Acta Universitatis Apulensis No. 24/2010
ISSN: 1582-5329 pp. 295-306

A SUBCLASS OF UNIFORMLY CONVEX FUNCTIONS ASSOCIATED WITH A FRACTIONAL CALCULUS OPERATOR INVOLVING CAPUTO'S FRACTIONAL DIFFERENTIATION

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Abstract. In this paper, we introduce a new class of functions which are analytic and univalent with negative coefficients defined by using certain fractional operators descibed in the Caputo sense. Characterization property, the results on modified Hadamard product and integral transforms are discussed. Further, distortion theorem and radii of starlikeness and convexity are also determined here.

Keywords: Caputo's differentiation operator; uniformly starlike; Hadamard product

1. Introduction

In recent years, considerable interest in fractional calculus operators has been stimulated due to their applications in the theory of analytic functions. There are many definitions of fractional integration and differentiation can be found in various books ([11]-[14]). For the purpose of this paper, the Caputo's definition of fractional differentiation will be used to introduce a new operator.

Denote by A the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disk $U = \{z : |z| < 1\}$. Also denote by T the subclass of A consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0. \tag{2}$$

Perhaps, Silverman [10] was the first to define the class T. For further properties for the class T, see also [9].

A function $f \in A$ is said to be in the class of uniformly convex functions of order α, denoted by $UCV(\alpha)$ if

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta \left|\frac{zf''(z)}{f'(z)} - 1\right|. \tag{3}$$

And is said to be in a corresponding subclass of $UCV(\alpha)$ denoted by $S_p(\alpha)$ if

$$\Re\left\{\frac{zf'(z)}{f(z)}-\alpha\right\}\geq\beta\left|\frac{zf'(z)}{f(z)}-1\right|, \tag{4}$$

$-1\leq\alpha\leq 1$ and $z\in U$.

The class of uniformly convex and uniformly starlike functions has been studied by Goodman, see ([2],[3]) and Ma and Minda [5]. If f of the form (1) and $g(z)=z+\sum_{n=2}^{\infty}b_nz^n$ are two functions in A. Then the Hadamard product of f and g is denoted by $f*g$ and is given by

$$(f*g)(z)=z+\sum_{n=2}^{\infty}a_nb_nz^n \tag{5}$$

Now we look at the Caputo's definition which shall be used throughout the paper.

Definition 1.*[15] Caputo's definition of the fractional-order derivative is defined as*

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}}d\tau \tag{6}$$

where $n-1<Re(\alpha)\leq n, n\in N$, and the parameter α is allowed to be real or even complex, a is the initial value of the function f .

Definition 2. *The generalization operator of Salagean[17] derivative operator and Libera integral operator [16], was given by Owa [6].*

$\Omega^{\lambda}f(z)=\Gamma(2-\lambda)z^{\lambda}D_z^{\lambda}f(z)=z+\sum_{n=2}^{\infty}\frac{\Gamma(2-\lambda)\Gamma(n+1)}{\Gamma(n-\lambda+1)}a_nz^n$, *for any real* λ.

Remark 1. We note that

$$\Omega^0f(z)=f(z)=z+\sum_{n=2}^{\infty}a_nz^n,$$

$$\Omega^1f(z)=\Omega f(z)=zf'(z)=z+\sum_{n=2}^{\infty}na_nz^n$$

and

$$\Omega^kf(z)=\Omega(\Omega^{k-1}f(z))=zf'(z)=z+\sum_{n=2}^{\infty}n^ka_nz^n \quad (k=1,2,3...)$$

and this is the Salagean derivative operator [17].

Remark 2. We also note that

$$\Omega^{-1}f(z)=\frac{2}{z}\int_0^z f(t)dt=z+\sum_{n=2}^{\infty}\frac{2}{n+1}a_nz^n$$

and

$$\Omega^{-k}f(z)=\Omega^{-1}(\Omega^{-k+1}f(z))=zf'(z)=z+\sum_{n=2}^{\infty}(\frac{2}{n+1})^ka_nz^n \quad (k=1,2,3...)$$

and this is the Libera integral operator [16].

Now, using the previous definitions we can introduce the following operator:

$$J_{\eta,\lambda}f\left(z\right)=\frac{\Gamma\left(2+\eta-\lambda\right)}{\Gamma\left(\eta-\lambda\right)}z^{\lambda-\eta}\int_0^z\frac{\Omega^{\eta}f\left(\xi\right)}{\left(z-\xi\right)^{\lambda+1-\eta}}d\xi \tag{7}$$

where η(real number) and $(\eta-1<\lambda\leq\eta<2)$.

