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CONVOLUTION PROPERTIES OF HARMONIC UNIVALENT FUNCTIONS PRESERVED BY SOME INTEGRAL OPERATOR

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Abstract. A complex valued function $f = u + iv$ defined in a domain $D \subset \mathbb{C}$, is harmonic in D, if u and v are real harmonic. Such functions can be represented as $f(z) = h(z) + \overline{g(z)}$, where h an g are analytic in D. In this paper we study some convolution properties preserved by the integral operator $I^n_{H,\lambda} f, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \lambda > 0$, where the functions f are univalent harmonic and sense-preserving in the open unit disc $E = \{z : |z| < 1\}, I^n_{H,\lambda} f(z) = I^n_\lambda h(z) + \overline{I^n_\lambda g(z)}$, and $I^n_h h(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{[1+\lambda(k-1)]^n} z^k, I^n_\lambda g(z) = \sum_{k=1}^{\infty} \frac{b_k}{[1+\lambda(k-1)]^n} z^k$. Relevant connections of the results presented here with those obtained in earlier works are pointed out.

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1. Introduction

Let A denote the class of analytic functions in the open unit disc $E = \{z : |z| < 1\}$, with the normalization

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let $K(\alpha), R_\alpha$, and $C(\alpha)$ denote the classes of functions $f \in A$, which are, respectively, convex, prestarlike and close to convex of order α.

A continuous complex-valued function $f = u + iv$ defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D. There is a close inter-relation between analytic functions and harmonic functions. For example, for real harmonic functions u and v there exist analytic functions U and V so that $u = Re\,U$ and $v = Im\,V$. Then

$$f(z) = h(z) + \overline{g(z)},$$

where h and g are, respectively, the analytic functions $(U + V)/2$ and $(U - V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by

$$J_{f(z)} = |h'(z)|^2 - |g'(z)|^2,$$

The mapping $z \to f(z)$ is sense preserving and locally univalent in D if and only if $J_f > 0$ in D. See also Clune and Sheil-Small [4]. The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \to f(z)$ is sense preserving harmonic and univalent in D. We call h the analytic part and g the co-analytic part of $f = h + \overline{g}$.

Let S_H denote the family of functions $f = h + \overline{g}$ which are harmonic univalent and sense-preserving in E with the normalization

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k. \tag{1.1}$$

Let K_H and C_H denote the subclasses of S_H consisting of harmonic functions which are, respectively. convex and close to convex in E.

Finally, we define convolution of two complex-valued harmonic functions $f_1(z) = z + \sum_{k=2}^{\infty} a_{1k} z^k + \sum_{k=1}^{\infty} \overline{b_{1k}}\overline{z}^k$ and $f_2(z) = z + \sum_{k=2}^{\infty} a_{2k} z^k + \sum_{k=1}^{\infty} \overline{b_{2k}}\overline{z}^k$ by

$$f_1(z) * f_2(z) = z + \sum_{k=2}^{\infty} a_{1k} a_{2k} z^k + \sum_{k=1}^{\infty} \overline{b}_{1k}\overline{b}_{2k}\overline{z}^k.$$

The above convolution formula reduces to the Hadamard product if the co-analytic parts of f_1 and f_2 are zero.

In this paper we define and give some properties of the integral operator $I^n_{H,\lambda} f, n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda > 0$, constructed as the convolution $f \,\widetilde{*}\, \psi_n$ of harmonic univalent and sense-preserving functions f in the unit disc E with the prestarlike function ψ_n. We mainly study some convolution properties preserved by this operator.

2. Definitions and Preliminary Results

In the following we define the integral operator $I^n_{H,\lambda} f$ and derive some of its basic properties. We need the following definition.

Definition 1. *Let $f \in S_H$, and assume ψ is analytic in E. Then* $(f \,\widetilde{*}\, \psi)(z) = (\psi \,\widetilde{*}\, f)(z) = (h * \psi)(z) + \overline{(g * \psi)(z)}$.

