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ON APPLICATIONS OF GENERALIZED INTEGRAL OPERATOR TO A SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. Making use of generalized integral operator, we define a subclass of analytic functions with negative coefficients. The main object of this paper is to obtain coefficient estimates, closure theorems and extreme points. Also we obtain radii of close-to-convexity, starlikeness and convexity and neighbourhood results for functions in the generalized class$R^*(\alpha,\beta,\mu,\eta)$.

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1.Introduction and Preliminaries

Denote by $\mathcal{A}$ the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

that are analytic and univalent in the open disc $U = \{z : |z| < 1\}$. Also denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad a_n \geq 0, z \in U. \tag{2}$$

was introduced and studied by Silverman [21].

For functions $\Phi \in \mathcal{A}$ given by

$$\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \tag{3}$$

and $\Psi \in \mathcal{A}$ given by

$$\Psi(z) = z + \sum_{n=2}^{\infty} \psi_n z^n, \tag{4}$$

we define the Hadamard product (or convolution) of Φ and Ψ by

$$(\Phi * \Psi)(z) = z + \sum_{n=2}^{\infty} \phi_n \psi_n z^n, \quad z \in U. \tag{5}$$

We recall here a general Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ defined in [23] by

$$\Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{6}$$

$$(a \in C \setminus \{Z_0^-\}; s \in C, \text{when } \ |z| < 1 \ \ R(s) > 1 \ \text{ and } \ |z| = 1)$$

where, as usual, $Z_0^- := Z \setminus \{N\}$, $(Z := \{0, \pm 1, \pm 2, \pm 3, ...\}); N := \{1, 2, 3, ...\}$.

Several interesting properties and characteristics of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ can be found in the recent investigations by Choi and Srivastava [5], Ferreira and Lopez [7], Garg et al. [9], Lin and Srivastava [12], Lin et al. [13], and others. Srivastava and Attiya [22] (see also Raducanu and Srivastava [19], and Prajapat and Goyal [17]) introduced and investigated the linear operator:

$$\mathcal{J}_{\mu,b} : \mathcal{A} \to \mathcal{A}$$

defined in terms of the Hadamard product by

$$\mathcal{J}_{\mu,b} f(z) = \mathcal{G}_{b,\mu} * f(z) \tag{7}$$

$(z \in U; b \in C \setminus \{Z_0^-\}; \mu \in C; f \in \mathcal{A})$, where, for convenience,

$$G_{\mu,b}(z) := (1+b)^{\mu}[\Phi(z, \mu, b) - b^{-\mu}] \ \ (z \in U). \tag{8}$$

We recall here the following relationships (given earlier by [19]) which follow easily by using (1), (7) and (8)

$$\mathcal{J}_b^{\mu} f(z) = z + \sum_{n=2}^{\infty} \left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n. \tag{9}$$

Motivated essentially by the Srivastava-Attiya operator, Al-Shaqsi and Darus [?] introduced and investigated the integral operator

$$\mathcal{J}_{\mu,b}^{\eta;k} f(z) = z + \sum_{n=2}^{\infty} C_n^{\eta}(b, \mu) a_n z^n. \tag{10}$$

where

$$C_n^{\eta}(b, \mu) = | \left(\frac{1+b}{n+b}\right)^{\mu} \frac{\eta!(n+k-2)!}{(k-2)!(n+\eta-1)!} | \tag{11}$$

and (throughout this paper unless otherwise mentioned) the parameters μ, b are constrained as $b \in C \setminus \{Z_0^-\}; \mu \in C, k \geq 2$ and $\eta > -1$. Further note that $J_{\mu,b}^{1,2}$is the Srivastava-Attiya operator, and $J_{0,b}^{\eta,k}$ is the well-known Choi-Saigo- Srivastava operator (see [6]). Assuming $\eta = 1$ and $k = 2$,we state the following integral operators by specializing μ and b.

