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THE UNIVERSALITY OF OSBORN LOOPS

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Abstract. Recently, two identities that characterize universal(left and right universal) Osborn loops were established. In this study, Kinyon's conjecture that 'every CC-quasigroup is isotopic to an Osborn loop' is shown to be true for universal(left and right universal) Osborn loops if and only if every CC-quasigroup obeys any of the two identities. An Osborn loop is proved to be universal if and only if any of its f, g-principal isotopes is isomorphic to some principal isotopes of the loop, left universal if and only if any of its f, g-principal isotopes is isomorphic to a left principal isotopes of the loop and right universal if and only if any of its f, e-right principal isotopes is isomorphic to some principal isotopes of the loop. The existence of a bi-mapping in the Bryant-Schneider group of a left universal Osborn loop is shown and the consequences of this is discussed for extra loops using some existing results in literature. It is established that there is no non-trivial: universal Osborn loop that can form a special class of G-loop or right G-loop (e.g extra loops, CC-loops or VD-loops) under a tri-mapping, left universal Osborn loop that can form a special class of G-loop under a bi-mapping and a right universal Osborn loop that can form a special class of right G-loop under a bi-mapping.

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1. Introduction

The isotopic invariance of varieties of quasigroups and loops described by one or more equivalent identities, especially those that fall in the class of Bol-Moufang type loops as first named by Fenyves [24] and [23] in the 1960s and later on in this 21^{st} century by Phillips and Vojtěchovský [41], [42] and [35] have been of interest to researchers in loop theory in the recent past. Among such is Etta Falconer's Ph.D [21] and her paper [22] which investigated isotopy invariants in quasigroups. Loops such as Bol loops, Moufang loops, central loops and extra loops are the most popular loops of Bol-Moufang type whose isotopic invariance have been considered. For a good background in loop theory, see [12],[15],[20],[25],[27],[44].

Consider $(G, \cdot)$ and $(H, \circ)$ been two distinct groupoids(quasigroups, loops). Let A, B and C be three bijective mappings, that map G onto H. The triple $\alpha = (A, B, C)$ is called an isotopism of $(G, \cdot)$ onto $(H, \circ)$ if and only if

$$xA \circ yB = (x \cdot y)C \ \forall \ x, y \in G.$$

So, $(H, \circ)$ is called a groupoid(quasigroup, loop) isotope of $(G, \cdot)$. If $C = I$, the identity map on G so that $H = G$, then the triple $\alpha = (A, B, I)$ is called a principal isotopism of $(G, \cdot)$ onto $(G, \circ)$ and $(G, \circ)$ is called a principal isotope of $(G, \cdot)$. Eventually, the equation of relationship now becomes

$$x \cdot y = xA \circ yB \ \forall \ x, y \in G$$

which is easier to work with. But taken $A = R_g$ and $B = L_f$ where $R_x \ : G \to G$ is defined by $yR_x = y \cdot x$ and $L_x \ : G \to G$ is defined by $yL_x = x \cdot y$ for all $x, y \in G$, for some $f, g \in G$, the relationship now becomes $x \cdot y = xR_g \circ yL_f \ \forall \ x, y \in G$ or $x \circ y = xR_g^{-1} \cdot yL_f^{-1} \ \forall \ x, y \in G$. With this new form, the triple $\alpha = (R_g, L_f, I)$ is called an f, g-principal isotopism of $(G, \cdot)$ onto $(G, \circ)$, f and g are called translation elements of G or at times written in the pair form (g, f), while $(G, \circ)$ is called an f, g-principal isotope of $(G, \cdot)$. The last form of α above gave rise to an important result in the study of loop isotopes of loops.

Theorem 1 *(Bruck [12]) Let $(G, \cdot)$ and $(H, \circ)$ be two distinct isotopic loops. For some $f, g \in G$, there exists an f, g-principal isotope $(G, *)$ of $(G, \cdot)$ such that $(H, \circ) \cong (G, *)$.*

With this result, to investigate the isotopic invariance of an isomorphic invariant property in loops, one simply needs only to check if the property in consideration is true in all f, g-principal isotopes of the loop. A property is isotopic invariant if whenever it holds in the domain loop i.e $(G, \cdot)$ then it must hold in the co-domain loop i.e $(H, \circ)$ which is an isotope of the formal. In such a situation, the property in consideration is said to be a universal property hence the loop is called a universal loop relative to the property in consideration as often used by Nagy and Strambach [38] in their algebraic and geometric study of the universality of some types of loops. For instance, if every isotope of a "certain" loop is a "certain" loop, then the former is called a universal "certain" loop. So, we can now restate Theorem 1 as :

Theorem 2 *Let $(G, \cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G, \cdot)$ is a universal "certain" loop if and only if every f, g-principal isotope $(G, *)$ of $(G, \cdot)$ has the "certain" loop property.*

From the earlier discussions, if $(H, \circ) = (G, \cdot)$ then the triple $\alpha = (A, B, C)$ is called an autotopism where $A, B, C \in SYM(G, \cdot)$, the set of all bijections on $(G, \cdot)$ called the symmetric group of $(G, \cdot)$. Such triples form a group $AUT(G, \cdot)$ called the autotopism group of $(G, \cdot)$. Furthermore, if $A = B = C$, then $A \in AUM(G, \cdot) = AUM(G)$, the automorphism group of G.

