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ON SOME CLASSES OF MEROMORPHIC FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

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Abstract. For $p \in \mathbb{N}^*$ let Σ_p denote the class of meromorphic functions of the form $g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots$, $z \in \dot{U}$, $a_{-p} \neq 0$. In the present paper we introduce some new subclasses of the class Σ_p, denoted by $\Sigma S^n_{p,\lambda}(\alpha)$ and $\Sigma S^n_{p,\lambda}(\alpha, \delta)$, which are defined by a multiplier transformation, and we study some properties of these subclasses.

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1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{f : U \to \mathbb{C} : f \text{ is holomorphic in } U\}$, $\mathbb{Z} = \{\ldots -1, 0, 1, \ldots\}$, $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

For $p \in \mathbb{N}^*$ let Σ_p denote the class of meromorphic functions in $\dot{U}$ of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots, \ z \in \dot{U}, \ a_{-p} \neq 0.$$

We will also use the following notations:
$\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$,
$\Sigma^*_p(\alpha) = \left\{ g \in \Sigma_p : \text{Re}\left[-\frac{zg'(z)}{g(z)} \right] > \alpha, \ z \in U \right\}$, where $\alpha < p$,
$\Sigma^*_p(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \text{Re}\left[-\frac{zg'(z)}{g(z)} \right] < \delta, \ z \in U \right\}$, where $\alpha < p < \delta$,
$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots\}$ for $a \in \mathbb{C}, \ n \in \mathbb{N}^*$,
$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots\}$, $n \in \mathbb{N}^*$, and for $n = 1$ we denote A_1 by A and this set is called *the class of analytic functions normalized at the origin.*

We know that $\Sigma_1^*(\alpha)$ is the class of meromorphic starlike functions of order α, when $0 \le \alpha < 1$.

For $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, let the operator $J_{p,\lambda}^n$ on Σ_p be defined as

$$J_{p,\lambda}^n g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} \left(\frac{\lambda - p}{k+\lambda}\right)^n a_k z^k, \text{ where } g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k.$$

Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. We have:

1. $J_{p,\lambda}^{-1} g(z) = \dfrac{1}{\lambda - p} z g'(z) + \dfrac{\lambda}{\lambda - p} g(z), \; g \in \Sigma_p,$
2. $J_{p,\lambda}^0 g(z) = g(z), \; g \in \Sigma_p,$
3. $J_{p,\lambda}^1 g(z) = \dfrac{\lambda - p}{z^\lambda} \displaystyle\int_0^z t^{\lambda-1} g(t)dt = J_{p,\lambda}(g)(z), \; g \in \Sigma_p,$
4. If $g \in \Sigma_p$ with $J_{p,\lambda}^n g \in \Sigma_p$, then $J_{p,\lambda}^m (J_{p,\lambda}^n g) = J_{p,\lambda}^{n+m} g$, for $m, n \in \mathbb{Z}$.

Remark 1. Let $p \in \mathbb{N}^*$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$. We know from [7] that if $g \in \Sigma_p$, then $J_{p,\lambda}(g) \in \Sigma_p$, hence, using item 4 and the induction, we obtain

$$J_{p,\lambda}^n g \in \Sigma_p \quad \text{for all } n \in \mathbb{N}^*.$$

We notice from item 1 that for $g \in \Sigma_p$ we have $J_{p,\lambda}^{-1} g \in \Sigma_p$, so

$$J_{p,\lambda}^{-n} g \in \Sigma_p \quad \text{for all } n \in \mathbb{N}^*.$$

Therefore, for $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, we have $J_{p,\lambda}^n : \Sigma_p \to \Sigma_p$.