1. For $\mu = 0$
$$\mathcal{J}_b^0(f)(z) := f(z). \tag{12}$$
2. For $\mu = 1$
$$\mathcal{J}_b^1(f)(z) := \int_0^z \frac{f(t)}{t} dt := \mathcal{L}_b f(z). \tag{13}$$
3. For $\mu = 1$ and $b = \nu(\nu > -1)$
$$\mathcal{J}_\nu^1(f)(z) := \mathcal{F}_\nu(f)(z) = \frac{1+\nu}{z^\nu} \int_0^z t^{\nu-1} f(t) dt := z + \sum_{n=2}^{\infty} \left(\frac{1+\nu}{n+\nu}\right) a_n z^n. \tag{14}$$
4. For $\mu = \sigma(\sigma > 0)$ and $b = 1$
$$\mathcal{J}_1^\sigma(f)(z) := z + \sum_{n=2}^{\infty} \left(\frac{2}{n+1}\right)^\sigma a_n z^n = \mathcal{I}^\sigma(f)(z), \tag{15}$$

where $\mathcal{L}_b(f)$ and $\mathcal{F}_\nu$ are the integral operators introduced by Alexandor [1] and Bernardi [4], respectively, and $\mathcal{I}^\sigma(f)$ is the Jung-Kim-Srivastava integral operator [11] closely related to some multiplier transformation studied by Fleet [8]. Making use of the operator $\mathcal{J}_{\mu,b}^{\lambda,k}$, we introduce a new subclass of analytic functions with negative coefficients and discuss some usual properties of the geometric function theory of this generalized function class.

For $\alpha(\alpha \geq 0)$, $\beta(0 \leq \beta < 1)$, and $\eta(\eta > -1)$, we let $R(\alpha, \beta, \mu, \eta)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1) and satisfying the analytic criterion
$$\mathrm{Re}\left\{(1-\alpha)\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)' + \alpha\left(z\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)'\right)'\right\} > \beta, \quad z \in U. \tag{16}$$

We also let $R^*(\alpha, \beta, \mu, \eta) = R(\alpha, \beta, \mu, \eta) \cap \mathcal{T}$.

We note that, by suitably specializing the parameters α, β, μ, η, the class $R^*(\alpha, \beta, \mu, \eta)$ reduces to the classes studied in [2,3,20].

Motivated by the earlier works of Murugusundaramoorthy [14,15,16] and Prajapat et.al.,[17] we obtain the necessary and sufficient conditions for the functions $f(z) \in R^*(\alpha, \beta, \mu, \eta)$, and to study the extreme points, closure properties, radii of close-to-convexity, starlikness and convexity, $\delta-$ neighbourhoods for $f(z) \in R^*(\alpha, \beta, \mu, \eta)$.

2. Main Results

Theorem 1. *Let the function* $f(z)$ *be defined by (7). Then* $f(z) \in R^*(\alpha, \beta, \mu, \eta)$, *if and only if*

$$\sum_{n=2}^{\infty} n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)a_n \leq 1-\beta, \tag{17}$$

$\alpha(\alpha \geq 0)$, $\beta(0 \leq \beta < 1)$, *and* $\eta(\eta > -1)$.

Proof. Assume that inequality (17) holds and let $|z| < 1$. Then we have

$$\begin{aligned}
& \left|(1-\alpha)\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)' + \alpha\left(z\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)'\right)' - 1\right| \\
= & \ |(1-\alpha)\left(1-\sum_{n=2}^{\infty} nC_n^{\eta}(b,\mu)a_n z^{n-1}\right) \\
& +\alpha\left(1-\sum_{n=2}^{\infty} n^2 C_n^{\eta}(b,\mu)a_n z^{n-1}\right) - 1 \ | \\
\leq & \left|\sum_{n=2}^{\infty} n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)a_n z^{n-1}\right| \\
\leq & \ 1-\beta.
\end{aligned}$$

This shows that the values of $(1-\alpha)\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)' + \alpha\left(z\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)'\right)'$ lies in a circle centered at $w = 1$ whose radius is $1-\beta$. Hence $f(z)$ satisfies the condition (16).

Conversely, assume that the function $f(z)$ defined by (7), is in the class $R^*(\alpha, \beta, \mu, \eta)$. Then

$$\begin{aligned}
& \text{Re}\left\{(1-\alpha)\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)' + \alpha\left(z\left(\mathcal{J}_{\mu,b}^{\eta,k} f(z)\right)'\right)'\right\} \\
= & \ \text{Re}\ \{(1-\alpha)\left(1-\sum_{n=2}^{\infty} nC_n^{\eta}(b,\mu)a_n z^{n-1}\right) \\
& +\alpha\left(1-\sum_{n=2}^{\infty} n^2 C_n^{\eta}(b,\mu)a_n z^{n-1}\right)\} \\
> & \ \beta, z \in U.
\end{aligned}$$

Letting $z \to 1$ along the real axis, we obtain the desired inequality.