By simple calculations we can write

$$J_{\eta,\lambda}f\left(z\right)=z+\sum_{n=2}^{\infty}\frac{\left(\Gamma\left(n+1\right)\right)^2\Gamma\left(2+\eta-\lambda\right)\Gamma\left(2-\eta\right)}{\Gamma\left(n+\eta-\lambda+1\right)\Gamma\left(n-\eta+1\right)}a_nz^n \tag{8}$$

for $f\left(z\right)\in A$ and has the form (1).

Further, note that $J_{0,0}f\left(z\right)=f\left(z\right)$ and $J_{1,1}f\left(z\right)=zf^{'}\left(z\right)$.

Now using the operator introduced in (7), we can define the following subclass of analytic function. For $-1\leq\alpha<1$, a function $f\in A$ is said to be in the class $S^*_{\eta,\lambda}\left(\alpha\right)$ if and only if

$$\Re\left\{\frac{z\left(J_{\eta,\lambda}f\left(z\right)\right)^{'}}{J_{\eta,\lambda}f\left(z\right)}-\alpha\right\}\geq\left|\frac{z\left(J_{\eta,\lambda}f\left(z\right)\right)^{'}}{J_{\eta,\lambda}f\left(z\right)}-1\right|, z\in U. \tag{9}$$

Now let's write $TR\left(\eta,\lambda,\alpha\right)=S^*_{\eta,\lambda}\left(\alpha\right)\bigcap T$.

Note that if $\eta=\lambda=0$, we have

$$S^*_{\eta,\lambda}\left(\alpha\right)=S_p\left(\alpha\right).$$

And for $\eta = \lambda = 1$, we have

$$S^{*}_{\eta,\lambda}(\alpha) = UCV(\alpha).$$

The classes $S_p(\alpha)$and $UCV(\alpha)$ are introduced and studied by various authors including [1],[7],and [8].

2. Characterization Property

Definition 3. *A function f is in $TR(\eta,\lambda,\alpha)$if $f$satisfies the analytic characterization*

$$\Re\left\{\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - \alpha\right\} > \left|\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right| \tag{10}$$

where $0 \le \alpha < 1$,$\eta - 1 < \lambda \le \eta < 2$.

Theorem 1. *A function f defined by (2) is in the class $TR(\eta,\lambda,\alpha)$ if and only if*

$$\sum_{n=2}^{\infty} \frac{(\Gamma(n+1))^2\,\Gamma(2-\eta)\,\Gamma(2+\eta-\lambda)}{\Gamma(n+\eta-\lambda+1)\,\Gamma(n-\eta+1)} \cdot \frac{2n-1-\alpha}{1-\alpha}\,|a_n| \le 1. \tag{11}$$

Proof. It suffices to show that

$$\left|\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right| \le \Re\left\{\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - \alpha\right\}$$

and we have

$$\left|\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right| \le \Re\left\{\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right\} + (1-\alpha)$$

that is

$$\left|\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right| - \Re\left\{\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right\} \le 2\left|\frac{z\left(J_{\eta,\lambda}f(z)\right)'}{J_{\eta,\lambda}f(z)} - 1\right| \le \frac{\sum_{n=2}^{\infty}(n-1)\phi(n)|a_n|}{1-\sum_{n=2}^{\infty}\phi(n)|a_n|}$$

where

$\phi(n) = \frac{(\Gamma(n+1))^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$.

The above expression is bounded by $(1-\alpha)$ and hence the assertion of the result.

Now we want to show that $f \in TR(\eta,\lambda,\alpha)$ satisfies (11) if $f \in TR(\eta,\lambda,\alpha)$, then (10) yields

$\frac{1-\sum_{n=2}^{\infty} n\phi(n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \phi(n)a_n z^{n-1}} - \alpha \geq \frac{1-\sum_{n=2}^{\infty}(n-1)\phi(n)a_n z^{n-1}}{1-\sum_{n=2}^{\infty} \phi(n)a_n z^{n-1}}.$

Letting $z \to 1$ along the real axis leads to the inequality $\sum_{n=2}^{\infty} (2n-1-\alpha)\,\phi(n)\,a_n \leq 1-\alpha$.

