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A SUBCLASS OF M-W-STARLIKE FUNCTIONS

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Abstract. In 1999, Kanas and Ronning introduced the classes of functions starlike and convex, which are normalized with $f(w) = f'(w) - 1 = 0$ and w is a fixed point in U. The aim of this paper is to continue the investigation of the univalent normalized with $f(w) = f'(w) - 1 = 0$, where w is a fixed point in U by using the method of Briot-Bouquet differentail subordination.

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1. Introduction

Let $H(U)$ be the class of functions which are regular in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, $A = \{f \in H(U) : f(0) = f'(0) - 1 = 0\}$ and $S = \{f \in A : f$ is univalent in $U\}$.

We recall here the definitions of the well-known classes of starlike and convex functions:

$$S^* = \left\{ f \in A : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \right\},$$

$$S^c = \left\{ f \in A : \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in U \right\}.$$

Let w be a fixed point in U and $A(w) = \{f \in H(U) : f(w) = f'(w) - 1 = 0\}$.

In [7], Kanas and Ronning introduced the following classes:

$$S(w) = \{f \in A(w) : f \text{ is univalent in } U\},$$

$$ST(w) = S^*(w) = \left\{ f \in S(w) : \operatorname{Re} \frac{(z-w)f'(z)}{f(z)} > 0,\ z \in U \right\},$$

$$CV(w) = S^c(w) = \left\{ f \in S(w) : 1 + \operatorname{Re} \frac{(z-w)f''(z)}{f'(z)} > 0, z \in U \right\}.$$

It is obvious that the natural " Alexander relation " between the classes $S^*(w)$ and $S^c(w)$ is as follows:

$$g \in S^c(w) \ \Leftrightarrow \ f(z) = (z-w)g'(z) \in S^*(w). \tag{1.1}$$

It is easy to see that a function $f \in A(w)$ has the series of expansion:

$$f(z) = (z-w) + a_2(z-w)^2 + \dots . \tag{1.2}$$

In [2], Acu and Owa defined the following classes:

$$D(w) = \left\{ z \in U : \operatorname{Re}\left(\frac{w}{z}\right) < 1 \text{ and } \operatorname{Re}\left[\frac{z(1+z)}{(z-w)(1-z)}\right] > 0 \right\}, \text{for } D(0) = U;$$

$$s(w) = \{f : D(w) \longrightarrow \mathbb{C}\} \cap S(w), s^*(w) = S^*(w) \cap s(w)$$

where w is a fixed point in U.

Also Acu and Owa [2] considerd the integral operator $L_a : A(w) \longrightarrow A(w)$ defined by

$$f(z) = L_a F(a) = \frac{1+a}{(z-w)^a} \int_w^z F(t)(t-w)^{a-1} dt, \quad a \in \mathbb{R},\ a \geq 0. \tag{1.3}$$

Let $f \in A(w)$, w *be a fixed point in* U, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\}$, $\lambda \geq 0$ and $l \geq 0$, we define the following differential operator $I_w^m(\lambda, l) : A(w) \longrightarrow A(w)$ as follows:

$$I_w^0(\lambda, l) f(z) = f(z), \tag{1.4}$$

$$I_w^1(\lambda, l) f(z) = I_w(\lambda, l) f(z) = I_w^0(\lambda, l) f(z) \frac{(1-\lambda+l)}{(1+l)} + \left(I_w^0(\lambda, l) f(z)\right)' \frac{\lambda(z-w)}{(1+l)}$$

$$= (z-w) + \sum_{n=2}^{\infty} \left(\frac{1+\lambda(n-1)+l}{1+l}\right) a_n (z-w)^n, \tag{1.5}$$

$$I_w^2(\lambda, l) f(z) = I_w(\lambda, l) f(z) \frac{(1-\lambda+l)}{(1+l)} + \left(I_w(\lambda, l) f(z)\right)' \frac{\lambda(z-w)}{(1+l)}$$

$$= (z-w) + \sum_{n=2}^{\infty} \left(\frac{1+\lambda(n-1)+l}{1+l}\right)^2 a_n (z-w)^n \tag{1.6}$$

and (in general)

$$I_w^m(\lambda,l)f(z) = I_w(\lambda,l)(I_w^{m-1}(\lambda,l)f(z))$$
$$= (z-w) + \sum_{n=2}^{\infty}\left(\frac{1+\lambda(n-1)+l}{1+l}\right)^m a_n(z-w)^n \ (m\in\mathbb{N}_0;\lambda\geq 0; l\geq 0). \tag{1.7}$$

