Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 307-312

SOME PROPERTIES OF THE GENERALIZED CLASS OF NON-BAZILEVIC FUNCTIONS

Ali Muhmmad

Abstract. In this paper, we define a new class $N_k(n, \lambda, \alpha, \rho)$ in the open unit disk. The object of the present paper is to derive some interesting properties of functions belonging to the class $N_k(n, \lambda, \alpha, \rho)$.

2000 *Mathematics Subject Classification*: 30C45, 30C50.

1. Introduction

Let $\mathcal{A}(n)$, $n \in \mathbb{N}$, denote the class of functions of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k \, z^k, \tag{1.1}$$

which are analytic in the unit disc $E = \{z : z \in \mathbb{C}, |z| < 1\}$. Let $P_k(\rho)$ be the class of functions $h(z)$ analytic in E satisfying the properties $h(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\mathrm{Re} h(z) - \rho}{1-\rho} \right| d\theta \leq k\pi, \tag{1.2}$$

where $z = re^{i\theta}, k \geq 2$ and $0 \leq \rho < 1$. This class has been introduced in [3]. We note, for $\rho = 0$, we obtain the class P_k defined and studied in [4], and for $\rho = 0, k = 2$, we have the well-known class P of functions with positive real part. The case $k = 2$ gives the class $P(\rho)$ of functions with positive real part greater than ρ. From (1.2) we can easily deduce that $h \in P_k(\rho)$ if and only if, there exists $h_1, h_2 \in P(\rho)$ such that for $z \in E$,

$$h(z) = \left(\frac{k}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) h_2(z). \tag{1.3}$$

where $h_i(z) \in P(\rho)$, $i = 1, 2$ and $z \in E$.

Assume that $0 < \alpha < 1$, a function $f \in \mathcal{A}$ is in the class $N(\alpha)$ if and only if

$$\operatorname{Re}\left\{ f'(z) \left(\frac{z}{f(z)} \right)^{1+\alpha} \right\} > 0, \quad z \in E. \tag{1.4}$$

$N(\alpha)$ was introduced by Obradovic [2] recently, he called this class of functions to be of non-Bazilevic type.Until now, this class was studied in a direction of finding necessary conditions over α that embeds this class into the class of univalent functions or its subclass, which is still an open problem.

Definition 1.1. *Let $f \in \mathcal{A}$.Then $f \in N_k(n, \lambda, \alpha, \rho)$ if and only if*

$$\left\{ (1+\lambda) \left(\frac{z}{f(z)} \right)^{\alpha} - \lambda \frac{zf'(z)}{f(z)} \left(\frac{z}{f(z)} \right)^{\alpha} \right\} \in P_k(\rho), \quad z \in E,$$

where $0 < \alpha < 1$, $\lambda \in \mathbb{C}$, $k \geq 2$ and $0 \leq \rho < 1$.The powers are understood as principal values. For $k = 2$ and with different choices of n, λ, α, ρ, these classes have been studied in [2, 5]. *In particular $N_2(1, -1, \alpha, 0)$ is the class of non-Bazilevic functions studied in* [2].

We shall need the following result.

Lemma 1.1 [1]. *Let $u = u_1 + iu_2$, $v = v_1 + iv_2$ and $\Psi(u, v)$ be a complex valued function satisfying the conditions:*
(i). $\Psi(u, v)$ *is continuous in a domain* $D \subset \mathbb{C}^2$,
(ii). $(1, 0) \in D$ *and* $Re\Psi(1, 0) > 0$,
(iii). $Re\Psi(iu_2, v_1) \leq 0$, *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leq -\frac{n}{2}\left(1 + u_2^2\right)$.

If $h(z) = 1 + c_n z + c_{n+1} z^{n+1} + \cdots$ is a function analytic in E such that $(h(z), zh'(z)) \in D$ and $Re\Psi(h(z), zh'(z)) > 0$ for $z \in E$, then $Reh(z) > 0$ in E.