Corollary 1. *Let a function f, defined by (2) belongs to the class $TR(\eta,\lambda,\alpha)$. Then*

$$a_n \leq \frac{\Gamma(n+\eta-\lambda+1)\,\Gamma(n-\eta+1)}{(\Gamma(n+1))^2\,\Gamma(2+\eta-\lambda)\,\Gamma(2-\eta)} \cdot \frac{1-\alpha}{2n-1-\alpha}, \text{for } n \geq 2.$$

Next we consider the growth and distortion theorems for the class $TR(\eta,\lambda,\alpha)$. We shall omit the proof as the techniques are similar to various other papers.

Theorem 2. *Let the function f, defined by (2) be in the class $TR(\eta,\lambda,\alpha)$. Then*

$$|z| - |z|^2 \frac{(2+\eta-\lambda)(2-\eta)(1-\alpha)}{4(3-\alpha)} \leq |J_{\eta,\lambda} f(z)| \leq |z| + |z|^2 \frac{(2+\eta-\lambda)(2-\eta)(1-\alpha)}{4(3-\alpha)} \quad (12)$$

$$1 - |z| \frac{(2+\eta-\lambda)(2-\eta)(1-\alpha)}{2(3-\alpha)} \leq \left|(J_{\eta,\lambda} f(z))'\right| \leq 1 + |z| \frac{(2+\eta-\lambda)(2-\eta)(1-\alpha)}{2(3-\alpha)} \quad (13)$$

The bounds (12) and (13) are attained for the functions given by

$$f(z) = z - \frac{(2+\eta-\lambda)(2-\eta)(1-\alpha)}{4(3-\alpha)} z^2. \quad (14)$$

Theorem 3. *Let a function f, be defined by (2) and*

$$g(z) = z - \sum_{n=2}^{\infty} b_n z^n \quad (15)$$

be in the class $TR(\eta,\lambda,\alpha)$. then the function h, defined by

$$h(z) = (1-\beta) f(z) + \beta g(z) = z - \sum_{n=2}^{\infty} c_n z^n \quad (16)$$

where $c_n = (1-\beta)\, a_n + \beta b_n$,*and* $0 \leq \beta \leq 1$,*is also in the class* $TR(\eta, \lambda, \alpha)$.

Proof. The result follows easily by using (11) and (16).

Now we define the following functions $f_j(z)$,$(j = 1, 2, 3, ...m)$ of the form

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,j} z^n, a_{n,j} \geq 0, z \in U. \tag{17}$$

Theorem 4. (Closure theorem)*Let the functions* $f_j(z), (j = 1, 2, 3, ...m)$ *defined by (16), be in the class* $TR(\eta, \lambda, \alpha_j)$ $(j = 1, 2, 3, ...m)$ *respectively. Then the function* $h(z)$ *defined by* $h(z) = z - \frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m} a_{n,j}\right) z^n$

is in the class $TR(\eta, \lambda, \alpha)$ *where*

$$\alpha = \min_{1 \leq j \leq m} \{\alpha_j\}\, with 0 \leq \alpha_j < 1. \tag{18}$$

Proof. Since $f_j \in TR(\eta, \lambda, \alpha_j)$ $(j = 1, 2, 3, ...m)$ by applying Theorem 1, we observe that

$\sum_{n=2}^{\infty} \phi(n)(2n-1-\alpha)\left(\frac{1}{m}\sum_{j=1}^{m} a_{n,j}\right) = \frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}\phi(n)(2n-1-\alpha)\, a_{n,j}\right).$

In view of Theorem 1 again implies that $h(z) \in TR(\eta, \lambda, \alpha)$.

3. Results involving convolution

Theorem 5. *For functions* $f_j(z)$ $(j = 1, 2)$ *defined by (17) let* $f_1(z) \in TR(\eta, \lambda, \alpha)$ *and* $f_2(z) \in TR(\eta, \lambda, \beta)$*. then* $f_1 * f_2 \in TR(\eta, \lambda, \gamma)$*, where*

$$\gamma \leq 1 - \frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\,\phi(n) - (1-\alpha)(1-\beta)}, \tag{19}$$

$n \geq 2$ *and* $\phi(n) = \frac{(\Gamma(n+1))^2\Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1)}$

Proof. In view of Theorem 1, it suffices to prove that

$$\sum_{n=2}^{\infty} \frac{2n-1-\gamma}{1-\gamma}\phi(n)\, a_{n,1} a_{n,2} \leq 1.$$