Definition 2. *Let $f \in S_H$. Then the integral operator $I^n_{H,\lambda} f, n \in \mathbb{N}_0,\ \lambda > 0$, is defined by*

$$I^n_{H,\lambda} f(z) = I^n_\lambda h(z) + \overline{I^n_\lambda g(z)}, \tag{2.1}$$

where $I_\lambda^n f, \lambda > 0$ *is the multiplier integral operator defined [3] on* $f \in A$ *as follows*

$$\begin{aligned} I_\lambda^0 f(z) =& f(z) \\ I_\lambda^1 f(z) =& \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} f(t)dt \\ I_\lambda^2 f(z) =& \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^1 f(t)dt \\ & \cdots \\ I_\lambda^n f(z) =& \frac{1}{\lambda z^{\frac{1}{\lambda}-1}} \int_0^z t^{\frac{1}{\lambda}-2} I_\lambda^{n-1} f(t)dt, n \in \mathbb{N}. \end{aligned} \tag{2.2}$$

Remark 1. If the co-analytic part of $I_{H,\lambda}^n f$ is zero, then $I_{H,\lambda}^n f \equiv I_\lambda^n f$, [3]. When $\lambda = \dfrac{1}{1+\gamma}$, $\gamma > -1, I_\lambda^1 f$ is Bernardi integral operator. When $\lambda = 1$, $I_1^n f$ is Salagean integral operator [8].

Remark 2. The integral operator $I_{H,\lambda}^n f$ satisfies the following

(i) $I_{H,\lambda}^n (D_{H,\lambda}^n f(z)) = f(z)$,

where $D_{H,\lambda}^n f$ is harmonic differential operator defined by Li Shuai and Liu Peide [9].

(ii) $I_{H,\lambda}^n f(z) = (f \widetilde{*} \psi_n)(z)$ where

$$\begin{aligned} \psi_n(z) &= z + \sum_{k=2}^{\infty} \frac{1}{[1+\lambda(k-1)]^n} z^k, \\ &= \underbrace{(\psi * \psi \cdots * \psi)}_{n\text{-times}}(z) \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} \psi(z) &= z + \sum_{k=2}^{\infty} \frac{1}{[1+\lambda(k-1)]} z^k, \\ &= z {}_2F_1\left(1, \frac{1}{\lambda}; 1 + \frac{1}{\lambda}; z\right), \end{aligned} \tag{2.4}$$

where the function ${}_2F_1(a, b; c; z) = 1 + \dfrac{a \cdot b}{1 \cdot c} z + \dfrac{a(a+1) \cdot b(b+1)}{2! \cdot c(c+1)} z^2 + \cdots$, for any real or complex numbers a, b and c ($c \neq 0, -1, -2, \ldots$), is the well-known hypergeometric series which represents an analytic function E

(iii) $I^n_{H,\lambda} f(z) = z + \sum_{k=2}^{\infty} \frac{a_k}{[(1+\lambda(k-1)]^n} z^k + \sum_{k=1}^{\infty} \overline{\frac{b_k}{[1+\lambda(k-1)]^n} z^k}$,

(iv) $I^n_{H,\lambda} f(z) = (1-\lambda) I^{n+1}_{H,\lambda} f(z) + \lambda[z(I^{n+1}_{H,\lambda} f(z))_z + \overline{z}(I^{n+1}_{H,\lambda} f(z))_{\overline{z}}]$.

(v) The operator $I^n_{H,\lambda} f(z)$, given by (2.1), is sense-preserving, but may not be univalent in E. Consider $I^n_{H,\lambda} f(z) = H(z) + \overline{G(z)}$, then $|G'| = |(I^n_\lambda g)'| = |(g * \psi_n)'| = \left|\frac{1}{z}\psi_n * g'\right| < \left|\frac{1}{z}\psi_n * h'\right| = |(h * \psi_n)'| = |(I^n_\lambda h)'| = |H'|$, which shows that $I^n_{H,\lambda} f(z)$ is sense-preserving. To show that $I^n_{H,\lambda} f(z)$, may not be univalent in E, consider the case where the co-analytic part of $I^n_{H,\lambda} f$ is zero, and $n = 1, \lambda = \frac{1}{2}$. The function $f(z) = (1+i)[(1-z)^{-1+i} - 1] = (1+i)(e^{(-1+i)\ln(1-z)} - 1)$, given by Campbell and V. Singh [5] is normalized univalent in E, with this function f, equation (2.2) gives

$$I^1_{\frac{1}{2}} f(z) = \frac{2}{z}\int_0^z f(t)dt = 2i(1+i)\frac{(1-z)^i - 1}{z} - 2(1-i).$$

If we set $z_m = 1 - e^{-2\pi m}$, we find that $I^0_{\frac{1}{2}} f(z_m) = -2(1+i)$ for $m = 1, 2, \ldots$ Hence $I^0_{\frac{1}{2}} f(z)$ is infant-valent [5].

