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INTEGRAL MEANS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS DEFINED BY DZIOK-SRIVASTAVA OPERATOR

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Abstract. In this paper, we introduce the subclass $UT_{q,s}([\alpha_1];\alpha,\beta)$ of analytic functions defined by Dziok-Srivastava operator. The object of the present paper is to determine the Silvermen's conjecture for the integral means inequality to this class.

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1. Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \qquad (1.1)$$

which are analytic in the open unit disc $U = \{z : \ |z| < 1\}$. Let $K(\alpha)$ and $S^*(\alpha)$ denote the subclasses of A which are, respectively, convex and starlike functions of order $\alpha,\ 0 \leq \alpha < 1$. For convenience, we write $K(0) = K$ and $S^*(0) = S^*$ (see [18]). The Hadamard product (or convolution) $(f * g)(z)$ of the functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z).$$

If f and g are analytic functions in U, we say that f is subordinate to g, written $f \prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z)),\ z \in U$.
For positive real parameters $\alpha_1, ..., \alpha_q$ and $\beta_1, ..., \beta_s$ $(\beta_j \in \mathbb{C}\backslash\mathbb{Z}_0^-, \mathbb{Z}_0^- = 0, -1, -2, ...; j = 1, 2, ..., s)$, the generalized hypergeometric function ${}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s;\ z)$ is defined by

$$ {}_qF_s(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n...(\alpha_q)_n}{(\beta_1)_n...(\beta_s)_n n!} z^n $$

$$ (q \leq s+1; s, q \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1, 2, ...\};\ z \in U), $$

where $(\theta)_n$, is the Pochhammer symbol defined in terms of the Gamma function Γ, by

$$ (\theta)_n = \frac{\Gamma(\theta+n)}{\Gamma(\theta)} = \begin{cases} 1 & (n=0) \\ \theta(\theta+1)....(\theta+n-1) & (n \in \mathbb{N}). \end{cases} $$

For the function $h(\alpha_1, ..., \alpha_q; \beta_1, ...\beta_s; z) = z_q F_s(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s;\ z)$, the Dziok-Srivastava linear operator (see [5] and [6]) $H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ...$ $..., \beta_s) : A \longrightarrow A$, is defined by the Hadamard product as follows:

$$ \begin{aligned} H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s) f(z) &= h(\alpha_1, ..., \alpha_q; \beta_1, ...\beta_s; z) * f(z) \\ &= z + \sum_{n=2}^{\infty} \Psi_n(\alpha_1)\, a_n z^n \quad (z \in U), \end{aligned} \tag{1.2} $$

where

$$ \Psi_n(\alpha_1) = \frac{(\alpha_1)_{n-1} ....(\alpha_q)_{n-1}}{(\beta_1)_{n-1}...(\beta_s)_{n-1}\ (n-1)!}. \tag{1.3} $$

For brevity, we write

$$ H_{q,s}(\alpha_1, ..., \alpha_q;\ \beta_1, ..., \beta_s; z) f(z) = H_{q,s}(\alpha_1) f(z). \tag{1.4} $$

For $0 \leq \alpha < 1, \beta \geq 0$ and for all $z \in U$, let $US_{q,s}([\alpha_1]; \alpha, \beta)$ denote the subclass of A consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$ Re\left\{ \frac{H_{q,s}(\alpha_1) f(z)}{z(H_{q,s}(\alpha_1) f(z))'} - \alpha \right\} > \beta \left| \frac{H_{q,s}(\alpha_1) f(z)}{z(H_{q,s}(\alpha_1) f(z))'} - 1 \right|. \tag{1.5} $$

Denote by T the subclass of A consisting of functions of the form:

$$ f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0), \tag{1.6} $$

which are analytic in U. We define the class $UT_{q,s}([\alpha_1]; \alpha, \beta)$ by:

$$ UT_{q,s}([\alpha_1]; \alpha, \beta) = US_{q,s}([\alpha_1]; \alpha, \beta) \cap T. \tag{1.7} $$

We note that for suitable choices of q, s, α and β, we obtain the following subclasses studied by various authors.
(1) For $q = 2$ *and* $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([1]\,; \alpha, \beta)$ reduces to the class $ST(\alpha, \beta)$

$$= \left\{ f \in T : Re\left\{ \frac{f(z)}{zf'(z)} - \alpha \right\} > \beta \left| \frac{f(z)}{zf'(z)} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, z \in U \right\}$$

and the class $ST(\alpha, 0) = ST(\alpha)$ is the family of functions $f(z) \in T$ which satisfy the following condition (see [7] and [19])

$$ST(\alpha) = Re\left\{ \frac{f(z)}{zf'(z)} \right\} > \alpha \quad (0 \leq \alpha < 1);$$