Bol-Moufang type of quasigroups(loops) are not the only quasigroups(loops) that are isomorphic invariant and whose universality have been considered. Some others are weak inverse property loops(WIPLs) and cross inverse property loops(CIPLs). The universality of WIPLs and CIPLs have been addressed by OSborn [39] and Artzy [2] respectively. In 1970, Basarab [5] later continued the work of Osborn of 1961 on universal WIPLs by studying isotopes of WIPLs that are also WIPLs after he had studied a class of WIPLs([3]) in 1967. Osborn [39], while investigating the universality of WIPLs discovered that a universal WIPL $(G, \cdot)$ obeys the identity

$$yx \cdot (zE_y \cdot y) = (y \cdot xz) \cdot y \ \forall \ x, y, z \in G \tag{1}$$

where $E_y = L_y L_{y^\lambda} = R_{y^\rho}^{-1} R_y^{-1} = L_y R_y L_y^{-1} R_y^{-1}$. Eight years after Osborn's [39] 1960 work on WIPL, in 1968, Huthnance Jr. [26] studied the theory of generalized Moufang loops. He named a loop that obeys (1) a generalized Moufang loop and later on in the same thesis, he called them M-loops. On the other hand, he called a universal WIPL an Osborn loop and this same definition was adopted by Chiboka [16]. Basarab dubbed a loop $(G, \cdot)$ satisfying the identity:

$$x(yz \cdot x) = (x \cdot yE_x) \cdot zx \ \forall \ x, y, z \in G \tag{2}$$

an Osborn loop where $E_x = R_x R_{x^\rho} = (L_x L_{x^\lambda})^{-1} = R_x L_x R_x^{-1} L_x^{-1}$. It will look confusing if both Basarab's and Huthnance's definitions of an Osborn loop are both adopted because an Osborn loop of Basarab is not necessarily a universal WIPL(Osborn loop of Huthnance). So in this work, Huthnance's definition of an Osborn loop will be dropped while we shall stick to that of Basarab which was actually adopted by Kinyon [32] and the open problem we intend to solve is relative to Basarab's definition of an Osborn loop and not that of Huthnance. Huthnance [26] was able to deduce some properties of E_x relative to (1). $E_x = E_{x^\lambda} = E_{x^\rho}$. So, since $E_x = R_x R_{x^\rho}$, then $E_x = E_{x^\lambda} = R_{x^\lambda} R_x$ and $E_x = (L_{x^\rho} L_x)^{-1}$. So, we now have two identities equivalent to identities (1) and (2) defining an Osborn loop.

$$\mathrm{OS}_0 \ : \ x(yz \cdot x) = x(yx^\lambda \cdot x) \cdot zx \tag{3}$$

$$\mathrm{OS}_1 \ : \ x(yz \cdot x) = x(yx \cdot x^\rho) \cdot zx \tag{4}$$

Although Basarab [8] and [11] considered universal Osborn loops but the universality of Osborn loops was raised as an open problem by Michael Kinyon in 2005 at a conference tagged "Milehigh Conference on Loops, Quasigroups and Non-associative Systems" held at the University of Denver, where he presented a talk titled "A Survey of Osborn Loops". He also stated a conjecture.

Kinyon's Conjecture: Every CC-quasigroup is isotopic to an Osborn. And he mentioned that CC-quasigroups include CC-loops, quasigroups that are isotopic to groups and trimedial quasigroups. Trimedial quasigroups have been shown to be isotopic to commutative Moufang loops in Kepka [31]. In Jaiyéọlá and Adéníran [28], two identities that characterize universal(left and right universal) Osborn loops were established while in Jaiyéọlá and Adéníran [29], it is established that not all Osborn loops are universal. In this study, Kinyon's conjecture that 'every CC-quasigroup is isotopic to an Osborn loop' is shown to be true for universal(left and right universal) Osborn loops if and only if every CC-quasigroup obeys any of the two identities. An Osborn loop is proved to be universal if and only if any of its f,g-principal isotopes is isomorphic to some principal isotopes of the loop, left universal if and only if any of its f,g-principal isotopes is isomorphic to a left principal isotopes of the loop and right universal if and only if any of its f,e-right principal isotopes is isomorphic to some principal isotopes of the loop. The existence of a bi-mapping in the Bryant-Schneider group of a left universal Osborn loop is shown and the consequences of this is discussed for extra loops using some existing results in literature. It is established that there is no non-trivial: universal Osborn loop that can form a special class of G-loop or right G-loop (e.g extra loops, CC-loops or VD-loops) under a tri-mapping, left universal Osborn loop that can form a special class of G-loop under a bi-mapping and a right universal Osborn loop that can form a special class of right G-loop under a bi-mapping.

2. Preliminaries

Let G be a non-empty set. Define a binary operation $(\cdot)$ on G. If $x \cdot y \in G$ for all $x, y \in G$, then the pair $(G, \cdot)$ is called a groupoid or Magma. If the system of equations:

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions in G for x and y respectively, then $(G, \cdot)$ is called a quasigroup. A quasigroup is therefore an algebra having a binary multiplication $x \cdot y$ usually written xy which satisfies the conditions that for any a, b in the quasigroup the equations

$$a \cdot x = b \qquad \text{and} \qquad y \cdot a = b$$

have unique solutions for x and y lying in the quasigroup. If there exists a unique element $e \in G$ called the identity element such that for all $x \in G$, $x \cdot e = e \cdot x = x$, $(G, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. Let x be a fixed element in a groupoid $(G, \cdot)$. The left and right translation maps of G, L_x and R_x respectively can be defined by

$$yL_x = x \cdot y \qquad \text{and} \qquad yR_x = y \cdot x.$$

It can now be seen that a groupoid $(G, \cdot)$ is a quasigroup if it's left and right translation mappings are bijections or permutations. Since the left and right translation mappings of a loop are bijective, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} = y\mathbb{L}_x \qquad \text{and} \qquad x/y = xR_y^{-1} = x\mathbb{R}_y$$

and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

Hence, $(G, \backslash)$ and $(G, /)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be stated as follows.

