappa_\lambda^n f(z))' + \beta z\left(\kappa_\lambda^n f(z)\right)'' = \left(1-\frac{\beta}{\alpha}\right)\left(\kappa_\lambda^n f(z)\right)' + \frac{\beta}{\alpha}\left[(\kappa_\lambda^n f(z))' + \alpha z\left(\kappa_\lambda^n f(z)\right)''\right].$$

Since ${}^{\beta}\!/\!_{\alpha} < 1$ and $h(U)$ is convex,

$$(\kappa_\lambda^n f(z))' + \beta z\left(\kappa_\lambda^n f(z)\right)''(z) \in h(U).$$

Thus $f \in R_\lambda^n(h,\beta)$.

## 3. Convolution results and its applications

**Theorem 17.** *Let $h \in N$, $n \in \mathbb{N}_0$ and $0 \leqslant \beta \leqslant 1$. If $g \in K$ and $f \in P_\lambda^n(h,\beta)$ then $f * g \in P_\lambda^n(h,\beta)$.*

*Proof.* We begin by assuming $f \in P_\lambda^n(h,\beta)$ and $g \in K$. In the proof we use the same idea as in the proof of Theorem 2. Let

$$\frac{z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''}{(1-\beta)\,\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'} = h(w(z)),$$

and

$$p(z) = (1-\beta)\,\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'.$$

Using the following equalities:

$$z\left(\phi_\lambda^n * f\right)'(z) = \left(\phi_\lambda^n * zf'\right)(z) \quad \text{and} \quad z^2\left(\phi_\lambda^n * f\right)''(z) = \left(\phi_\lambda^n * z^2 f''\right)(z),$$

we write

$$\frac{z(\kappa_\lambda^n\left(f*g\right)(z))' + \beta z^2(\kappa_\lambda^n\left(f*g\right)(z))''}{(1-\beta)\,\kappa_\lambda^n\left(f*g\right)(z) + \beta z(\kappa_\lambda^n\left(f*g\right)(z))'} =$$

$$= \frac{z(\phi_\lambda^n * f * g)'(z) + \beta z^2(\phi_\lambda^n * f * g)''(z)}{(1-\beta)\left(\phi_\lambda^n * f * g\right)(z) + \beta z\left(\phi_\lambda^n * f * g\right)'(z)} =$$

$$= \frac{g * \left[z(\kappa_\lambda^n f(z))' + \beta z^2(\kappa_\lambda^n f(z))''\right]}{g * \left[(1-\beta)\,\kappa_\lambda^n f(z) + \beta z(\kappa_\lambda^n f(z))'\right]} = \frac{g * h(w(z))p(z)}{g * p(z)} \prec h(z).$$

Consequently $f * g \in P_\lambda^n(h,\beta)$.

**Theorem 18.** *Let $h \in N$, $n \in \mathbb{N}_0$, $\alpha \geq 0$ and $\operatorname{Re}\left(\frac{g(z)}{z}\right) > {}^{1}\!/\!_{2}$. If $g \in K$ and $f \in T_\lambda^n(h,\alpha)$ then $f * g \in T_\lambda^n(h,\alpha)$.*

*Proof.* By observing that

$$(1-\alpha)\frac{\kappa_\lambda^n\,(f*g)\,(z)}{z}+\alpha(\kappa_\lambda^n\,(f*g)\,(z))'=\frac{g(z)}{z}*\left[(1-\alpha)\frac{\kappa_\lambda^n f(z)}{z}+\alpha(\kappa_\lambda^n f(z))'\right]$$

and by applying the same methods in the proof of Theorem 7 we get Theorem 18.

**Theorem 19.** *Let $h \in N$, $n \in \mathbb{N}_0$, $\alpha \geq 0$ and* $\operatorname{Re}\left(\frac{g(z)}{z}\right) > 1/2$. *If $g \in K$ and $f \in R_\lambda^n(h,\alpha)$ then $f * g \in R_\lambda^n(h,\alpha)$.*

*Proof.* If $f \in R_\lambda^n(h,\alpha)$ then, from Theorem 10 we have $zf' \in T_\lambda^n(h,\alpha)$ and using Theorem 18, we obtain $zf' * g \in T_\lambda^n(h,\alpha)$. Therefore

$$zf'(z) * g(z) = z(f*g)'(z) \in T_\lambda^n(h,\alpha).$$

By applying Theorem 10 again, we conclude that $f * g \in R_\lambda^n(h,\alpha)$. The proof is complete.

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