We will need the following lemmas.

Lemma 1. *[6] Let φ and ψ be convex analytic in E. Then*

(i) $\varphi * \psi$ is convex analytic in E.

(ii) $\varphi * f$ is close to convex analytic in E, if f is close to convex analytic in E.

Lemma 2. *[7] (i) Let $\alpha \le 1$ and $f, g \in R_\alpha$. Then $f * g \in R_\alpha$.*
(ii) For $\alpha < \beta \le 1$, we have $R_\alpha \subset R_\beta$.
(iii) For $\alpha < 1$, let $f \in C(\alpha)$ and $g \in R_\alpha$. Then $f * g \in C(\alpha)$.

Lemma 3. *[4] Let $h \in A$ and $g \in A$.*

(i) if $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close to convex in E, for each $\epsilon(|\epsilon| = 1)$, then $f = h + \overline{g} \in C_H$.

(ii) If $f = h + \overline{g}$ is harmonic and locally univalent in E, and if $h + \epsilon g$ is convex analytic in E for some $\epsilon(|\epsilon| = 1)$, then $f = h + \overline{g} \in C_H$.

Lemma 4. *Let ψ be as in (2.4). Then*

(i) $\psi \in R_{\left(1-\frac{1}{\lambda}\right)}$.

(ii) $\dfrac{\psi(z)}{z}$ *is convex univalent in E.*

For (i) we refer to [7] and for (ii) see [2].

From Lemma 2 and Lemma 4(i) we obtain

Corollary 1. $\psi_n \in R_{\left(1-\frac{1}{\lambda}\right)}$.

3. Main Results

We now state and prove our main results.

Theorem 1. *Let $f = h+\overline{g}$, where h and g are given by (1.1). If $|g'(0)| < |h'(0)|$ and $(h+\epsilon g) \in C\left(1-\frac{1}{\lambda}\right)$, $\lambda \geq 1$ for each $\epsilon(\epsilon| = 1)$, then $I^n_{H,\lambda} f \in C_H$.*

Proof.
$$\begin{aligned} I^n_\lambda(h+\epsilon g) &= (h+\epsilon g) * \psi_n \\ &= I^n_\lambda h + \epsilon I^n_\lambda g. \end{aligned} \tag{3.1}$$

Since $(h+\epsilon g) \in C\left(1-\dfrac{1}{\lambda}\right)$, and $\psi_n \in R_{\left(1-\frac{1}{\lambda}\right)}$, applying Lemma 2(ii), we obtain $(I^n_\lambda h + \epsilon I^n_\lambda g) \in C\left(1-\dfrac{1}{\lambda}\right)$. Since $1-\dfrac{1}{\lambda} \geq 0$, for $\lambda \geq 1$ then $I^n_\lambda h + \epsilon I^n_\lambda g) \in C(0)$, the class of analytic close to convex functions in E. Applying Lemma 3(i), we get the required result.

Theorem 2. *If $\lambda \leq 1, I^n_{H,\lambda} f = I^n_\lambda h + \overline{I^n_\lambda g}$ is harmonic and locally univalent in E, and if $h+\epsilon g$ is convex analytic in E for some $\epsilon(|\epsilon| = 1)$, then $I^n_{H,\lambda} f \in C_H$.*

Proof. Since $\psi_n \in R_{\left(1-\frac{1}{\lambda}\right)}$, and $\lambda \leq 1$, implies $1-\dfrac{1}{\lambda} \leq 0$, then applying Lemma 2(ii), we get $\psi_n \in R_0 \equiv K(0)$. Since $h+\epsilon g \in K(0)$, then from (3.1) and Lemma 1(i), we deduce that $I^n_\lambda h + \epsilon I^n_\lambda g$, is convex analytic in E. Applying Lemma 3(ii), we get the desired result.