Now it is easy to see that we have the next properties for $J_{p,\lambda}^n$, when $\operatorname{Re}\lambda > p$:

1. $J_{p,\lambda}^n (J_{p,\lambda}^m g(z)) = J_{p,\lambda}^{n+m} g(z)$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$,
2. $J_{p,\gamma}^n (J_{p,\lambda}^m g(z)) = J_{p,\lambda}^m (J_{p,\gamma}^n g(z))$, $n, m \in \mathbb{Z}$, $g \in \Sigma_p$, $\operatorname{Re}\gamma > p$,
3. $J_{p,\lambda}^n (g_1 + g_2)(z) = J_{p,\lambda}^n g_1(z) + J_{p,\lambda}^n g_2(z)$, for $g_1, g_2 \in \Sigma_p$, $n \in \mathbb{Z}$,
4. $J_{p,\lambda}^n (cg)(z) = c J_{p,\lambda}^n g(z)$, $c \in \mathbb{C}^*$, $n \in \mathbb{Z}$,
5. $J_{p,\lambda}^n (zg'(z)) = z(J_{p,\lambda}^n g(z))' = (\lambda - p) J_{p,\lambda}^{n-1} g(z) - \lambda J_{p,\lambda}^n g(z)$, $n \in \mathbb{Z}$, $g \in \Sigma_p$.

Remark 2. 1. When $\lambda = 2$ and $p = 1$, we have

$$J^n_{1,2}g(z) = \frac{a_{-1}}{z} + \sum_{k=0}^{\infty}(k+2)^{-n}a_k z^k,$$

and this operator was studied by Cho and Kim [1] for $n \in \mathbb{Z}$ and by Uralegaddi and Somanatha [8] for $n < 0$.

2. We also have

$$z^2 J^n_{1,2}g(z) = D^n(z^2 g(z)),\ g \in \Sigma_{1,0}\,,$$

where D^n is the well-known Sălăgean differential operator of order n [5], defined by $D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k$, $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$.

3. $J^n_{p,\lambda}$ is an extension to the meromorphic functions of the operator K^n_p, defined on $A(p) = \left\{ f \in H(U) : f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \right\}$, introduced in [6]. Also, for $n \geq 0$ we find that K^n_p is the Komatu linear operator, defined in [2].

4. It is easy to see that for $n > 0$, $J^n_{p,\lambda}$ is an integral operator while $J^{-n}_{p,\lambda}$ is a differential operator with the property $J^{-n}_{p,\lambda}(J^n_{p,\lambda}g(z)) = g(z)$.

Lemma 1. *[7] Let $n \in \mathbb{N}^*$, $\alpha, \beta \in \mathbb{R}$, $\gamma \in \mathbb{C}$ with $\operatorname{Re}\gamma - \alpha\beta > 0$. If $P \in H[P(0), n]$ with $P(0) \in \mathbb{R}$ and $P(0) > \alpha$, then we have*

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] > \alpha \Rightarrow \operatorname{Re} P(z) > \alpha,\ z \in U.$$

Definition 1. *[3, pg. 46], [4, pg. 228] Let $c \in \mathbb{C}$ with $\operatorname{Re} c > 0$ and $n \in \mathbb{N}^*$. We consider*

$$C_n = C_n(c) = \frac{n}{\operatorname{Re} c}\left[|c|\sqrt{1 + \frac{2\operatorname{Re} c}{n}} + \operatorname{Im} c\right].$$

If the univalent function $R : U \to \mathbb{C}$ is given by $R(z) = \frac{2C_n z}{1 - z^2}$, then we denote by $R_{c,n}$ the "Open Door" function, defined as

$$R_{c,n}(z) = R\left(\frac{z+b}{1+\bar{b}z}\right) = 2C_n \frac{(z+b)(1+\bar{b}z)}{(1+\bar{b}z)^2 - (z+b)^2}\ ,$$

where $b = R^{-1}(c)$.

Theorem 1. *[7] Let* $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1,p]$ *with* $\Phi(z)\varphi(z) \neq 0$, $z \in U$. *Let* $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ *with* $\beta \neq 0$, $\delta + p\beta = \gamma + p\alpha$ *and* $\operatorname{Re}(\gamma - p\beta) > 0$. *Let* $g \in \Sigma_p$ *and suppose that*

$$\alpha \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta \prec R_{\delta - p\alpha, p}(z).$$

If $G = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)$ *is defined by*

$$G(z) = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[\frac{\gamma - p\beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt\right]^{\frac{1}{\beta}}, \tag{1}$$

then $G \in \Sigma_p$ *with* $z^p G(z) \neq 0$, $z \in U$, *and*

$$\operatorname{Re}\left[\beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma\right] > 0,\ z \in U.$$

All powers in (1) are principal ones.