Definition 1 *A loop $(G, \cdot, /, \backslash, e)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that*

(i) $x \cdot (x\backslash y) = y$, $(y/x) \cdot x = y$ *for all* $x, y \in G$,

(ii) $x\backslash(x \cdot y) = y$, $(y \cdot x)/x = y$ *for all* $x, y \in G$ *and*

(iii) $x\backslash x = y/y$ *or* $e \cdot x = e$ *for all* $x, y \in G$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x \cdot (y\backslash z)$ respectively. In a loop $(G, \cdot)$ with identity element e, the left inverse element of $x \in G$ is the element $x^\lambda \in G$ such that

$$x^\lambda \cdot x = e$$

while the right inverse element of $x \in G$ is the element $x^\rho \in G$ such that

$$x \cdot x^\rho = e$$

The identities describing the most popularly known varieties of Osborn loops such as CC-loops, Moufang loops, VD-loops and universal WIPLs are given in Definition 2.4 of Jaiyéọlá and Adéníran [28]. All these four varieties of Osborn loops are universal. CC-loops, and VD-loops are G-loops. G-loops are loops that are isomorphic to all their loop isotopes. Kunen [37] has studied them. A conjugacy closed quasigroup(CC-quasigroup) is a quasigroup that obeys the identities

$$x \cdot (yz) = \{[x \cdot (y \cdot (x\backslash x))]/x\} \cdot (xz) \text{ and } (zy) \cdot x = (zx) \cdot \{x\backslash[((x/x) \cdot y) \cdot x]\}.$$

For the definitions of left isotopes, right isotopes, left principal isotopes and right principal isotopes, see Definition 2.7 of Jaiyéọlá and Adéníran [28]

Theorem 3 *Let $(G, \cdot)$ and $(H, \circ)$ be two distinct left(right) isotopic loops with the former having an identity element e. For some $g(f) \in G$, there exists an $e, g(f, e)$-principal isotope $(G, *)$ of $(G, \cdot)$ such that $(H, \circ) \cong (G, *)$.*

A loop is a left(right) universal "certain" loop if and only if all its $e, g(f, e)$-left(right) principal isotopes are "certain" loops. A loop is called a right G-loop(G_ρ-loop) if and only if it is isomorphic to all its f, e-right principal loop isotopes. A loop is called a left G-loop(G_λ-loop) if and only if it is isomorphic to all its e, g-left principal loop isotopes. A loop is a G-loop if and only if it is a G_ρ-loop and a G_λ-loop. Kunen [37] demonstrated the use of G_ρ-loops and G_λ-loops. We shall treat the G-loops and G_ρ-loops of some universal and right universal Osborn loops(respectively) in the following manner.

Definition 2 *Let $(L, \cdot, \backslash, /)$ be an Osborn loop with a mapping $\Theta \in SYM(L, \cdot)$. Suppose Θ is an element of the multiplication group $\mathcal{M}ult(L)$ of L such that $\Theta(x, y, z)$, i.e Θ is the product of right, left translation mappings $R_{\alpha(x,y,z)}, L_{\beta(x,y,z)}$ and their inverses $\mathbb{R}_{\alpha(x,y,z)}, \mathbb{L}_{\beta(x,y,z)}$ such that $\alpha(x, y, z)$ and $\beta(x, y, z)$ are words in L in terms of arbitrary elements $x, y, z \in L$ with a minimum of length one. Then Θ is called a tri-mapping of L.*

1. *L is called a $G(\Theta_3)$-loop if it is a G-loop such that there exists a tri-mapping Θ which is the isomorphism from L to all its f, g-principal isotopes.*
2. *L is called a $G_\rho(\Theta_2)$-loop if it is a G_ρ-loop such that there exists a bi-mapping Θ which is the isomorphism from L to all its f, e-principal isotopes.*

Remark 1 *Some popular examples of bi-mappings are the right and left inner mappings $R(x, y)$ and $L(x, y)$ respectively. The middle inner mapping $T(x)$ is a familiar mono-mapping. Tri-mappings, tetra-mappings e.t.c can be obtained by multiplying bi-mappings and mono-mappings. A demonstration of this can be seen in Bruck and Paige [13] and Kinyon et. al. [34]. In fact, according to Kinyon et. al. [33], in a CC-loop, $R(x, y)$ and $L(u, v)$ all commute with each other. So, it is sensible to consider tetra-mappings in some universal Osborn loops.*

Theorem 4 *Let $(G,\cdot)$ be a "certain" loop where "certain" is an isomorphic invariant property. $(G,\cdot)$ is a left(right) universal "certain" loop if and only if every $e,g(f,e)$-principal isotope $(G,*)$ of $(G,\cdot)$ has the "certain" loop property.*

Theorem 5 *(Chiboka and Solarin [18], Kunen [36]) Let $(G,\cdot)$ be a loop.*

1. *G is called a G_ρ-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta,\theta L_y^{-1},\theta)\in AUT(G,\cdot)\ \forall\ y\in G$.*
2. *G is called a G_λ-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta R_x^{-1},\theta,\theta)\in AUT(G,\cdot)\ \forall\ x\in G$.*
3. *G is called a G-loop if and only if there exists $\theta\in SYM(G,\cdot)$ such that $(\theta R_x^{-1},\theta L_y^{-1},\theta)\in AUT(G,\cdot)\ \forall\ x,y\in G$.*

Definition 3 *(Robinson [43]) Let $(G,\cdot)$ be a loop.*

1. *A mapping $\theta\in SYM(G,\cdot)$ is a right special map for G means that there exist $f\in G$ so that $(\theta,\theta L_f^{-1},\theta)\in AUT(G,\cdot)$.*
2. *A mapping $\theta\in SYM(G,\cdot)$ is a left special map for G means that there exist $g\in G$ so that $(\theta R_g^{-1},\theta,\theta)\in AUT(G,\cdot)$.*
3. *A mapping $\theta\in SYM(G,\cdot)$ is a special map for G means that there exist $f,g\in G$ so that $(\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)$.*

From Definition 3, it can be observed that θ is a left or right special map for a loop $(G,\cdot)$ with identity element e if and only if θ is an isomorphism of $(G,\cdot)$ onto some e,g- or f,e- principal isotope $(G,\circ)$ of $(G,\cdot)$. More so, θ is a special map for a loop $(G,\cdot)$ if and only if θ is an isomorphism of $(G,\cdot)$ onto some f,g-principal isotope $(G,\circ)$ of $(G,\cdot)$. Robinson [43] went further to show that if

$$BS(G,\cdot)=\{\theta\in SYM(G,\cdot)\ :\ \exists\ f,g\in G\ \ni\ (\theta R_g^{-1},\theta L_f^{-1},\theta)\in AUT(G,\cdot)\}$$