2. Main results

Theorem 2.1. *Let $Re\lambda > 0, 0 < \alpha < 1, 0 \leq \rho < 1$ and $f \in N_k(n, \lambda, \alpha, \rho)$. Then*

$$\left(\frac{z}{f(z)} \right)^{\alpha} \in P_k(\rho_1),$$

where ρ_1 is given by

$$\rho_1 = \frac{2\alpha\rho + n\lambda}{2\alpha + n\lambda}. \tag{2.1}$$

Proof. Let

$$\begin{aligned}\left(\frac{z}{f(z)}\right)^{\alpha} &= (1-\rho_1)h(z)+\rho_1 \\ &= (\frac{k}{4}+\frac{1}{2})\left\{(1-\rho_1)h_1(z)+\rho_1\right\}-(\frac{k}{4}-\frac{1}{2})\left\{(1-\rho_1)h_2(z)+\rho_1\right\}. \quad (2.2)\end{aligned}$$

Then $h_i(z)$ is analytic in E with $h_i(0)=1$, $i=1,2$. Differentiating of (2.2) and some computation gives us

$$\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\}=\left\{(1-\rho_1)h(z)+\rho_1+\frac{\lambda(1-\rho_1)zh'(z)}{\alpha}\right\}$$

$\in P_k(\rho),\ z\in E$. This implies that

$$\frac{1}{(1-\rho)}\left\{(1-\rho_1)h_i(z)+\rho_1-\rho+\frac{\lambda(1-\rho_1)zh_i'(z)}{\alpha}\right\}\in P,\quad i=1,2,\ z\in E.$$

We form the functional $\Psi(u,v)$ by choosing $u=h_i(z)$, $v=zh_i'(z)$.

$$\Psi(u,v)=\left\{(1-\rho_1)u+\rho_1-\rho+\frac{\lambda(1-\rho_1)v}{\alpha}\right\}.$$

The first two conditions of Lemma 1.1 are clearly satisfied.We verify the condition (iii) as follows:

$$\begin{aligned}\operatorname{Re}\left\{\Psi(iu_2,v_1)\right\} &= \rho_1-\rho+\operatorname{Re}\left\{\frac{\lambda(1-\rho_1)v_1}{\alpha}\right\} \\ &\le \rho_1-\rho-\frac{n\lambda(1-\rho_1)(1+u_2^2)}{2\alpha} \\ &= \frac{A+Bu_2^2}{2C},\end{aligned}$$

where

$$\begin{aligned}A &= 2\alpha(\rho_1-\rho)-n\lambda(1-\rho_1), \\ B &= -n\lambda(1-\rho_1)\ \text{and}\ \ C=\alpha>0.\end{aligned}$$

We notice that $\operatorname{Re}\{\Psi(iu_2,v_1)\}\le 0$ if and only if $A\le 0$, $B\le 0$. From $A\le 0$, we obtain $\rho_1$as given by (2.1) and $B\le 0$ gives us $0\le\rho_1<1$. Therefore applying Lemma 1.1, $h_i\in P$, $i=1,2$ and consequently $h\in P_k(\rho_1)$ for $z\in E$.This completes the proof.

Corollary 2.2. *If* $f(z) \in N_2(n, 0, \alpha, \rho)$,*then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > \rho,\ \ z \in E. \tag{2.3}$$

Corollary 2.3. *If* $f(z) \in N_2(n, -1, \alpha, \rho)$, *then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > \frac{2\alpha\rho - n}{2\alpha - n}, \quad z \in E. \tag{2.4}$$

Corollary 2.4. *If* $f(z) \in N_2(1, -1, \alpha, 0)$,*then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > \frac{2\alpha\rho - 1}{2\alpha - 1}, \quad z \in E. \tag{2.5}$$

Corollary 2.5. *If* $f(z) \in N_2(n, \lambda, \frac{1}{2}, \rho)$,*then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > \frac{\rho + n\lambda}{1 + n\lambda}, \quad z \in E. \tag{2.6}$$

Theorem 2.6. *Let* $Re\lambda > 0, 0 < \alpha < 1, 0 \leq \rho < 1$ *and* $f \in N_k(n, \lambda, \alpha, \rho)$. *Then*

$$\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\} \in P_k(\gamma),$$

where

$$\gamma = \frac{\lambda n + \sqrt{(\lambda n)^2 + 4(\alpha + \lambda n)\rho\alpha}}{2(\alpha + \lambda n)}. \tag{2.7}$$