(2) For $q = 2, s = 1, \alpha_1 = a\,(a > 0), \alpha_2 = 1$ *and* $\beta_1 = c\,(c > 0)$ in (1.5), the class $UT_{2,1}([a, 1; c]\,; \alpha, \beta)$ reduces to the class $T(a, c; \alpha, \beta)$

$$= \left\{ f \in T : Re\left\{ \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - \alpha \right\} > \beta \left| \frac{L(a,c)f(z)}{z(L(a,c)f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, z \in U \right\},$$

where $L(a, c)$ is the Carlson - Shaffer operator (see [2]);
(3) For $q = 2, s = 1, \alpha_1 = \lambda + 1 (\lambda > -1)$ *and* $\alpha_2 = \beta_1 = 1$ in (1.5), the class $UT_{2,1}([\lambda + 1]; \alpha, \beta)$ reduces to the class $W_\lambda(\alpha, \beta)$

$$= \left\{ f \in T : Re\left\{ \frac{D^\lambda f(z)}{z(D^\lambda f(z))'} - \alpha \right\} > \beta \left| \frac{D^\lambda f(z)}{z(D^\lambda f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, \lambda > -1, z \in U \right\} \ (\textit{ see } [11]\ ),$$

where $D^\lambda (\lambda > -1)$ is the Ruscheweyh derivative operator (see [15]);
(4) For $q = 2, s = 1, \alpha_1 = v + 1\,(v > -1), \alpha_2 = 1$ *and* $\beta_1 = v + 2$ in (1.5), the class $UT_{2,1}([v + 1, 1; v + 2]\,; \alpha, \beta)$ reduces to the class $\zeta T(v; \alpha, \beta)$

$$= \left\{ f \in T : Re\left\{ \frac{J_v f(z)}{z(J_v f(z))'} - \alpha \right\} > \beta \left| \frac{J_v f(z)}{z(J_v f(z))'} - 1 \right|,\ 0 \leq \alpha < 1, \beta \geq 0, v > -1, z \in U \right\},$$

where $J_v f(z)$ is the generalized Bernardi - Libera - Livingston operator (see [1], [8] and [10]);
(5) For $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = 2 - \mu\,(\mu \neq 2, 3, ...)$ in (1.5), the class $UT_{2,1}([2, 1; 2 - \mu]; \alpha, \beta)$ reduces to the class $\mathcal{F}T(\mu; \alpha, \beta)$

$$= \left\{f \in T : Re\left\{\frac{\Omega_z^{\mu} f(z)}{z(\Omega_z^{\mu} f(z))'} - \alpha\right\} > \beta\left|\frac{\Omega_z^{\mu} f(z)}{z(\Omega_z^{\mu} f(z))'} - 1\right|,\ 0 \leq \alpha < 1, \beta \geq 0, \mu \neq 2, 3, ..., z \in U\right\},$$

where $\Omega_z^{\mu} f(z)$ is the Srivastava - Owa fractional derivative operator (see [13] and [14]);
(6) For $q = 2, s = 1, \alpha_1 = \mu\,(\mu > 0)\,, \alpha_2 = 1$ *and* $\beta_1 = \lambda + 1\,(\lambda > -1)$ in (1.5), the class $UT_{2,1}([\mu, 1; \lambda + 1]; \alpha, \beta)$ reduces to the class $\pounds T(\mu, \lambda; \alpha, \beta)$

$$= \left\{f \in T : Re\left\{\frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - \alpha\right\} > \beta\left|\frac{I_{\lambda,\mu} f(z)}{z(I_{\lambda,\mu} f(z))'} - 1\right|,\ -1 \leq \alpha < 1, \beta \geq 0, \mu > 0, \lambda > -1, z \in U\right\},$$

where $I_{\lambda,\mu} f(z)$ is the Choi-Saigo-Srivastava operator (see [4]);
(7) For $q = 2, s = 1, \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = k + 1\,(k > -1)$ in (1.5), the class $UT_{2,1}([2, 1; k + 1]; \alpha, \beta)$ reduces to the class $AT(k; \alpha, \beta)$

$$= \left\{f \in T : Re\left\{\frac{I_k f(z)}{z(I_k f(z))'} - \alpha\right\} > \beta\left|\frac{I_k f(z)}{z(I_k f(z))'} - 1\right|,\ 0 \leq \alpha < 1, \beta \geq 0, k > -1, z \in U\right\},$$

where $I_k f(z)$ is the Noor integral operator (see [12]);
(8) For $q = 2, s = 1, \alpha_1 = c\,(c > 0)\,, \alpha_2 = \lambda + 1\,(\lambda > -1)$ *and* $\beta_1 = a\,(a > 0)$ in (1.5), the class $UT_{2,1}([c, \lambda + 1; a]\,; \alpha, \beta)$ reduces to the class $\digamma T(c, a, \lambda; \alpha, \beta)$