Taking $\beta = \alpha = 1, \delta = \gamma$, $\Phi = \varphi \equiv 1$, in the above theorem, and using the notation $J_{p,\gamma}$ instead of $J^{1,1}_{p,1,1,\gamma,\gamma}$, we obtain the corollary:

Corollary 1. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *with* $\operatorname{Re}\gamma > p$ *and let the function* $g \in \Sigma_p$ *satisfying the condition*

$$\frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma-p,p}(z),\ z \in U.$$

Then

$$G(z) = J_{p,\gamma}(g)(z) = \frac{\gamma - p}{z^\gamma} \int_0^z g(t)t^{\gamma-1}dt \in \Sigma_p,$$

with $z^pG(z) \neq 0$, $z \in U$, *and* $\operatorname{Re}\left[\gamma + \frac{zG'(z)}{G(z)}\right] > 0$, $z \in U$.

2. Main results

Definition 2. *For* $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$, $\operatorname{Re}\lambda > p$ *and* $\alpha < p < \delta$ *we define*

$$\Sigma S^n_{p,\lambda}(\alpha) = \left\{ g \in \Sigma_p : \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda}g(z)\right)'}{J^n_{p,\lambda}g(z)}\right] > \alpha,\ z \in U \right\},$$

$$\Sigma S^n_{p,\lambda}(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda}g(z)\right)'}{J^n_{p,\lambda}g(z)}\right] < \delta,\ z \in U \right\}.$$

Remark 3. 1. We have $g \in \Sigma S^n_{p,\lambda}(\alpha)$ if and only if $J^n_{p,\lambda} g \in \Sigma^*_p(\alpha)$, respectively $g \in \Sigma S^n_{p,\lambda}(\alpha, \delta)$ if and only if $J^n_{p,\lambda} g \in \Sigma^*_p(\alpha, \delta)$.

2. Using the equality $z(J^n_{p,\lambda} g(z))' = (\lambda - p) J^{n-1}_{p,\lambda} g(z) - \lambda J^n_{p,\lambda} g(z)$, we can easy see that for $\operatorname{Re}\lambda > p$ the condition

$$\alpha < \operatorname{Re}\left[-\frac{z\left(J^n_{p,\lambda} g(z)\right)'}{J^n_{p,\lambda} g(z)}\right] < \delta,\ z \in U,$$

is equivalent to

$$\operatorname{Re}\lambda - \delta < \operatorname{Re}\left[(\lambda - p)\frac{J^{n-1}_{p,\lambda} g(z)}{J^n_{p,\lambda} g(z)}\right] < \operatorname{Re}\lambda - \alpha,\ z \in U. \tag{2}$$

3. We have

$$\Sigma S^0_{p,\lambda}(\alpha, \delta) = \Sigma^*_p(\alpha, \delta),$$

$$\Sigma S^1_{p,\lambda}(\alpha, \delta) = \left\{ g \in \Sigma_p : G(z) = \frac{\lambda - p}{z^\lambda} \int_0^z t^{\lambda - 1} g(t) dt \in \Sigma^*_p(\alpha, \delta) \right\}.$$

The following theorem gives us a connection between the sets $\Sigma S^n_{p,\lambda}(\alpha)$ and $\Sigma S^{n-1}_{p,\lambda}(\alpha)$, respectively between $\Sigma S^n_{p,\lambda}(\alpha, \delta)$ and $\Sigma S^{n-1}_{p,\lambda}(\alpha, \delta)$.