Remark 1. For $n = 0$, Theorems 1 and 2 reduce to Clune and Sheil-Small results, given in Lemma 3.

Theorem 3. *Let $f = h + \overline{g}$, where h and g are given by (1.1), such that $|g'(0)| < |h'(0)|$ and $h + \epsilon g$ is close to convex analytic in E, for each $\epsilon(|\epsilon| = 1)$. If φ is convex analytic in E, then for $\lambda \geq 1$,*

$$(\varphi + \overline{\sigma\varphi}) * I^n_{H,\lambda} f \in C_H, |\sigma| = 1.$$

Proof. Let $(\varphi + \overline{\sigma\varphi}) * (I^n_\lambda h(z) + \overline{I^n_\lambda g(z)}) = \varphi * I^n_\lambda h + \overline{\sigma\varphi * I^n_\lambda g} = H + \overline{G}$. Now

$$\begin{aligned} H + \gamma G &= \varphi * I^n_\lambda h + \gamma\sigma\varphi * I^n_\lambda g \\ &= \varphi * (I^n_\lambda h + \epsilon I^n_\lambda g), \epsilon = \gamma\sigma. \end{aligned}$$

From the proof of Theorem 1, we see that $(I^n_\lambda h(z) + \epsilon I^n_\lambda g(z))$, is close to convex analytic function for $\lambda \geq 1$ and for each ϵ $(|\epsilon| = 1)$. Applying Lemma 3(i), we deduce that $H + \gamma G$ is close to convex analytic function for each $\gamma(|\gamma| = 1)$. Next we show that $|G'(0)| < |H'(0)|$,

$$\begin{aligned} |G'(0)| &= |(\sigma\varphi * I^n_\lambda g)'|_{z=0} = \left|\frac{1}{z}\varphi * \sigma(I^n_\lambda g)'\right|_{z=0} \\ &< \left|\frac{1}{z}\sigma\varphi * (I^n_\lambda h)'\right|_{z=0} = |(\varphi * I^n_\lambda h)'|_{z=0} = |H'(0)|. \end{aligned}$$

By Theorem 1, we obtain $H + \overline{G} = (\varphi + \overline{\sigma\varphi}) * I^n_{H,\lambda} f \in C_H$.

Remark 2. For $n = 0$, Theorem 3 reduces to the results of Ahuja and Jahangiri [1].

Theorem 4. *Let $f = h + \overline{g}$, where h and g are given by (1.1). Then $\dfrac{I^n_{H,\lambda} f(z)}{z} = \displaystyle\int_0^1 t^{-\lambda} I^{n-1}_{H,\lambda} f(zt^\lambda)dt$.*

Proof. From (2.4), ψ can be written as $\psi(z) = \displaystyle\int_0^z \frac{z\,dt}{1-zt^\lambda}$. Since $\left(h * \dfrac{z}{1-zt^\lambda}\right)(z) = \dfrac{h(t^\lambda z)}{t^\lambda}$ and $\left(g * \dfrac{z}{1-zt^\lambda}\right)(z) = \dfrac{g(t^\lambda z)}{t^\lambda}$, then $(I^{n-1}_\lambda h * \psi)(z) = \displaystyle\int_0^1 zt^{-\lambda} I^{n-1}_\lambda h(t^\lambda z)dt$, and $(I^{n-1}_\lambda g * \psi)(z) = \displaystyle\int_0^1 zt^{-\lambda} I^{n-1}_\lambda g(t^\lambda z)dt$. Therefore (2.1) gives

$$I^n_{H,\lambda} f(z) = \int_0^1 zt^{-\lambda}(I^{n-1}_\lambda h(t^\lambda z) + I^{n-1}_\lambda g(t^\lambda z))dt.$$

Hence

$$\frac{I^n_{H,\lambda} f(z)}{z} = \int_0^1 t^{-\lambda} I^{n-1}_{H,\lambda} f(t^\lambda z) dt.$$

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