Corollary 1. *Let the function* $f(z)$ *defined by (7) be in the class* $R^*(\alpha, \beta, \mu, \eta)$. *Then we have*

$$a_n \leq \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)}. \tag{18}$$

The equation (18) is attained for the function

$$f(z) = z - \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)} z^n \qquad (n \geq 2). \tag{19}$$

Let the functions $f_j(z)$ be defined, for $j = 1, 2, \dots m$, by

$$f_j(z) = z - \sum_{n=2}^{\infty} a_{n,\ j} z^n \quad a_{n,\ j} \geq 0, \quad z \in U. \tag{20}$$

We shall prove the following results for the closure of functions in the class $R^*(\alpha, \beta, \mu, \eta)$.

Theorem 2. *(Closure Theorem) Let the functions $f_j(z)(j = 1, 2, \dots m)$ defined by (20) be in the classes $R^*(\alpha, \beta_j, \lambda, \eta)$ $(j = 1, 2, \dots m)$ respectively. Then the function $h(z)$ defined by*

$$h(z) = z - \frac{1}{m} \sum_{n=2}^{\infty} \left(\sum_{j=1}^{m} a_{n,\ j} \right) z^n$$

is in the class $R^*(\alpha, \beta, \mu, \eta)$, where $\beta = \min_{1 \leq j \leq m} \{\beta_j\}$ where $0 \leq \beta_j \leq 1$.

Proof. Since $f_j(z) \in R^*(\alpha, \beta_j, \mu, \eta)$ $(j = 1, 2, \dots m)$ by applying Theorem 1, to (20) we observe that

$$\begin{aligned} & \sum_{n=2}^{\infty} n\ [1+(n-1)\alpha]C_n^{\eta}(b,\mu) \ \left(\frac{1}{m} \sum_{j=1}^{m} a_{n,\ j} \right) \\ = \ & \frac{1}{m} \sum_{j=1}^{m} \left(\sum_{n=2}^{\infty} n\ [1+(n-1)\alpha]\ C_n^{\eta}(b,\mu) a_{n,\ j} \right) \\ \leq \ & \frac{1}{m} \sum_{j=1}^{m} (1-\beta_j) \\ \leq \ & 1-\beta \end{aligned}$$

which in view of Theorem 1, $h(z) \in R^*(\alpha, \beta, \mu, \eta)$ and hence the proof is complete.

Theorem 3. *Let $f(z)$ defined by (7) and $g(z)$ defined by $g(z) = z - \sum_{n=2}^{\infty} b_n z^n$ be in the class $R^*(\alpha, \beta, \mu, \eta)$. Then the function $h(z)$ defined by*

$$h(z) = (1-\lambda)f(z) + \lambda g(z) = z - \sum_{n=2}^{\infty} q_n z^n$$

where $q_n = (1-\lambda)a_n + \lambda b_n$; $0 \leq \lambda < 1$ *is also in the class* $R^*(\alpha, \beta, \mu, \eta)$.

Theorem 4. *(Extreme Points)* *Let*

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)} z^n \quad (n \geq 2) \end{aligned} \tag{21}$$

for $\alpha \geq 0$, $0 \leq \beta < 1$ *and* $\eta > -1$. *Then* $f(z)$ *is in the class* $R^*(\alpha, \beta, \mu, \eta)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z) \tag{22}$$

where $\lambda_n \geq 0$ $(n \geq 1)$ *and* $\sum\limits_{n=1}^{\infty} \lambda_n = 1$.

Proof. Suppose that

$$f(z) = \lambda_1 f_1(z) + \sum_{n=2}^{\infty} \lambda_n f_n(z) = z - \sum_{n=2}^{\infty} \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)} \lambda_n z^n.$$

Then it follows that

$$\sum_{n=2}^{\infty} \frac{n[1+(n-1)\alpha]}{(1-\beta)} C_n^{\eta}(b,\mu)\lambda_n \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)} \leq 1$$

by Theorem 1, $f(z) \in R^*(\alpha, \beta, \mu, \eta)$.