i.e the set of all special maps in a loop, then $BS(G,\cdot)\le SYM(G,\cdot)$ called the Bryant-Schneider group of the loop $(G,\cdot)$ because its importance and motivation stem from the work of Bryant and Schneider [14]. Since the advent of the Bryant-Schneider group, some studies by Adeniran [1] and Chiboka [17] have been done on it relative to CC-loops and extra loops. Let

$$BS_\lambda(G,\cdot)=\{\theta\in SYM(G,\cdot)\ :\ \exists\ g\in G\ \ni\ (\theta R_g^{-1},\theta,\theta)\in AUT(G,\cdot)\}$$

i.e the set of all left special maps in a loop, then $BS_\lambda(G,\cdot) \le BS(G,\cdot)$ called the left Bryant-Schneider group of the loop $(G,\cdot)$ and

$$BS_\rho(G,\cdot) = \{\theta \in SYM(G,\cdot)\ :\ \exists\ f \in G\ \ni\ (\theta, \theta L_f^{-1}, \theta) \in AUT(G,\cdot)\}$$

i.e the set of all right special maps in a loop, then $BS_\rho(G,\cdot) \le BS(G,\cdot)$ called the right Bryant-Schneider group of the loop $(G,\cdot)$. We shall make a judicious use of these three groups as earlier predicted by Robinson [43]. We shall be making a judicious use of the following recently proven results of Jaiyéọlá and Adéníran [28].

Theorem 6 *A loop $(Q,\cdot,\backslash,/)$ is a universal Osborn loop if and only if it obeys the identity*

$$\underbrace{x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\} = (x\cdot u\backslash\{[y(u\backslash([(uv)/(u\backslash(xv))]v))]/v\cdot[u\backslash(xv)]\})/v\cdot u\backslash[((uz)/v)(u\backslash(xv))]}_{OS_0'}$$

or

$$\underbrace{x\cdot u\backslash\{(yz)/v\cdot[u\backslash(xv)]\} = \{x\cdot u\backslash\{[y(u\backslash(xv))]/v\cdot[x\backslash(uv)]\}\}/v\cdot u\backslash[((uz)/v)(u\backslash(xv))].}_{OS_1'}$$

Lemma 1 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. Q is a universal Osborn loop if and only if the triple $\big(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v)\big) \in AUT(Q)$ or the triple $\Big(R_{[u\backslash(xv)]}\mathbb{R}_vR_{[x\backslash(uv)]}\mathbb{R}_{[u\backslash(xv)]}R_v\gamma(x,u,v)\mathbb{R}_v,\beta(x,u,v),\gamma(x,u,v)\Big)\ \in\ AUT(Q)$ for all $x,u,v \in Q$ where $\alpha(x,u,v) = R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_v$, $\beta(x,u,v) = L_u\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_u$ and $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ are elements of $\mathcal{M}ult(Q)$.*

Theorem 7 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. If Q is a universal Osborn loop, then the triple $\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\beta(x,u,v),\gamma(x,u,v)\Big) \in AUT(Q)$ for all $x,u,v \in Q$ where $\alpha(x,u,v) = R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x\mathbb{R}_v$, $\beta(x,u,v) = L_u\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_u$ and $\gamma(x,u,v) = \mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ are elements of $\mathcal{M}ult(Q)$.*

Theorem 8 *A loop $(Q,\cdot,\backslash,/)$ is a left universal Osborn loop if and only if it obeys the identity*

$$\underbrace{x\cdot[(y\cdot zv)/v\cdot(xv)] = (x\cdot\{[y([v/(xv)]v)]/v\cdot(xv)\})/v\cdot[z\cdot xv]}_{OS_0^\lambda} \qquad or$$

$$\underbrace{x\cdot[(y\cdot zv)/v\cdot(xv)] = \{(x\cdot[(y\cdot xv)/v\cdot(x\backslash v)]\}/v\cdot[z(xv)].}_{OS_1^\lambda}$$

Lemma 2 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. Q is a left universal Osborn loop if and only if the triple $\big(\alpha(x,v),\beta(x,v),\gamma(x,v)\big)\in AUT(Q)$ or $\big(R_{[xv]}\mathbb{R}_vR_{[x\backslash v]}\mathbb{R}_{[xv]}R_v\gamma(x,v)\mathbb{R}_v,\beta(x,v),\gamma(x,v)\big)\in AUT(Q)$ for all $x,v\in Q$ where $\alpha(x,v)=R_{([v/(xv)]v)}\mathbb{R}_vR_{[xv]}L_x\mathbb{R}_v$, $\beta(x,v)=\mathbb{R}_vR_{[xv]}$ and $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$ are elements of $\mathcal{M}ult(Q)$.*

Theorem 9 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. If Q is a left universal Osborn loop, then the triple $\Big(\gamma(x,v)\mathbb{R}_{[v^\lambda\cdot xv]},\beta(x,v),\gamma(x,v)\Big)\in AUT(Q)$ for all $x,v\in Q$ where $\alpha(x,v)=R_{([v/(xv)]v)}\mathbb{R}_vR_{(xv)}L_x\mathbb{R}_v$, $\beta(x,v)=\mathbb{R}_vR_{(xv)}$ and $\gamma(x,v)=\mathbb{R}_vR_{(xv)}L_x$ are elements of $\mathcal{M}ult(Q)$.*

Theorem 10 *A loop $(Q,\cdot,\backslash,/)$ is a right universal Osborn loop if and only if it obeys the identity*

$$\underbrace{(ux)\cdot u\backslash\{yz\cdot x\}=((ux)\cdot u\backslash\{[y(u\backslash[u/x])]\cdot x\})\cdot u\backslash[(uz)x].}_{OS_0^\rho}\qquad or$$

$$\underbrace{(ux)\cdot u\backslash\{(yz)\cdot x\}=\{(ux)\cdot u\backslash[yx\cdot(ux)\backslash u]\}\cdot u\backslash[(uz)x].}_{OS_1^\rho}$$