Theorem 2. *Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$, $\alpha < p < \delta$ and $g \in \Sigma_p$. Then*

$$g \in \Sigma S^n_{p,\lambda}(\alpha) \Leftrightarrow J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha),$$

respectively

$$g \in \Sigma S^n_{p,\lambda}(\alpha, \delta) \Leftrightarrow J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha, \delta),$$

where $J_{p,\lambda}(g)(z) = \dfrac{\lambda - p}{z^\lambda} \displaystyle\int_0^z t^{\lambda - 1} g(t) dt.$

Proof . We know that $g \in \Sigma S^n_{p,\lambda}(\alpha, \delta)$ is equivalent to $J^n_{p,\lambda} g \in \Sigma^*_p(\alpha, \delta)$. Since $J^n_{p,\lambda} g = J^{n-1}_{p,\lambda}(J^1_{p,\lambda} g)$, we have $J^{n-1}_{p,\lambda}(J^1_{p,\lambda} g) \in \Sigma^*_p(\alpha, \delta)$, which is equivalent to

$$J^1_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha, \delta).$$

Since $J^1_{p,\lambda}(g) = J_{p,\lambda}(g)$, we obtain $J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha,\delta)$. Therefore,

$$g \in \Sigma S^n_{p,\lambda}(\alpha,\delta) \Leftrightarrow J_{p,\lambda}(g) \in \Sigma S^{n-1}_{p,\lambda}(\alpha,\delta).$$

The proof for the first equivalence is similar, so we omit it.

□

A corollary presented in [7] holds that:

Corollary 2. *[7] Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \delta \leq \operatorname{Re}\gamma$. If $g \in \Sigma^*_p(\alpha,\delta)$, then*

$$G = J_{p,\gamma}(g) \in \Sigma^*_p(\alpha,\delta).$$

Theorem 3. *Let $p \in \mathbb{N}^*$, $n \in \mathbb{Z}, \lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \delta \leq \operatorname{Re}\gamma$. Then*

$$g \in \Sigma S^n_{p,\lambda}(\alpha,\delta) \Rightarrow J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha,\delta).$$

Proof . Because $g \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$ we have $J^n_{p,\lambda}(g) \in \Sigma^*_p(\alpha,\delta)$, hence, from Corollary 2, we obtain

$$J_{p,\gamma}(J^n_{p,\lambda}(g)) \in \Sigma^*_p(\alpha,\delta).$$

Using the fact that $J^1_{p,\gamma}(J^n_{p,\lambda}(g)) = J^n_{p,\lambda}(J^1_{p,\gamma}(g))$, where $J^1_{p,\gamma}(g) = J_{p,\gamma}(g)$, we obtain

$$J^n_{p,\lambda}(J_{p,\gamma}(g)) \in \Sigma^*_p(\alpha,\delta),$$

which is equivalent to $J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$.

□

Corollary 3. *Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*, \lambda \in \mathbb{C}$ and $\alpha < p < \delta \leq \operatorname{Re}\lambda$. Then we have*

$$\Sigma S^n_{p,\lambda}(\alpha,\delta) \subset \Sigma S^{n+1}_{p,\lambda}(\alpha,\delta).$$

Proof . Let $g \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$. Taking $\gamma = \lambda$ in Theorem 3 we have $J_{p,\lambda}(g) \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$, which, from Theorem 2, is equivalent to $g \in \Sigma S^{n+1}_{p,\lambda}(\alpha,\delta)$. And the result follows.

□

A theorem presented in [7] holds that:

Theorem 4. *[7] Let* $p \in \mathbb{N}^*$, $\beta > 0, \gamma \in \mathbb{C}$ *and* $\alpha < p < \dfrac{\operatorname{Re}\gamma}{\beta} \leq \delta$.
If $g \in \Sigma_p^*(\alpha, \delta)$, *with*

$$\beta \frac{zg'(z)}{g(z)} + \gamma \prec R_{\gamma - p\beta, p}(z),\ z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta)$.

Using this theorem, for $\beta = 1$, we get the next result.

Theorem 5. *Let* $n \in \mathbb{Z}$, $p \in \mathbb{N}^*, \lambda, \gamma \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > p$ *and* $\alpha < p < \operatorname{Re}\gamma \leq \delta$. *If* $g \in \Sigma S_{p,\lambda}^n(\alpha, \delta)$ *and satisfies the condition*

$$\frac{z\left[J_{p,\lambda}^n(g)(z)\right]'}{J_{p,\lambda}^n(g)(z)} + \gamma \prec R_{\gamma - p, p}(z),\ z \in U,$$

then $J_{p,\gamma}(g) \in \Sigma S_{p,\lambda}^n(\alpha, \delta)$.