From (1.7) it is easy to verify that

$$\lambda\,(z-w)\,(I_w^m\,(\lambda,l)\,f(z))' = (1+l)I_w^{m+1}(\lambda,l)f(z) - (1-\lambda+l)\,I_w^m\,(\lambda,l)\,f(z) \ \ (\lambda>0), \tag{1.8}$$
$$I_w^{m_1}(\lambda,l)(I_w^{m_2}(\lambda,l)f(z)) = I_w^{m_2}(\lambda,l)(I_w^{m_1}(\lambda,l)f(z)),$$

for all integers $m_1$and m_2.

Remark 1. *(i) For* $\lambda=1$ and $l=0$, *the operator* $D_w^m = I_w^m(1,0)$ *was introduced and studied by Acu and Owa [3];*

(ii) For $w=0$ *the operator* $I^m(\lambda,l) = I_0^m(\lambda,l)$ *was introduced and studied by Cătaş et al. [4];*

(iii) For $w=0, l=0$ *and* $\lambda\geq 0$, *the operator* $D_\lambda^m = I_0^m(\lambda,0)$ *was introduced and studied by Al-Oboudi [1];*

(iv) For $w=0, l=0$ *and* $\lambda=1$, *the operator* $D^m = I_0^m(1,0)$ *was introduced and studied by Sălăgean [11];*

(v) For $w=0$ *and* $\lambda=1$, *The operator* $I^m(l) = I_0^m(1,l)$ *was studied recently by Cho and Kim [5] and Cho and Srivastava [6];*

(vi) For $w=0$ *and* $l=\lambda=1$, *The operator* $I_m = I_0^m(1,1)$ *was studied by Uralegaddi and Somanatha [12].*

Definition 1. *Let* w *be a fixed point in* U, $m\in\mathbb{N}_0$, $\lambda\geq 0$, $l\geq 0$ *and* $f\in S(w)$. *Then the function* $f(z)$ *is said to be an* l-λ-m-w-*starlike function if*

$$\operatorname{Re}\left\{\frac{I_w^{m+1}(\lambda,l)f(z)}{I_w^m(\lambda,l)f(z)}\right\} > 0, \ z\in U. \tag{1.9}$$

The class of all these functions is denoted by $S_m^*(\lambda,l,w)$.

Remark 2. *(i)* $S_m^*(1,0,w) = S_m^*(w), m\in\mathbb{N}_0$, *where* $S_m^*(w)$ *is the class of* $m-w-$*starlike functions introduced by Acu and Owa [3];*
(ii) $S_0^*(1,0,w) = S^*(w)$ *and* $S_m^*(1,0,0) = S_m^*$, $m\in\mathbb{N}_0$, *where* S_m^* *is the class of* $m-$*starlike functions introduced by Sălăgean [11];*
(iii) If $f\in S_m^*(\lambda,l,w)$ *and we denote* $I_w^m(\lambda,l)f(z) = g(z)$, *we obtain* $g(z)\in S^*(w)$;
(iv) Using the class $s(w)$, *we obtain* $s_m^*(\lambda,l,w) = S_m^*(\lambda,l,w)\cap s(w)$.

Also we note that:
(i) $S_m^*(\lambda,0,w)=P_m^*(\lambda,w)$

$$=\left\{f\in S(w):\mathrm{Re}\left\{\frac{I_w^{m+1}(\lambda)f(z)}{I_w^m(\lambda)f(z)}\right\}>0,\lambda\geq 0,m\in\mathbb{N}_0,z\in U\right\};\qquad(1.10)$$

where

$$I_w^m(\lambda)f(z)=(z-w)+\sum_{n=2}^{\infty}[1+\lambda(n-1)]^m\,a_n(z-w)^n;$$

(ii) $S_m^*(1,l,w)=P_m^*(l,w)$

$$=\left\{f\in S(w):\mathrm{Re}\left\{\frac{I_w^{m+1}(l)f(z)}{I_w^m(l)f(z)}\right\}>0\quad,l\geq 0,m\in\mathbb{N}_0,z\in U\right\};\qquad(1.11)$$

where

$$I_w^m(l)f(z)=(z-w)+\sum_{n=2}^{\infty}\left(\frac{n+l}{1+l}\right)^m a_n(z-w)^n.$$

2. Main results

In order to prove our main results, we shall need the following lemmas.