Lemma 3 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. Q is a right universal Osborn loop if and only if the triple $\big(\alpha(x,u),\beta(x,u),\gamma(x,u)\big)\in AUT(Q)$ or the triple $\Big(R_{[u\backslash x]}R_{[x\backslash u]}\mathbb{R}_{[u\backslash x]}\gamma(x,u),\beta(x,u),\gamma(x,u)\Big)\in AUT(Q)$ for all $x,u\in Q$ where $\alpha(x,u)=R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_uL_x$, $\beta(x,u)=L_uR_{[u\backslash x]}\mathbb{L}_u$ and $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ are elements of $\mathcal{M}ult(Q)$.*

Theorem 11 *Let Q be a loop with multiplication group $\mathcal{M}ult(Q)$. If Q is a right universal Osborn loop, then the triple $\Big(\gamma(x,u)\mathbb{R}_{(u\backslash x)},\beta(x,u),\gamma(x,u)\Big)\in AUT(Q)$ for all $x,u\in Q$ where $\alpha(x,u)=R_{(u\backslash[u/(u\backslash x)])}R_{[u\backslash x]}\mathbb{L}_uL_x$, $\beta(x,u)=L_uR_{[u\backslash x)]}\mathbb{L}_u$ and $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ are elements of $\mathcal{M}ult(Q)$.*

3. Main Results

Lemma 4 *A quasigroup is isotopic to a universal Osborn loop if and only if it obeys the identity OS_0' or OS_1'.*

Proof Let Q be a quasigroup that is isotopic to a universal Osborn loop L i.e every loop isotope G of L is an Osborn loop. Then, the isotopisms $Q\longrightarrow L$ and $L\longrightarrow G$ imply the isotopism $Q\longrightarrow G$. Let H be any loop isotope of Q, then $H\longrightarrow G$ is an isotopism and so $H \longrightarrow L$ is an isotopism, hence, H is an Osborn loop. Let $H = (Q, \blacktriangle)$ be a principal loop isotope of $(Q, \cdot)$ such that

$$x\blacktriangle y = xR_v^{-1} \cdot yL_u^{-1} = (x/v) \cdot (u\backslash y)\ \forall\ u, v \in Q.$$

Then, thinking in line with the proof of Theorem 6, H obeys identity OS_0 or OS_1 if and only if Q obeys identity OS_0' or OS_1'. The proof of the conversely is as follows. If Q obeys identity OS_0' or OS_1', then every f, g-principal loop isotope of Q is an Osborn loop, hence, all loop isotopes of Q are Osborn loops. Let L be a loop isotope of Q with arbitrary loop isotope L'. So L' is a loop isotope of Q, hence L' is an Osborn loop. Therefore, Q is isotopic to a universal Osborn loop.

Corollary 1 *A quasigroup is isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL implies it obeys the identity OS_0' or OS_1'.*

Remark 2 *Not all CC-quasigroups are isotopic to groups or Moufang loops or VD-loops.*

Theorem 12 *An Osborn loop is universal if and only if any of its x, v-principal isotopes is isomorphic to some particular principal isotopes.*

Proof Let $(Q, \cdot, \backslash, /)$ be a universal Osborn loop. We shall use Lemma 1. The triple

$$\big(\alpha(x,u,v), \beta(x,u,v), \gamma(x,u,v)\big) = \Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\gamma\mathbb{R}_v, L_u\gamma\mathbb{L}_x, \gamma\Big)$$

can be written as the following compositions $\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}, L_u, I\Big)(\gamma,\gamma,\gamma)(\mathbb{R}_v, \mathbb{L}_x, I)$. Let $(Q, \circ)$ be an arbitrary x, v-principal isotope of $(Q, \cdot)$ and $(Q, *)$ a particular principal isotope of $(Q, \cdot)$. Let $\phi(x, u, v) = (u\backslash([(uv)/(u\backslash(xv))]v))$, then the composition above can be expressed as:

$$(Q,\cdot) \xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u,v)}, L_u, I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v, \mathbb{L}_x, I)} (Q,\cdot).$$

This means that any x, v-principal isotope $(Q, \circ)$ of $(Q, \cdot)$ is isomorphic to some particular principal isotope $(Q, *)$ of $(Q, \cdot)$.

Theorem 13 *An Osborn loop is universal if and only if the existence of the principal autotopism $(R_{\phi(x,u,v)}, L_u, I)$, $\phi(x, u, v) = (u\backslash([(uv)/(u\backslash(xv))]v))$ in the loop implies the triple $\big(\gamma(x,u,v)\mathbb{R}_v, \gamma(x,u,v)\mathbb{L}_x, \gamma(x,u,v)\big)$, where $\gamma(x,u,v) = \mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ is an autotopism in the loop, and vice versa.*

Proof The proof is in line with Theorem 12 with a slight adjustment to the composition of the triple

$$\big(\alpha(x,u,v),\beta(x,u,v),\gamma(x,u,v)\big)=\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))}\gamma\mathbb{R}_v,L_u\gamma\mathbb{L}_x,\gamma\Big)$$

which can be re-written as the following compositions $\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))},L_u,I\Big)(\gamma\mathbb{R}_v,\gamma\mathbb{L}_x,\gamma)$. Hence, the conclusion follows.

Theorem 14 *If an Osborn loop is universal then, any of its u,e-right principal isotopes is isomorphic to some principal isotopes.*

Proof By Theorem 7, if Q is a universal Osborn loop, then

$$\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\beta(x,u,v),\gamma(x,u,v)\Big)=$$

$$\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},L_u\gamma\mathbb{L}_x,\gamma(x,u,v)\Big)\in AUT(Q)$$

for all $x,u,v\in Q$. Writing

$$\Big(\gamma(x,u,v)\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},L_u\gamma\mathbb{L}_x,\gamma(x,u,v)\Big)=(I,L_u,I)\big(\gamma(x,u,v),\gamma(x,u,v),\gamma(x,u,v)\big)$$

$$\Big(\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\mathbb{L}_x,I\Big)\in AUT(Q)$$

such that

$$(Q,\cdot)\xrightarrow[\text{right principal isotopism}]{(I,L_u,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{\Big(\mathbb{R}_{(u\backslash[(u/v)(u\backslash(xv))])},\mathbb{L}_x,I\Big)}(Q,\cdot)$$

where $(Q,*)$ is a u,e-right principal isotope of $(Q,\cdot)$ and $(Q,\circ)$ are some particular principal isotope of $(Q,\cdot)$, the conclusion of the theorem follows.