We omit the proof because it is similar with that of Theorem 3.
If we consider in Theorem 5 that $\delta \to \infty$ we get:

Theorem 6. *Let* $n \in \mathbb{Z}$, $p \in \mathbb{N}^*, \lambda, \gamma \in \mathbb{C}$ *with* $\operatorname{Re}\lambda > p$ *and* $\alpha < p < \operatorname{Re}\gamma$. *If* $g \in \Sigma S_{p,\lambda}^n(\alpha)$ *and satisfies the condition*

$$\frac{z\left[J_{p,\lambda}^n(g)(z)\right]'}{J_{p,\lambda}^n(g)(z)} + \gamma \prec R_{\gamma - p, p}(z),\ z \in U,$$

then $J_{p,\gamma}(g) \in \Sigma S_{p,\lambda}^n(\alpha)$.

Theorem 7. *Let* $n \in \mathbb{N}$, $p \in \mathbb{N}^*, \lambda, \gamma \in \mathbb{C}$ *and* $\alpha < p < \operatorname{Re}\gamma \leq \operatorname{Re}\lambda \leq \delta$. *If* $h \in \Sigma S_{p,\lambda}^n(\alpha, \delta)$ *and satisfies the condition*

$$\frac{zh'(z)}{h(z)} + \gamma \prec R_{\gamma - p, p}(z),\ z \in U,$$

then $J_{p,\gamma}(h) \in \Sigma S_{p,\lambda}^n(\alpha, \delta)$.

Proof . Let us denote $H = J_{p,\gamma}(h)$. Because the conditions of Corollary 1 are fullfilled, we have $H \in \Sigma_p$ with $z^p H(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{zH'(z)}{H(z)} + \gamma\right] > 0,\ z \in U. \tag{3}$$

Since $\operatorname{Re}\gamma \leq \operatorname{Re}\lambda$, we obtain from (3) that $\operatorname{Re}\left[\frac{zH'(z)}{H(z)} + \lambda\right] > 0,\ z \in U.$
Using now Corollary 1, we have $J_{p,\lambda}H \in \Sigma_p$ with $z^p J_{p,\lambda}H(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{z(J_{p,\lambda}H)'(z)}{J_{p,\lambda}H(z)} + \lambda\right] > 0,\ z \in U.$$

By induction, we obtain $z^p J^n_{p,\lambda}H(z) \neq 0$, $z \in U$, and

$$\operatorname{Re}\left[\frac{z(J^n_{p,\lambda}H)'(z)}{J^n_{p,\lambda}H(z)} + \lambda\right] > 0,\ z \in U. \tag{4}$$

Since $\operatorname{Re}\lambda \leq \delta$, we get from (4) that

$$\operatorname{Re}\left[-\frac{z(J^n_{p,\lambda}H)'(z)}{J^n_{p,\lambda}H(z)}\right] < \delta,\ z \in U. \tag{5}$$

From the definition of H we have

$$\gamma H(z) + zH'(z) = (\gamma - p)h(z),\ z \in \dot{U}. \tag{6}$$

We apply the operator $J^n_{p,\lambda}$ to (6) and after using the properties of $J^n_{p,\lambda}$, we obtain

$$(\gamma - p)J^n_{p,\lambda}h(z) = \gamma J^n_{p,\lambda}H(z) + z(J^n_{p,\lambda}H(z))'. \tag{7}$$

From $h \in \Sigma S^n_{p,\lambda}(\alpha,\delta)$, we have

$$\alpha < \operatorname{Re}\left[-\frac{z(J^n_{p,\lambda}h(z))'}{J^n_{p,\lambda}h(z)}\right] < \delta,$$

which is equivalent to (see Remark 3, item 2)

$$\operatorname{Re}\lambda - \delta < \operatorname{Re}\left[(\lambda - p)\frac{J^{n-1}_{p,\lambda}h(z)}{J^n_{p,\lambda}h(z)}\right] < \operatorname{Re}\lambda - \alpha. \tag{8}$$