Conversely, suppose that $f(z) \in R^*(\alpha, \beta, \mu, \eta)$. Then $a_n \leq \frac{(1-\beta)}{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)}$ $(n \geq 2)$ we set $\lambda_n = \frac{n[1+(n-1)\alpha]}{(1-\beta)} C_n^{\eta}(b,\mu)\ a_n$ $(n \geq 2)$ and $\lambda_1 = 1 - \sum\limits_{n=2}^{\infty} \lambda_n$. We obtain $f(z) = \lambda_1 f_1(z) + \sum\limits_{n=2}^{\infty} \lambda_n f_n(z)$. This completes the proof of Theorem 4.

3.Radius of Starlikeness and Convexity

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $R^*(\alpha, \beta, \mu, \eta)$.

Theorem 5. *Let the function* $f(z)$ *defined by (7)belong to the class* $R^*(\alpha, \beta, \mu, \eta)$. *Then* $f(z)$ *is close-to-convex of order* σ $(0 \leq \sigma < 1)$ *in the disc* $|z| < r_1$, *where*

$$r_1 = r_1(\alpha, \beta, \mu, \eta, \sigma) := \inf_{n \geq 2} \left[\left(\frac{1-\sigma}{1-\beta} \right) [1+(n-1)\alpha] C_n^{\eta}(b,\mu) \right]^{\frac{1}{n-1}} . \tag{23}$$

The result is sharp, with extremal function $f(z)$ given by (19).

Proof. Given $f \in \mathcal{T}$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma. \tag{24}$$

For the left hand side of (24) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\sigma} a_n |z|^{n-1} < 1.$$

Using the fact, that $f \in R^*(\alpha, \beta, \mu, \eta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n[1+(n-1)\alpha]}{(1-\beta)} C_n^{\eta}(b, \mu) a_n \leq 1,$$

We can say (24) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{n[1+(n-1)\alpha]}{(1-\beta)} C_n^{\eta}(b, \mu)$$

Or, equivalently,

$$|z|^{n-1} = \left[\left(\frac{1-\sigma}{1-\beta}\right)[1+(n-1)\alpha] C_n^{\eta}(b, \mu)\right]$$

Which completes the proof.

Theorem 6. *Let $f \in R^*(\alpha, \beta, \mu, \eta)$. Then*

1. *f is starlike of order $\sigma (0 \leq \sigma < 1)$ is the disc $|z| < r_2$; that is, $Re \left\{\frac{zf'(z)}{f(z)}\right\} > \sigma$, where*

$$r_2 = \inf_{n \geq 2} \left[\left(\frac{n(1-\sigma)}{(n-\sigma)(1-\beta)}\right)[1+(n-1)\alpha] C_n^{\eta}(b, \mu)\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{25}$$

2. *f is convex of order σ $(0 \leq \sigma < 1)$ in the unit disc $|z| < r_3$, that is $Re \left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \sigma$, $(|z| < r_3; 0 \leq \sigma < 1)$, where*

$$r_3 = \inf_{n \geq 2} \left[\left(\frac{(1-\sigma)}{(n-\sigma)(1-\beta)}\right)[1+(n-1)\alpha] C_n^{\eta}(b, \mu)\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{26}$$

Each of these results are sharp for the extremal function $f(z)$ given by (21).

Proof. Given $f \in \mathcal{T}$, and f is starlike of order σ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \sigma. \tag{27}$$

For the left hand side of (27) we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1 - \sum\limits_{n=2}^{\infty} a_n\ |z|^{n-1}}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n-\sigma}{1-\sigma} a_n\ |z|^{n-1} < 1.$$

Using the fact, that $f \in R^*(\alpha, \beta, \mu, \eta)$ if and only if

$$\sum_{n=2}^{\infty} \frac{n[1+(n-1)\alpha]}{(1-\beta)} C_n^{\eta}(b,\mu) a_n \leq 1.$$

We can say (27) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1} < \frac{n[1+(n-1)\alpha]C_n^{\eta}(b,\mu)}{(1-\beta)}$$

Or, equivalently,

$$|z|^{n-1} = \left[\ \frac{n(1-\sigma)}{(n-\sigma)(1-\beta)}[1+(n-1)\alpha]C_n^{\eta}(b,\mu)\right]$$

which yields the starlikeness of the family.