Lemma 1 [7]. *Let $f\in S^*(w)$ and $f(z)=(z-w)+b_2(z-w)^2+\ldots$. Then*

$$|b_2|\leq\frac{2}{1-d^2},\qquad |b_3|\leq\frac{3+d}{(1-d^2)^2},$$
$$|b_4|\leq\tfrac{2}{3}.\frac{(2+d)\,(3+d)}{(1-d^2)^3},\qquad |b_5|\leq\tfrac{1}{6}.\frac{(2+d)\,(3+d)\,(3d+5)}{(1-d^2)^4},\qquad(2.1)$$

where $d=|w|$.

Remark 3. *It is clear that the above lemma provides bounds for the coefficients of functions in the class $S^c(w)$, due to the relation between $S^c(w)$ and $S^*(w)$.*

Lemma 2 (*[8], [9] and [10]*). *Let h be convex in U and $\mathrm{Re}[\beta h(z)+\gamma]>0$, $z\in U$. If $p\in H(U)$ with $p(0)=h(0)$ and p satisfied the Briot-Bouquet differentail subordination*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec h(z),\qquad(2.2)$$

then $p(z) \prec h(z)$.

Theorem 1. *Let* w *be a fixed point in* U *and* $m \in \mathbb{N}_0, l \geq 0$ and $\lambda > 0$. *If* $f \in s^*_{m+1}(\lambda, l, w)$ *then* $f \in s^*_m(\lambda, l, w)$. *This means*

$$s^*_{m+1}(\lambda, l, w) \subset s^*_m(\lambda, l, w). \tag{2.3}$$

Proof. Since $f \in s^*_{m+1}(\lambda, l, w)$, then we have $\operatorname{Re}\left\{\frac{I_w^{m+2}(\lambda,l)f(z)}{I_w^{m+1}(\lambda,l)f(z)}\right\} > 0, \ z \in U$. We denote $p(z) = \frac{I_w^{m+1}(\lambda,l)f(z)}{I_w^{m}(\lambda,l)f(z)}$, where $p(0) = 1$ and $p(z) \in H(U)$. By using (1.8), we obtain

$$\frac{I_w^{m+2}(\lambda,l)f(z)}{I_w^{m+1}(\lambda,l)f(z)} = \frac{I_w^1\left(I_w^{m+1}(\lambda,l)f(z)\right)}{I_w^1\left(I_w^{m}(\lambda,l)f(z)\right)}$$

$$= \frac{(1-\lambda+l)\, I_w^{m+1}(\lambda,l)\, f(z) + \lambda\,(z-w)\left(I_w^{m+1}(\lambda,l)\, f(z)\right)'}{(1-\lambda+l)\, I_w^{m}(\lambda,l)\, f(z) + \lambda\,(z-w)\left(I_w^{m}(\lambda,l)\, f(z)\right)'}$$

and

$$\begin{aligned} p'(z) &= \frac{\left(I_w^{m+1}(\lambda,l)f(z)\right)' I_w^{m}(\lambda,l)f(z) - \left(I_w^{m}(\lambda,l)f(z)\right)' I_w^{m+1}(\lambda,l)f(z)}{\left(I_w^{m}(\lambda,l)f(z)\right)^2}, \\ &= \frac{\left(I_w^{m+1}(\lambda,l)f(z)\right)'}{\left(I_w^{m}(\lambda,l)f(z)\right)'} . \frac{\left(I_w^{m}(\lambda,l)f(z)\right)'}{I_w^{m}(\lambda,l)f(z)} - p(z)\frac{\left(I_w^{m}(\lambda,l)f(z)\right)'}{I_w^{m}(\lambda,l)f(z)}. \end{aligned} \tag{2.4}$$