Theorem 15 *If an Osborn loop is universal then, the existence of the principal autotopism $(R_{\psi(x,u,v)},L_x,I)$, $\psi(x,u,v)=(u\backslash[(u/v)(u\backslash(xv))])$ in the loop implies the triple $\big(\gamma(x,u,v)^{-1},\gamma(x,u,v)^{-1}\mathbb{L}_u,\gamma(x,u,v)^{-1}\big)$, where $\gamma(x,u,v)=\mathbb{R}_vR_{[u\backslash(xv)]}\mathbb{L}_uL_x$ is an autotopism in the loop, and vice versa.*

Proof The proof is in line with Theorem 14 with a slight adjustment to the composition by simply considering the inverse composition and reasoning like we did in Theorem 13.

Theorem 16 *There does not exist a non-trivial universal Osborn loop that is a $G(\gamma_3)$-loop or a $G_\rho(\gamma_3^{-1})$-loop with the tri-mapping $\gamma(x,u,v)=\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$.*

Proof

(a) We shall show that if $\mathcal{Q}=(Q,\cdot,\backslash,/)$ is an Osborn loop such that the tri-mapping $\gamma(x,u,v)=\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$. Then, $\mathcal{Q}$ is a universal Osborn loop if and only if $\mathcal{Q}$ is a G(γ_3)-loop implies it obeys the identity

$$y(u\backslash([(uv)/(u\backslash(xv))]v))\cdot uz=yz\ \forall\ x,y,z,u,v\in Q \tag{5}$$

and vice versa.

The proof of this statement is based on Theorem 1 and is achieved by using the compositions $\Big(R_{(u\backslash([(uv)/(u\backslash(xv))]v))},L_u,I\Big)(\gamma\mathbb{R}_v,\gamma\mathbb{L}_x,\gamma)$ of Theorem 13 and hence following Theorem 5, it is a G-loop, and in particular a G(γ_3)-loop which implies it obeys identity (5) and vice versa.

(b) We shall also show that if $\mathcal{Q}=(Q,\cdot,\backslash,/)$ is an Osborn loop such that the tri-mapping $\gamma(x,u,v)=\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$ and $\mathcal{Q}$ is a universal Osborn loop then, $\mathcal{Q}$ is a G$_\rho$(γ_3^{-1})-loop implies it obeys the identity

$$y(u\backslash[(u/v)(u\backslash(xv))])\cdot xz=yz\ \forall\ x,y,z,u,v\in Q \tag{6}$$

and vice versa.

The proof of this statement is based on the composition used in Theorem 15. The reasoning used is similar to that in (a).

According to (a) or (b), if $\mathcal{Q}$ is a G(γ_3)-loop or G$_\rho$(γ_3^{-1}) then it obeys identity (5) or (6). Put $y=z=v=u=e$ in identity (5) or (6), then $x=e$. Which is a contradiction.

Remark 3 *There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop that is a $G(\gamma_3)$-loop or $G_\rho(\gamma_3^{-1})$ when $\gamma(x,u,v)=\mathbb{R}_v R_{[u\backslash(xv)]}\mathbb{L}_u L_x$*

Lemma 5 *A quasigroup is left isotopic to a left universal Osborn loop if and only if it obeys the identity OS_0^λ or OS_1^λ.*

Proof The method of the proof of this lemma is similar to the method used to prove Lemma 4 by using Theorem 3 and Theorem 4.

Corollary 2 *A quasigroup is left isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL implies it obeys the identity OS_0^λ or OS_1^λ.*

Remark 4 *Not all CC-quasigroups are left isotopic to groups or Moufang loops or VD-loops.*

Theorem 17 *An Osborn loop is left universal if and only if any of its v, x-principal isotopes is isomorphic to a left principal isotope.*

Proof Let $(Q, \cdot, \backslash, /)$ be a left universal Osborn loop. We shall use Lemma 2. The triple
$$\big(\alpha(x,v), \beta(x,v), \gamma(x,v)\big) = \big(R_{([v/(xv)]v)}\gamma\mathbb{R}_v, \gamma\mathbb{L}_x, \gamma\big)$$
can be written as the following compositions $\big(R_{([v/(xv)]v)}, I, I\big)(\gamma,\gamma,\gamma)(\mathbb{R}_v, \mathbb{L}_x, I)$. Let $(Q, \circ)$ be a x, v-principal isotope of $(Q, \cdot)$ and $(Q, *)$ a left principal isotope of $(Q, \cdot)$. Let $\phi(x,v) = ([v/(xv)]v)$, then the composition above can be expressed as:

$$(Q,\cdot) \xrightarrow[\text{left principal isotopism}]{(R_{\phi(x,v)}, I, I)} (Q,*) \xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)} (Q,\circ) \xrightarrow[\text{principal isotopism}]{(\mathbb{R}_v, \mathbb{L}_x, I)} (Q,\cdot).$$

This means that a x, v-principal isotope $(Q, \circ)$ of $(Q, \cdot)$ is isomorphic to a left principal isotope $(Q, *)$ of $(Q, \cdot)$.

Theorem 18 *An Osborn loop is left universal if and only if the existence of the principal autotopism $(R_{\phi(x,v)}, I, I)$, $\phi(x,v) = ([v/(xv)]v)$ in the loop implies the triple $\big(\gamma(x,v)\mathbb{R}_v, \gamma(x,v)\mathbb{L}_x, \gamma(x,v)\big)$, where $\gamma(x,v) = \mathbb{R}_v R_{(xv)} L_x$ is an autotopism in the loop and vice versa.*

Proof The proof is in line with Theorem 17 with a slight adjustment to the composition of the triple
$$\big(\alpha(x,v), \beta(x,v), \gamma(x,v)\big) = \Big(R_{([v/(xv)]v)}\gamma\mathbb{R}_v, \gamma\mathbb{L}_x, \gamma\Big)$$
which can be re-written as the following compositions $\Big(R_{([v/(xv)]v)}, I, I\Big)(\gamma\mathbb{R}_v, \gamma\mathbb{L}_x, \gamma)$. Hence, the conclusion follows.