It follows from Theorem 1 and the Cauchy-Shwarz inequality that

$$\sum_{n=2}^{\infty}\frac{\sqrt{2n-1-\alpha}\cdot\sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\phi(n)\sqrt{a_{n,1}a_{n,2}}\leq 1.$$

Thus we need to find γ such that

$$\sum_{n=2}^{\infty}\frac{2n-1-\gamma}{1-\gamma}\phi(n)\,a_{n,1}a_{n,2}\leq\sum_{n=2}^{\infty}\frac{\sqrt{2n-1-\alpha}\cdot\sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\phi(n)\sqrt{a_{n,1}a_{n,2}}\leq 1$$

or

$$\sqrt{a_{n,1}a_{n,2}}\leq\sum_{n=2}^{\infty}\frac{\sqrt{2n-1-\alpha}\cdot\sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\cdot\frac{1-\gamma}{2n-1-\gamma}$$

by virtue of (19), it suffices to find $\phi(n)$ such that

$$\frac{\sqrt{(1-\alpha)(1-\beta)}}{\sqrt{2n-1-\alpha}\sqrt{2n-1-\beta}\phi(n)}\leq\frac{\sqrt{2n-1-\alpha}\sqrt{2n-1-\beta}}{\sqrt{(1-\alpha)(1-\beta)}}\cdot\frac{1-\gamma}{2n-1-\gamma}$$

which yield

$$\gamma\leq 1-\frac{2(n-1)(1-\alpha)(1-\beta)}{(2n-1-\alpha)(2n-1-\beta)\phi(n)-(1-\alpha)(1-\beta)},n\geq 2.$$

Theorem 6. *Let the functions $f_j(z),(j=1,2)$ defined by (16) be in the class $TR(\eta,\lambda,\alpha)$. Then $(f_1*f_2)(z)\in TR(\eta,\lambda,\delta)$ where $\delta\leq 1-\frac{2(n-1)(1-\alpha)^2}{(2n-1-\alpha)^2\phi(n)-(1-\alpha)^2}$*

Proof. By taking $\beta=\alpha$ in the above theorem, the result follows.

Theorem 7. *Let the function f defined by (2) be in the class $TR(\eta,\lambda,\alpha)$, and let $g(z)=z-\sum_{n=2}^{\infty}b_nz^n$, for $|b_n|\leq 1$. Then $(f*g)(z)\in TR(\eta,\lambda,\alpha)$.*

Proof.

$$\sum_{n=2}^{\infty}\phi(n)(2n-1-\alpha)|a_nb_n|=\sum_{n=2}^{\infty}\phi(n)(2n-1-\alpha)a_n|b_n|$$

$$\leq\sum_{n=2}^{\infty}\phi(n)(2n-1-\alpha)a_n\leq 1-\alpha.$$

Hence it follows that $(f * g)(z) \in TR(\eta, \lambda, \alpha)$.

Corollary 2. *Let the function f defined by (2) be in the class $TR(\eta, \lambda, \alpha)$. Also let $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ for $0 \leq b_n \leq 1$. Then $(f * g)(z) \in TR(\eta, \lambda, \alpha)$.*

Theorem 8. *Let the functions $f_j(z), (j = 1, 2)$ defined by (16) be in the class $TR(\eta, \lambda, \alpha)$. Then the function h defined by $h(z) = z - \sum_{n=2}^{\infty} \left(a_{n,1}^2 + a_{n,2}^2\right) z^n$ is in the class $TR(\eta, \lambda, \mu)$ where*

$$\mu \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2 \phi(n) - 2(1-\alpha)^2}, n \geq 2 \tag{20}$$

Proof In view of Theorem 1, it suffices to prove that

$$\sum_{n=2}^{\infty} \phi(n) \frac{2n-1-\mu}{1-\mu} \left(a_{n,1}^2 + a_{n,2}^2\right) \leq 1 \tag{21}$$

From (16) and Theorem 1 we get,

$$\sum_{n=2}^{\infty} \left[\phi(n) \frac{2n-1-\alpha}{1-\alpha}\right]^2 a_{n,j}^2 \leq \sum_{n=2}^{\infty} \left[\phi(n) \frac{2n-1-\alpha}{1-\alpha} a_{n,j}\right]^2 \leq 1 \tag{22}$$

On comparing (21) and (22), it can be seen that inequality (20) will be satisfied if

$$\phi(n) \frac{2n-1-\mu}{1-\mu} \left(a_{n,1}^2 + a_{n,2}^2\right) \leq \frac{1}{2} \left[\phi(n) \frac{2n-1-\alpha}{1-\alpha}\right] \left(a_{n,1}^2 + a_{n,2}^2\right)$$

that is if

$$\mu \leq 1 - \frac{4(1-\alpha)^2}{(2n-1-\alpha)^2 \phi(n) - 2(1-\alpha)^2}$$

which completes the proof.