(2) Using the fact that f is convex if and only if zf' is starlike, we can prove (2), on lines similar to the proof of (1).

4. Inclusion relations involving $N_\delta(e)$

Following [10,18], we define the $\delta-$ neighborhood of function $f(z) \in \mathcal{T}$ by

$$N_\delta(f) := \left\{ g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \ \text{ and } \ \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{28}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ g \in \mathcal{T} : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \le \delta \right\}. \tag{29}$$

Theorem 7. *If*

$$\delta := \frac{(1-\beta)}{(1+\alpha)C_2^\eta(b,\mu)} \tag{30}$$

then $R^*(\alpha,\beta,\mu,\eta) \subset N_\delta(e)$.

Proof. For $f \in R^*(\alpha,\beta,\mu,\eta)$, Theorem 1 immediately yields

$$2C_2^\eta(b,\mu)(1+\alpha) \sum_{n=2}^{\infty} a_n \le 1-\beta,$$

so that

$$\sum_{n=2}^{\infty} a_n \le \frac{(1-\beta)}{2C_2^\eta(b,\mu)(1+\alpha)} \tag{31}$$

On the other hand, from (17) and (31) that

$$\begin{aligned} \sum_{n=2}^{\infty} n a_n &\le (1-\beta) - 2\alpha C_2^\eta(b,\mu) \sum_{n=2}^{\infty} a_n \\ &\le (1-\beta) - 2\alpha C_2^\eta(b,\mu) \frac{(1-\beta)}{2C_2^\eta(b,\mu)(1+\alpha)} \\ &\le \frac{1-\beta}{C_2^\eta(b,\mu)(1+\alpha)}, \end{aligned}$$

that is

$$\sum_{n=2}^{\infty} n a_n \le \frac{1-\beta}{C_2^\eta(b,\mu)(1+\alpha)} := \delta \tag{32}$$

which, in view of the definition (29) proves Theorem 7.

Now we determine the neighborhood for the class $R^{*(\rho)}(\alpha,\beta,\mu,\eta)$ which we define as follows. A function $f \in \mathcal{T}$ is said to be in the class $R^{*(\rho)}(\alpha,\beta,\mu,\eta)$ if there exists a function $g \in R^{*(\rho)}(\alpha,\beta,\mu,\eta)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1-\rho, \quad (z \in U, \quad 0 \le \rho < 1). \tag{33}$$

Theorem 8.*If* $g \in R^{*(\rho)}(\alpha,\beta,\mu,\eta)$ *and*

$$\rho = 1 - \frac{\delta C_2^\eta(b,\mu)(1+\alpha)}{2C_2^\eta(b,\mu)(1+\alpha) - (1-\beta)} \tag{34}$$

then

$$N_\delta(g) \subset R^{*(\rho)}(\alpha, \beta, \lambda, \eta). \tag{35}$$

Proof. Suppose that $f \in N_\delta(g)$ we then find from (28) that

$$\sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n - b_n| \leq \frac{\delta}{2}.$$

Next, since $g \in R^*(\alpha, \beta, \mu, \eta)$, we have

$$\sum_{n=2}^{\infty} b_n = \frac{(1-\beta)}{2C_2^{\eta}(b,\mu)(1+\alpha)}$$

so that

$$\begin{aligned}
\left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum\limits_{n=2}^{\infty} |a_n - b_n|}{1 - \sum\limits_{n=2}^{\infty} b_n} \\
&\leq \frac{\delta}{2} \times \frac{2C_2^{\eta}(b,\mu)(1+\alpha)}{2C_2^{\eta}(b,\mu)(1+\alpha) - (1-\beta)} \\
&\leq \frac{\delta C_2^{\eta}(b,\mu)(1+\alpha)}{2C_2^{\eta}(b,\mu)(1+\alpha) - (1-\beta)} \\
&= 1 - \rho.
\end{aligned}$$

provided that ρ is given precisely by (35). Thus by definition, $f \in R^{*(\rho)}(\alpha, \beta, \lambda, \eta)$ for ρ given by (35), which completes the proof.

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