Thus, we have

$$\frac{\lambda(z-w)}{1+l}p'(z) = \left[p(z) - \frac{(1-\lambda+l)}{1+l}\right]\frac{\left(I_w^{m+1}(\lambda,l)f(z)\right)'}{\left(I_w^{m}(\lambda,l)f(z)\right)'} - \left[p(z) - \frac{(1-\lambda+l)}{1+l}\right]p(z),$$

$$\frac{\left(I_w^{m+1}(\lambda,l)f(z)\right)'}{\left(I_w^{m}(\lambda,l)f(z)\right)'} = p(z) + \frac{\lambda(z-w)p'(z)}{p(z)(1+l) - (1-\lambda+l)}. \tag{2.5}$$

Since $\operatorname{Re}\left\{\frac{\left(I_w^{m+2}(\lambda,l)f(z)\right)}{\left(I_w^{m+1}(\lambda,l)f(z)\right)}\right\} > 0$, we obtain

$$p(z) + \frac{\lambda(z-w)}{p(z)(1+l)}p'(z) \prec \frac{1+z}{1-z}$$

or

$$p(z) + \frac{zp'(z)}{\{(1+l)/\lambda\,[1-(w/z)]\}\,p(z)} \prec \frac{1+z}{1-z} \equiv h(z), \text{ with } h(0) = 1. \tag{2.6}$$

By hypothesis, we have $\operatorname{Re}\left\{\frac{(1+l)}{\lambda[1-(w/z)]}p(z)\right\} > 0$, and from Lemma 2, we obtain that, $p(z) \prec h(z)$ or $\operatorname{Re}\{p(z)\} > 0$. This means $f \in s_m^*(\lambda, l, w)$. ■

Remark 4. *From Theorem 1, we obtain*

$$s_m^*(\lambda, l, w) \subset s_0^*(1, 0, w) \subset S^*(w) \ (m \in \mathbb{N}; l \geq 0; \lambda > 0).$$

Theorem 2. *If $F(z) \in s_m^*(\lambda, l, w)$ $(\lambda > 0)$, then $f(z) = L_a F(z) \in S_m^*(\lambda, l, w)$ $(\lambda > 0)$, where L_a is the integral operator defined by* (1.3).

Proof. From (1.3), we obtain

$$(1+a)F(z) = af(z) + (z-w)f'(z). \tag{2.7}$$

By means of the application of the operator $I_w^{m+1}(\lambda, l)$, we obtain

$$\lambda(1+a)I_w^{m+1}(\lambda, l) F(z) = [\lambda a - (1-\lambda+l)] I_w^{m+1}(\lambda, l)f(z) + (1+l)I_w^{m+2}(\lambda, l)f(z) \ (\lambda > 0). \tag{2.8}$$

Similarly, be means of application of the operator $I_w^m(\lambda, l)$, we obtain

$$\lambda(1+a)I_w^m(\lambda, l)F(z) = [\lambda a - (1-\lambda+l)] I_w^m(\lambda, l)f(z) + (1+l)I_w^{m+1}(\lambda, l)f(z) \ (\lambda > 0). \tag{2.9}$$

Thus, we have

$$\frac{I_w^{m+1}(\lambda, l) F(z)}{I_w^m(\lambda, l) F(z)} = \frac{(1+l)\left(\frac{I_w^{m+2}(\lambda, l)f(z)}{I_w^{m+1}(\lambda, l)f(z)}\right) \cdot \left(\frac{I_w^{m+1}(\lambda, l)f(z)}{I_w^m(\lambda, l)f(z)}\right) + [\lambda a - (1-\lambda+l)]\left(\frac{I_w^{m+1}(\lambda, l)f(z)}{I_w^m(\lambda, l)f(z)}\right)}{(1+l)\left(\frac{I_w^{m+1}(\lambda, l)f(z)}{I_w^m(\lambda, l)f(z)}\right) + [\lambda a - (1-\lambda+l)]}. \tag{2.10}$$