4. Integral Transform of the class $TR(\eta, \lambda, \alpha)$

We define the integral transform

$$V_\mu(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt$$

where $\mu(t)$ is a real valued, non-negative weight function normalized so that $\int_0^1 \mu(t)\,dt = 1$.

Special case of $\mu(t)$ is $\mu(t) = \frac{(c+1)^{\delta}}{\mu(\delta)} t^{c} \left(\log \frac{1}{t}\right)^{\delta-1}$, $c > -1, \delta \geq 0$, which gives the Komatu operator.

For further details on integral transform, see also [4].

Theorem 9. *Let $f \in TR(\eta, \lambda, \alpha)$. Then $V_{\mu}(f) \in TR(\eta, \lambda, \alpha)$.*

Proof By definition, we have

$$V_{\mu}(f) = \frac{(c+1)^{\delta}}{\mu(\delta)} \int_0^1 (-1)^{\delta-1} t^{c} (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1}\right) dt$$

$$= \frac{(-1)^{\delta-1} (c+1)^{\delta}}{\mu(\delta)} \lim_{r \to 0^+} \left[\int_r^1 t^{c} (\log t)^{\delta-1} \left(z - \sum_{n=2}^{\infty} a_n z^n t^{n-1}\right) dt\right]$$

with simple calculations, we get

$$V_{\mu}(f)(z) = z - \sum_{n=2}^{\infty} \left(\frac{c+1}{c+n}\right)^{\delta} a_n z^n.$$

We need to prove that

$$\sum_{n=2}^{\infty} \phi(n) \cdot \frac{2n-1-\alpha}{1-\alpha} \left(\frac{c+1}{c+n}\right)^{\delta} a_n < 1. \tag{23}$$

On the other hand $f \in TR(\eta, \lambda, \alpha)$ if and only if

$$\sum_{n=2}^{\infty} \phi(n) \cdot \frac{2n-1-\alpha}{1-\alpha} a_n < 1$$

hence $\frac{c+1}{c+n} < 1$. Therefore (23) holds and the proof is complete.

Theorem 10. **(Radius of starlikeness)** *Let $f \in TR(\eta, \lambda, \alpha)$.then $V_{\mu}(f)$is starlike of order $0 \leq \gamma \prec 1$ in $|z| \prec R_1$, where*

$$R_1 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^{\delta} \cdot \frac{1-\gamma(2n-1-\alpha)}{(n-\gamma)(1-\alpha)} \phi(n)\right]^{\frac{1}{n-1}}$$

and

$$\phi(n) = \frac{(\Gamma(n+1))^2 \Gamma(2+\eta-\lambda)\Gamma(2-\eta)}{\Gamma(n+\eta-\lambda+1)\Gamma(n-\eta+1).}$$

Proof. It suffices to prove that

$$\left| \frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1 \right| < 1-\gamma \tag{24}$$

we have

$$\left| \frac{z(V_\mu(f)(z))'}{V_\mu(f)(z)} - 1 \right| = \left| \frac{\sum_{n=2}^{\infty}(1-n)\left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)\left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}{1-\sum_{n=2}^{\infty}\left(\frac{c+1}{c+n}\right)^\delta a_n |z|^{n-1}}$$

the last expression is less that $1-\gamma$ since

$$|z|^{n-1} < \left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)(2n-1-\alpha)}{(n-\gamma)(1-\alpha)}\varphi(n)$$

so the proof is complete.

Theorem 11. *If $f \in TR(\eta,\lambda,\alpha)$. Then $V_\mu(f)$is convex of order $0 \leq \gamma < 1$,in $|z| < R_2$, where*

$$R_2 = \inf_n \left[\left(\frac{c+n}{c+1}\right)^\delta \frac{(1-\gamma)(2n-1-\alpha)}{n(n-\gamma)(1-\alpha)}\phi(n) \right]^{\frac{1}{n-1}}$$

using the fact that f is convex if and only if zf' is starlike, the proof can be easily derived.

AcknowledgementThe work presented here was supported by MOHE: UKM-ST-06-FRGS0107-2009

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