Proof. Let

$$\begin{aligned} \left(\frac{z}{f(z)}\right)^{\alpha} &= ((1-\gamma)h(z) + \gamma)^2 \\ &= (\frac{k}{4} + \frac{1}{2})\left\{(1-\gamma)h_1(z) + \gamma\right\}^2 - (\frac{k}{4} - \frac{1}{2})\left\{(1-\gamma)h_2(z) + \gamma\right\}^2, \end{aligned} \tag{2.8}$$

so $h_i(z)$ is analytic in E, with $h_i(0) = 1$, $i = 1, 2$.Differentiating (2.8) and some computation gives us

$$\begin{aligned} &\left\{(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha} - \lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\} \\ = &\left[\{(1-\gamma)h(z)+\gamma\} + \frac{2\lambda}{\alpha}\{(1-\gamma)h(z)+\gamma\}(1-\gamma)zh'(z)\right] \in P_k(\rho). \end{aligned}$$

This implies that

$$\frac{1}{(1-\rho)}\left[\{(1-\gamma)h_i(z)+\gamma\}(1-\gamma)zh_i'(z) - \rho\right] \in P, \quad z \in E, i = 1, 2.$$

We form the functional $\Psi(u, v)$ by choosing $u = h_i(z)$, $v = zh_i'(z)$.

$$\Psi(u,v) = \{(1-\gamma)u+\gamma\}^2 + \left[\frac{2\lambda}{\alpha}\{(1-\gamma)u+\gamma\}(1-\gamma)v - \rho\right].$$

$$\begin{aligned} \operatorname{Re}\{\Psi(iu_2, v_1)\} &= \gamma^2 - (1-\gamma)^2u_2^2 + \left[\frac{2\lambda}{\alpha}\gamma(1-\gamma)v_1 - \rho\right] \\ &\le \gamma^2 - \rho - (1-\gamma)^2u_2^2 - \frac{\lambda}{\alpha}\left[n\gamma(1-\gamma)(1+u_2^2\right] \\ &= \frac{A + Bu_2^2}{2C}, \end{aligned}$$

where

$$\begin{aligned} A &= (\alpha+\lambda n)\gamma^2 - n\lambda\gamma - \alpha\rho \\ B &= -\alpha(1-\gamma)^2 - n\lambda\gamma(1-\gamma) \quad \text{and } C = \frac{\alpha}{2} > 0. \end{aligned}$$

We notice that $\operatorname{Re}\{\Psi(iu_2, v_1)\} \le 0$ if and only if $A \le 0$, $B \le 0$. From $A \le 0$, we obtain γ as given by (2.7) and $B \le 0$ gives us $0 \le \gamma < 1$. Therefore applying Lemma 1.1, $h_i \in P$, $i = 1, 2$ and consequently $h \in P_k(\gamma)$ for $z \in E$. This completes the proof of Theorem 2.6.

Corollary 2.7. *If $f(z) \in N_2(1, \lambda, \alpha, \rho)$, then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\} > \frac{\lambda + \sqrt{\lambda^2 + 4(\alpha+\lambda)\rho\alpha}}{2(\alpha+\lambda)}, \quad z \in E.$$

Corollary 2.8. *If* $f(z) \in N_2(n, -1, \alpha, \rho)$, *then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\} > \frac{-n + \sqrt{n^2 + 4(\alpha - n)\rho\alpha}}{2(\alpha - n)}, \quad z \in E.$$

Corollary 2.9. *If* $f(z) \in N_2(n, -1, \alpha, 0)$,*then*

$$Re\left\{\left(\frac{z}{f(z)}\right)^{\frac{\alpha}{2}}\right\} > 0, \quad z \in E.$$

References

[1] S. S. Miller and P. T. Mocanu, *Second order differential inequalities in the complex plane*, J. Math. Anal. Appl., 65 (1978), 289-305.
[2] M. Obradovic, *A class of univalent functions*, Hokkaido Math. J., 27 (2) (1998), 329-355.
[3] K. Padmanabhan and R. Parvatham, *Properties of a class of functions with bounded boundary rotation*, Ann. Polan. Math., 31(1975), 311-323.
[4] B. Pinchuk, *Functions with bounded boundary rotation*, Isr. J. Math., 10(1971), 7-16.
[5] N. Tuneski and M. Darus, *Functional for non-Bazilevic functions*, Acta Mathematica. Academiae Paedagogicae. Nyiregyhaziensis, 18 (2002), 63-65.

Ali Muhammad
Department of Basic Sciences,
University of Engineering and Technology Peshawar, Pakistan.
E-mail: *ali7887@gmail.com*