From (7), we have

$$(\lambda - p)\frac{J_{p,\lambda}^{n-1}h(z)}{J_{p,\lambda}^{n}h(z)} = (\lambda - p)\frac{\gamma J_{p,\lambda}^{n-1}H(z) + z(J_{p,\lambda}^{n-1}H(z))'}{\gamma J_{p,\lambda}^{n}H(z) + z(J_{p,\lambda}^{n}H(z))'}. \tag{9}$$

Let us denote $P(z) = -\frac{z(J_{p,\lambda}^{n}H(z))'}{J_{p,\lambda}^{n}H(z)}$. Using the fact that

$$z(J_{p,\lambda}^{n}H(z))' = (\lambda - p)J_{p,\lambda}^{n-1}H(z) - \lambda J_{p,\lambda}^{n}H(z),$$

we obtain

$$P(z) = (p - \lambda)\frac{J_{p,\lambda}^{n-1}H(z)}{J_{p,\lambda}^{n}H(z)} + \lambda,\ z \in U.$$

Hence

$$\frac{J_{p,\lambda}^{n-1}H(z)}{J_{p,\lambda}^{n}H(z)} = \frac{P(z) - \lambda}{p - \lambda}. \tag{10}$$

If we apply the logarithmic differential to (10), we get

$$\frac{z(J_{p,\lambda}^{n-1}H(z))'}{J_{p,\lambda}^{n-1}H(z)} - \frac{z(J_{p,\lambda}^{n}H(z))'}{J_{p,\lambda}^{n}H(z)} = \frac{zP'(z)}{P(z) - \lambda},$$

so,

$$\frac{z(J_{p,\lambda}^{n-1}H(z))'}{J_{p,\lambda}^{n-1}H(z)} = -P(z) + \frac{zP'(z)}{P(z) - \lambda}. \tag{11}$$

Using (10) and (11) we obtain from (9),

$$(\lambda - p)\frac{J_{p,\lambda}^{n-1}h(z)}{J_{p,\lambda}^{n}h(z)} = (\lambda - p)\frac{J_{p,\lambda}^{n-1}H(z)}{J_{p,\lambda}^{n}H(z)} \cdot \frac{\gamma + \frac{z(J_{p,\lambda}^{n-1}H(z))'}{J_{p,\lambda}^{n-1}H(z)}}{\gamma + \frac{z(J_{p,\lambda}^{n}H(z))'}{J_{p,\lambda}^{n}H(z)}} =$$

$$= (\lambda - p)\frac{P(z) - \lambda}{p - \lambda} \cdot \frac{\gamma - P(z) + \frac{zP'(z)}{P(z) - \lambda}}{\gamma - P(z)} = \lambda - P(z) + \frac{zP'(z)}{P(z) - \gamma}. \tag{12}$$

From (8) and (12), we get

$$\operatorname{Re}\lambda - \delta < \operatorname{Re}\left[\lambda - P(z) + \frac{zP'(z)}{P(z) - \gamma}\right] < \operatorname{Re}\lambda - \alpha,$$

which is equivalent to

$$\alpha < \operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - P(z)}\right] < \delta. \tag{13}$$

Because we know from (5) that $\operatorname{Re} P(z) < \delta$, $z \in U$, we have only to verify that $\operatorname{Re} P(z) > \alpha$, $z \in U$. To show this we will use Lemma 1.

We know that $J^n_{p,\lambda}H \in \Sigma_p$ and since we have proved that $z^p J^n_{p,\lambda}H(z) \neq 0$, $z \in U$, we get that P is analytic in U. We also have $\operatorname{Re}\gamma - \alpha > 0$ and $P(0) = p > \alpha$. Since the hypothesis of Lemma 1 is fulfilled for $\beta = 1$, we obtain $\operatorname{Re} P(z) > \alpha$, $z \in U$, which is equivalent to

$$\operatorname{Re}\left[-\frac{z(J^n_{p,\lambda}H(z))'}{J^n_{p,\lambda}H(z)}\right] > \alpha,\ z \in U. \tag{14}$$

Since $H \in \Sigma_p$, we get from (5) and (14) that $H \in \Sigma S^n_{p,\lambda}(\alpha, \delta)$.