$$= \left\{f \in T : Re\left\{\frac{I^{\lambda}(a, c) f(z)}{z(I^{\lambda}(a, c) f(z))'} - \alpha\right\} > \beta\left|\frac{I^{\lambda}(a, c) f(z)}{z(I^{\lambda}(a, c) f(z))'} - 1\right|,\ 0 \leq \alpha < 1, \beta \geq 0, c > 0, \lambda > -1, a > 0, z \in U\right\},$$

where $I^{\lambda}(a, c) f(z)$ is the Cho-Kwon-Srivastava operator (see [3]).
In [16] Silverman found that the function $f_2 = z - \frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured and settled in [17]:

$$\int_0^{2\pi} \left|f(re^{i\theta})\right|^{\delta} d\theta \leq \int_0^{2\pi} \left|f_2(re^{i\theta})\right|^{\delta} d\theta,$$

for all $f \in T$, $\delta > 0$ and $0 < r < 1$. In [17], he also proved his conjecture for the subclasses $T^*(\alpha)$ and $C(\alpha)$ of T, where $C(\alpha)$ *and* $T^*(\alpha)$ *are* the classes of convex

and starlike functions of order $\alpha,\ 0 \leq \alpha < 1$, respectively.
In this paper, we prove Silverman's conjecture for functions in the class $US_{q,s}([\alpha_1]\,;\alpha,\beta)$. Also by taking appropriate choices of the parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$, we obtain the integral means inequalities for several known as well as new subclasses of uniformly convex and uniformly starlike functions in U.

2. Coefficient estimates

Unless otherwise mentioned, we shall assume in the reminder of this paper that, the parameters $\alpha_1,...,\alpha_q$ and $\beta_1,...,\beta_s$ are positive real numbers,$-1 \leq \alpha < 1,\ \beta \geq 0,\ n \geq 2,\ z \in U$ *and* $\Psi_n(\alpha_1)$ is defined by (1.3).

Theorem 1. *A function $f(z)$ of the form (1.6) is in the class $UT_{q,s}([\alpha_1]\,;\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)\,a_n \leq 1-\alpha. \quad (2.1)$$

Proof. Suppose that (2.1) is true. Since

$$\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{1-\alpha} - n\Psi_n(\alpha_1) = \frac{(n-1)(1+\beta)\Psi_n(\alpha_1)}{1-\alpha} > 0,$$

we deduce

$$\sum_{n=2}^{\infty} n\Psi_n(\alpha_1)\,a_n < \sum_{n=2}^{\infty}\frac{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)}{1-\alpha}a_n \leq 1.$$

It suffices to show that

$$\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right| - Re\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right) \leq 1-\alpha,$$

we have

$$\beta\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right| - Re\left(\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right)$$

$$\leq (1+\beta)\left|\frac{H_{q,s}(\alpha_1)f(z)}{z(H_{q,s}(\alpha_1)f(z))'}-1\right|$$

$$\leq \frac{(1+\beta)\sum\limits_{n=2}^{\infty}(n-1)\Psi_n(\alpha_1)\,a_n}{1-\sum\limits_{n=2}^{\infty}n\Psi_n(\alpha_1)\,a_n} < 1-\alpha.$$

This completes the proof of Theorem 1.
Unfortunately, the converse of the above Theorem 1 is not true. So we define the subclass $T_{q,s}([\alpha_1];\alpha,\beta)$ of $UT_{q,s}([\alpha_1];\alpha,\beta)$ consisting of functions $f(z)$ which satisfy (2.1).
Remark 1. Putting $q=2, s=1, \beta=0$ and $\alpha_1=\alpha_2=\beta_1=1$, in Theorem 1, we will obtain the result obtained by Yamakawa [19, Lemma 2.1, with $n=p=1$].
Corollary 1. *Let the function $f(z)$ defined by (1.6) be in the class $T_{q,s}([\alpha_1];\alpha,\beta)$, then*

$$a_n \leq \frac{(1-\alpha)}{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)} \ (n \geq 2). \quad (2.2)$$

The result is sharp for the function

$$f(z) = z - \frac{(1-\alpha)}{[2n-n(\alpha-\beta)-(\beta+1)]\Psi_n(\alpha_1)} z^n \ (n \geq 2). \quad (2.3)$$

Putting $q=2, s=1, \alpha_1=\lambda+1(\lambda>-1)$ *and* $\alpha_2=\beta_1=1$ in Theorem 1, we obtain the following corollary.
Corollary 2. *A function $f(z)$ of the form (1.6) is in the class $W_\lambda(\alpha,\beta)$ if*

$$\sum_{n=2}^{\infty}[2n-n(\alpha-\beta)-(\beta+1)]\frac{(\lambda+1)_{n-1}}{(n-1)!}a_n \leq 1-\alpha.$$

Remark 2. The result in Corollary 2 correct the result obtained by Najafzadeh and Kulkarni [11, Lemma 1.1].