Theorem 19 *If an Osborn loop is left universal then, the mapping $\gamma(x,v) = \mathbb{R}_v R_{[xv]} L_x$ is an element of the Bryant Schneider group of the loop for all elements x, v in the loop.*

Proof By Theorem 9, if Q is a left universal Osborn loop, then
$$\Big(\gamma(x,v)\mathbb{R}_{(v^\lambda \cdot xv)}, \beta(x,v), \gamma(x,v)\Big) = \big(\gamma(x,v)\mathbb{R}_{(v^\lambda \cdot xv)}, \gamma(x,v)\mathbb{L}_x, \gamma(x,v)\big) \in AUT(Q)$$
for all $x, u, v \in Q$. Hence, $\gamma(x,v) \in BS(Q)$.

Lemma 6 *If an Osborn loop Q is left universal then, the mapping $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x\in AUM(Q)$ if and only if Q obeys the identity $(v^\lambda\cdot xv)y\cdot xz=yz$ for all $x,v,y,z\in Q$. Hence, Q is an abelian group.*

Proof By Theorem 9, if Q is a left universal Osborn loop, then

$$\Big(\gamma(x,v)\mathbb{R}_{(v^\lambda\cdot xv)},\beta(x,v),\gamma(x,v)\Big)=\big(\gamma(x,v)\mathbb{R}_{(v^\lambda\cdot xv)},\gamma(x,v)\mathbb{L}_x,\gamma(x,v)\big)\in AUT(Q)$$

for all $x,u,v\in Q$. By breaking this triple appropriately into two, the claim follows. In the equation $(v^\lambda\cdot xv)y\cdot xz=yz$, if $v=y=z=e$, then $x^2=e$ which means Q is an Osborn loop of exponent 2, thence, an abelian group following Basarab [8].

Theorem 20 *There does not exist a non-trivial left universal Osborn loop that is a $G(\gamma_2)$-loop with bi-mapping $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$.*

Proof It can be shown that if $\mathcal{Q}=(Q,\cdot,\backslash,/)$ is an Osborn loop such that the bi-mapping $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$. $\mathcal{Q}$ is a left universal Osborn loop if and only if $\mathcal{Q}$ is a G(γ_2)-loop implies it obeys the identity

$$y([v/(xv)]v)\cdot z=yz\ \forall\ x,y,z,v\in Q\tag{7}$$

and vice versa. The proof is based on Theorem 2 and is achieved by using the compositions $\Big(R_{([v/(xv)]v)},I,I\Big)(\gamma\mathbb{R}_v,\gamma\mathbb{L}_x,\gamma)$ of Theorem 18 and hence following Theorem 5, it is a G-loop, and in particular a G(γ_2)-loop which implies it obeys identity (7) and vice versa. From the statement above, if $\mathcal{Q}$ is a G(γ_2)-loop then it obeys identity (7). Put $y=z=v=e$ in identity (7), then $x=e$. Which is a contradiction.

Remark 5 *There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop that is a $G(\gamma_2)$-loop when $\gamma(x,v)=\mathbb{R}_vR_{[xv]}L_x$.*

Lemma 7 *A quasigroup is right isotopic to a right universal Osborn loop if and only if it obeys the identity OS_0^ρ or OS_1^ρ.*

Proof The method of the proof of this lemma is similar to the method used to prove Lemma 4 by using Theorem 3 and Theorem 4.

Corollary 3 *A quasigroup is right isotopic to a Moufang loop or CC-loop or VD-loop or universal WIPL if and only if it obeys the identity OS_0^ρ or OS_1^ρ.*

Remark 6 *Not all CC-quasigroups are right isotopic to groups or Moufang loops or VD-loops.*

Theorem 21 *An Osborn loop is right universal if and only if any of its x,e-right principal isotopes is isomorphic to some principal isotopes.*

Proof Let $(Q,\cdot,\backslash,/)$ be a right universal Osborn loop. We shall use Lemma 3. The triple

$$\big(\alpha(x,u),\beta(x,u),\gamma(x,u)\big)=\Big(R_{(u\backslash[u/(u\backslash x)])}\gamma,L_u\gamma\mathbb{L}_x,\gamma\Big)$$

can be written as the following compositions $\big(R_{(u\backslash[u/(u\backslash x)])},L_u,I\big)(\gamma,\gamma,\gamma)(I,\mathbb{L}_x,I)$. Let $(Q,\circ)$ be an arbitrary right principal isotope of $(Q,\cdot)$ and $(Q,*)$ a principal isotope of $(Q,\cdot)$. Let $\phi(x,u)=(u\backslash[u/(u\backslash x)])$, then the composition above can be expressed as:

$$(Q,\cdot)\xrightarrow[\text{principal isotopism}]{(R_{\phi(x,u)},L_u,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\text{right principal isotopism}]{(I,\mathbb{L}_x,I)}(Q,\cdot).$$

This means that a x,e-right principal isotope $(Q,\circ)$ of $(Q,\cdot)$ is isomorphic to some principal isotopes $(Q,*)$ of $(Q,\cdot)$.

Theorem 22 *An Osborn loop is right universal if and only if the existence of the principal autotopism $(R_{\phi(x,u)},L_u,I)$, $\phi(x,u)=(u\backslash[u/(u\backslash x)])$ in the loop implies the triple $(\gamma,\gamma\mathbb{L}_x,\gamma)$ where $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ is an autotopism in the loop and vice versa.*

Proof The proof is in line with Theorem 21 with a slight adjustment to the composition of the triple

$$\big(\alpha(x,u),\beta(x,u),\gamma(x,u)\big)=\Big(R_{(u\backslash[u/(u\backslash x)])}\gamma,L_u\gamma\mathbb{L}_x,\gamma\Big)$$

which can be re-written as the following compositions $\Big(R_{(u\backslash[u/(u\backslash x)])},L_u,I\Big)(\gamma,\gamma\mathbb{L}_x,\gamma)$. Hence, the conclusion follows.