Using the notation $p(z) = \frac{I_w^{m+1}(\lambda,l)f(z)}{I_w^m(\lambda,l)f(z)}$, with $p(0) = 1$, we have

$$\frac{\lambda(z-w)p'(z)}{p(z)(1+l)} = \frac{I_w^{m+2}(\lambda, l)f(z)}{I_w^{m+1}(\lambda, l)f(z)} - p(z) \tag{2.11}$$

or

$$\frac{I_w^{m+2}(\lambda, l)f(z)}{I_w^{m+1}(\lambda, l)f(z)} = p(z) + \frac{\lambda(z-w)p'(z)}{p(z)(1+l)}. \tag{2.12}$$

Thus, we have

$$\frac{I_w^{m+1}(\lambda, l) F(z)}{I_w^m(\lambda, l) F(z)} = p(z) + \frac{zp'(z)}{\frac{(1+l)}{\lambda[1-(w/z)]}p(z) + \frac{[\lambda a - (1-\lambda+l)]}{\lambda[1-(w/z)]}}. \tag{2.13}$$

Since $F(z) \in s_m^*(\lambda, l, w)$, we obtain

$$\frac{I_w^{m+1}(\lambda, l)\, F(z)}{I_w^m(\lambda, l)\, F(z)} \prec \frac{1+z}{1-z} \equiv h(z)$$

or

$$p(z) + \frac{zp'(z)}{\frac{(1+l)}{\lambda[1-(w/z)]}p(z) + \frac{[\lambda a-(1-\lambda+l)]}{\lambda[1-(w/z)]}} \prec h(z). \tag{2.14}$$

By hypothesis, we have $\operatorname{Re}\left\{\frac{(1+l)}{\lambda[1-(w/z)]}p(z) + \frac{[\lambda a-(1-\lambda+l)]}{\lambda[1-(w/z)]}\right\} > 0$ and from Lemma 2, we obtain $p(z) \prec h(z)$ or $\operatorname{Re}\left\{\frac{I_w^{m+1}(\lambda,l)f(z)}{I_w^m(\lambda,l)f(z)}\right\} > 0, z \in U$. This means $f(z) = L_a\ F(z) \in S_m^*(\lambda, l, w)$. ■

Remark 5. *(i) Putting $w = l = 0$ and $\lambda = 1$ in Theorem 2, we obtain that, the integral operator defined by (1.3) preserves the class of m-starlike functions;*

(ii) Putting $w = m = l = 0$ and $\lambda = 1$ in Theorem 2, we obtain the integral operator defined by (1.3) preserves the well known class of starlike functions.

Theorem 3. *Let w be a fixed point in U and $m \in \mathbb{N}_0, \lambda > 0,\ l \geq 0$ and $f \in S_m^*(\lambda, l, w)$ with $f(z) = (z-w) + \sum_{n=2}^{\infty} a_n(z-w)^n$. Then, we have*

$$\begin{aligned} |a_2| &\leq \frac{2}{(1-d^2)}\left(\frac{1+l}{1+\lambda+l}\right)^m, \\ |a_3| &\leq \frac{3+d}{(1-d^2)^2}\left(\frac{1+l}{1+2\lambda+l}\right)^m, \\ |a_4| &\leq \frac{2}{3}\frac{(2+d)(3+d)}{(1-d^2)^3}\left(\frac{1+l}{1+3\lambda+l}\right)^m, \\ |a_5| &\leq \frac{1}{6}\frac{(2+d)(3+d)(3d+5)}{(1-d^2)^4}\left(\frac{1+l}{1+4\lambda+l}\right)^m, \end{aligned} \tag{1}$$

where $d = |w|$.

Proof. From Remark 2 for $f \in S_m^*(\lambda, l, w)$, we obtain

$$D_{w,\lambda}^m f(z) = g(z) \in S^*(w). \tag{2.16}$$

If we consider $g(z) = (z-w) + \sum_{n=2}^{\infty} b_n(z-w)^n$, from (2.16) we obtain

$$\left(\frac{1+\lambda(n-1)+l}{1+l}\right)^m a_n = b_n,\ n = 2, 3, \ldots .$$

Thus, we have

$$a_n = \left(\frac{1+l}{1+\lambda(n-1)+l} \right)^m b_n, \ n = 2, 3, ...$$

and from the estimates (2.1) of Lemma 1 and Remark 1, we get the result. ■

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