□

If we consider $\gamma = \lambda$, in the above theorem, we obtain:

Corollary 4. *Let $n \in \mathbb{N}$, $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $h \in \Sigma S^n_{p,\lambda}(\alpha, \delta)$ with $\alpha < p < \operatorname{Re}\lambda \leq \delta$. If*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z),\ z \in U.$$

Then $J_{p,\lambda}(h) \in \Sigma S^n_{p,\lambda}(\alpha, \delta)$.

Taking $n = 0$ in Corollary 4, we get:

Corollary 5. *Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\lambda \leq \delta$. If $h \in \Sigma^*_p(\alpha, \delta)$ with*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z),\ z \in U,$$

then $J_{p,\lambda}(h) \in \Sigma^*_p(\alpha, \delta)$.

If in Corollary 5 we consider $\delta \mapsto \infty$ we have the next result:

Corollary 6. *Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ and $\alpha < p < \operatorname{Re}\lambda$. If $h \in \Sigma_p^*(\alpha)$ with*

$$\frac{zh'(z)}{h(z)} + \lambda \prec R_{\lambda-p,p}(z),\ z \in U,$$

then $J_{p,\lambda}(h) \in \Sigma_p^*(\alpha)$.

We remark that Corollary 5 and Corollary 6 were also obtained in [7].

Theorem 8. *Let $n \in \mathbb{Z}$, $p \in \mathbb{N}^*$, $\lambda, \gamma \in \mathbb{C}$ with $\operatorname{Re}\lambda > p$ and $\alpha < p < \operatorname{Re}\gamma$. If $g \in \Sigma S^n_{p,\lambda}(\alpha)$ with $J_{p,\gamma}(J^n_{p,\lambda}(g)(z)) \neq 0$, $z \in \dot{U}$, then*

$$J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha).$$

Proof. Let $G = J_{p,\gamma}(g)$. We know from Remark 1 that if $g \in \Sigma_p$ and $\operatorname{Re}\gamma > p$, then $G \in \Sigma_p$.
Let

$$P(z) = -\frac{z(J^n_{p,\lambda}G(z))'}{J^n_{p,\lambda}G(z)},\ z \in U.$$

We have $J^n_{p,\lambda}(G) = J^n_{p,\lambda}(J_{p,\gamma}(g)) = J_{p,\gamma}(J^n_{p,\lambda}(g)) \neq 0$ în $\dot{U}$. So $P \in H(U)$.
Making some calculations, similar to those from the proof of Theorem 7, we get:

$$-\frac{z(J^n_{p,\lambda}g(z))'}{J^n_{p,\lambda}g(z)} = P(z) + \frac{zP'(z)}{\gamma - P(z)},\ z \in U.$$

From $g \in \Sigma S^n_{p,\lambda}(\alpha)$ we have

$$\operatorname{Re}\left[-\frac{z(J^n_{p,\lambda}g(z))'}{J^n_{p,\lambda}g(z)}\right] > \alpha,\ z \in U,$$

which is equivalent to

$$\operatorname{Re}\left[P(z) + \frac{zP'(z)}{\gamma - P(z)}\right] > \alpha,\ z \in U.$$

Since $P \in H(U)$ with $P(0) = p > \alpha$ and $\operatorname{Re}\gamma > \alpha$, we have from Lemma 1 (when $\beta = 1$) that $\operatorname{Re} P(z) > \alpha$, $z \in U$, hence

$$G = J_{p,\gamma}(g) \in \Sigma S^n_{p,\lambda}(\alpha).$$

□

Taking $n = 0$ and $\lambda = \gamma$ in the above theorem we get the next corollary, which was also obtained in [7].

Corollary 7. *Let* $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ *and* $\alpha < p < \operatorname{Re}\gamma$.
If $g \in \Sigma_p^*(\alpha)$ *with* $z^p J_{p,\gamma}(g)(z) \neq 0$, $z \in U$, *then* $J_{p,\gamma}(g) \in \Sigma_p^*(\alpha)$.

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