Theorem 23 *If an Osborn loop is right universal then, any of its u,e-right principal isotopes is isomorphic to some principal isotopes.*

Proof By Theorem 11, if Q is a right universal Osborn loop, then

$$\Big(\gamma(x,u)\mathbb{R}_{(u\backslash x)},\beta(x,u),\gamma(x,u)\Big)=\big(\gamma(x,u)\mathbb{R}_{(u\backslash x)},L_u\gamma(x,u)\mathbb{L}_x,\gamma(x,u)\big)\in AUT(Q)$$

for all $x,u,v\in Q$. Writing the last triple as

$$(I,L_u,I)\big(\gamma(x,u),\gamma(x,u),\gamma(x,u)\big)\big(\mathbb{R}_{(u\backslash x)},\mathbb{L}_x,I\big).$$

These compositions mean

$$(Q,\cdot)\xrightarrow[\text{right principal isotopism}]{(I,L_u,I)}(Q,*)\xrightarrow[\text{isomorphism}]{(\gamma,\gamma,\gamma)}(Q,\circ)\xrightarrow[\text{principal isotopism}]{\left(\mathbb{R}_{(u\backslash x)},\mathbb{L}_x,I\right)}(Q,\cdot)$$

where $(Q,*)$ is a u,e-right principal isotope of $(Q,\cdot)$ and $(Q,\circ)$ some principal isotopes of $(Q,\cdot)$. Thus, the conclusion of the theorem follows.

Remark 7 *Although the statement of Theorem 23 can be deduced from the statement of Theorem 21. But the principal isotopes defer.*

Theorem 24 *If an Osborn loop is right universal then, the existence of the principal autotopism $(R_{(u\backslash x)},L_x,I)$ in the loop implies the triple $\left(\gamma(x,u)^{-1},\gamma(x,u)^{-1}\mathbb{L}_u,\gamma(x,u)^{-1}\right)$ where $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ is an autotopism in the loop, and vice versa.*

Proof The proof is in line with Theorem 23 with a slight adjustment to the composition by simply considering the inverse composition and reasoning like we did in Theorem 22.

Theorem 25 *There does not exist a non-trivial right universal Osborn loop that is a $G_\rho(\gamma_2)$-loop or $G_\rho(\gamma_2^{-1})$-loop with bi-mapping $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$.*

Proof

(a) It can be shown that if $\mathcal{Q}=(Q,\cdot,\backslash,/)$ is an Osborn loop such that the bi-mapping $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$. $\mathcal{Q}$ is a right universal Osborn loop if and only if $\mathcal{Q}$ is a $\mathrm{G}_\rho(\gamma_2)$-loop implies it obeys the identity

$$y(u\backslash[u/(u\backslash x)])\cdot uz=yz\ \forall\ x,y,z,u,\in Q \qquad (8)$$

and vice versa. The proof of this statement is based on Theorem 3 and is achieved by using the compositions $\Big(R_{(u\backslash[u/(u\backslash x)])},L_u,I\Big)(\gamma,\gamma\mathbb{L}_x,\gamma)$ of Theorem 22 and hence following Theorem 5, it is a G_ρ-loop, and in particular a $\mathrm{G}_\rho(\gamma_2)$-loop which implies it obeys identity (8) and vice versa.

(b) It can be shown that if $\mathcal{Q}=(Q,\cdot,\backslash,/)$ is an Osborn loop such that the bi-mapping $\gamma(x,u)=R_{[u\backslash x]}\mathbb{L}_uL_x$ and if $\mathcal{Q}$ is a right universal Osborn loop then, $\mathcal{Q}$ is a $\mathrm{G}_\rho(\gamma_2^{-1})$-loop implies it obeys the identity

$$y(u\backslash x)\cdot xz=yz\ \forall\ x,y,z,u,\in Q \qquad (9)$$

and vice versa. The proof of this statement is based on the composition used in Theorem 24. The reasoning used is similar to that in (a).

According to (a) or (b), if $\mathcal{Q}$ is a $G_\rho(\gamma_2)$-loop or $G_\rho(\gamma_2^{-1})$-loop then it obeys identity (8) or (9). Put $y = z = u = e$ or $x = y = z = e$ in identity (8) or (9), then $x = e$ or $u = e$. Which is a contradiction.

Remark 8 *There is no non-trivial group or Moufang loop or universal WIPL or VD-loop or CC-loop that is a $G_\rho(\gamma_2)$-loop or $G_\rho(\gamma_2^{-1})$-loop when $\gamma(x,u) = R_{[u\backslash x]}\mathbb{L}_u L_x$.*

4. Concluding Remarks and Future Studies

Using the bi-mapping $\gamma(x,v) = \mathbb{R}_v R_{[xv]} L_x$ of Theorem 19 in some existing results of Adeniran [2] and Chiboka [17] on the Bryant Schneider groups of left universal Osborn loops like CC-loops and extra loops respectively, more equations and information can be deduced. For example, Theorem 2.2 of Chiboka [17] claims that in an extra loop $(L,\cdot)$, corresponding to every mapping $\theta \in BS(L,\cdot)$ is a unique pair of right pseudo-automorphisms. So for the bi-mapping $\gamma(x,v) = \mathbb{R}_v R_{[xv]} L_x$, the mappings $\vartheta = \mathbb{R}_v R_{xv} L_x L_{v^{-1}x^{-1}vx^{-1}}$ and $\varphi = \mathbb{R}_v R_{xv} L_x R_{v^{-1}x^{-1}vx^{-1}}$ are right pseudo-automorphisms with companions $c_1 = (xv)^{-1}vxv^{-1}xv$ and $c_2 = xv^{-1}x^{-1}vx^{-1}$ respectively. Also, in Chiboka [19], the author showed that in an extra loop $(L,\cdot)$, the middle inner mapping $T(x) = R_x L_x^{-1} \in BS(L,\cdot)$ for all $x \in L$. $T(x)$ is a mono-mapping but $\gamma(x,v)$ is a bi-mapping. Multiplying them, more elements of Bryant Schneider group of an extra loop can be gotten. We need to identify the subgroup(s) of the multiplications group to which the bi-mappings and tri-mappings(which are not special mappings) of Theorem 16 and Theorem 25 belong to.

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