3.Integral Means

Lemma 1 [9]. *If the functions f and g are analytic in U with $g \prec f$, then for $\delta>0$ and $0<r<1$,*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^\delta d\theta \leq \int_0^{2\pi}\left|f(re^{i\theta})\right|^\delta d\theta.$$

Applying Theorem 1 and Lemma 1 we prove the following theorem.
Theorem 2. *Suppose $f(z) \in T_{q,s}([\alpha_1];\alpha,\beta), \delta>0$, the sequence $\{\Psi_n(\alpha_1)\}$ $(n \geq 2)$ is non-decrecing and $f_2(z)$ is defined by:*

$$f_2(z) = z - \frac{1-\alpha}{(3-2\alpha+\beta)\Psi_2(\alpha_1)} z^2, \quad (3.1)$$

then for $z = re^{i\theta}$, $0 < r < 1$, *we have*

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^\delta d\theta \leq \int_0^{2\pi} \left| f_2(re^{i\theta}) \right|^\delta d\theta. \quad (3.2)$$

Proof. For $f(z)$ of the form (1.6), (3.2) is equivalent to proving that

$$\int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{n-1} \right|^\delta d\theta \leq \int_0^{2\pi} \left| 1 - \frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)} z \right|^\delta d\theta.$$

By using Lemma 1, it suffices to show that

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} \prec 1 - \frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)} z. \quad (3.3)$$

Setting

$$1 - \sum_{n=2}^{\infty} a_n z^{n-1} = 1 - \frac{(1-\alpha)}{(3-2\alpha+\beta)\Psi_2(\alpha_1)} w(z), \quad (3.4)$$

and using (2.1) and the hypothises $\{\Psi_n(\alpha_1)\}$ $(n \geq 2)$ is non-decrecing, we obtain

$$\begin{aligned} |w(z)| &= \left| \frac{(3-2\alpha+\beta)\Psi_2(\alpha_1)}{(1-\alpha)} \sum_{n=2}^{\infty} a_n z^{n-1} \right| \\ &\leq |z| \sum_{n=2}^{\infty} \frac{(3-2\alpha+\beta)\Psi_2(\alpha_1)}{(1-\alpha)} a_n \\ &\leq |z| \sum_{n=2}^{\infty} \frac{[2n - n(\alpha-\beta) - (\beta+1)]\Psi_n(\alpha_1)}{(1-\alpha)} a_n \\ &\leq |z|. \end{aligned}$$

This completes the proof of Theorem 2.

Putting $q = 2$ *and* $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in Theorems 1 and 2, respectively, we obtain the following corollary:

Corollary 3. *If* $f(z) \in ST(\alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{(3-2\alpha+\beta)} z^2.$$

Putting $\beta = 0$ in Corollary 3, we obtain the following corollary:

Corollary 4. *If* $f(z) \in ST(\alpha), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{(3-2\alpha)} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = a\ (a > 0),\ \alpha_2 = 1$ *and* $\beta_1 = c\ (c > 0)$ in Theorems 1 and 2, respectively, we obtain the following corollary:
Corollary 5. *If* $f(z) \in \mathcal{L}T(a, c; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)c}{(3-2\alpha+\beta)a} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = \lambda + 1\ (\lambda > -1)$ *and* $\alpha_2 = \beta_1 = 1$ in Theorems 1 and 2, respectively, we obtain the following corollary:
Corollary 6. *If* $f(z) \in W_\lambda(\alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)}{(3-2\alpha+\beta)(\lambda+1)} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = v + 1\ (v > -1), \alpha_2 = 1$ *and* $\beta_1 = v + 2$ in Theorems 1 and 2, respectively, we obtain the following corollary:
Corollary 7. *If* $f(z) \in \zeta T(v; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(v+2)}{(3-2\alpha+\beta)(v+1)} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = 2 - \mu\ \ (\mu \neq 2, 3, ...)$ in Theorems 1 and 2, respectively, we obtain the following corollary:
Corollary 8. *If* $f(z) \in \mathcal{F}T(\mu; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(2-\mu)}{2(3-2\alpha+\beta)} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = \mu(\mu > 0), \alpha_2 = 1$ *and* $\beta_1 = \lambda + 1(\lambda > -1)$ in Theorems 1 and 2, respectively, we obtain the following corollary:
Corollary 9. *If* $f(z) \in \pounds T(\mu, \lambda; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(\lambda+1)}{\mu(3-2\alpha+\beta)} z^2.$$

Putting $q = 2,\ s = 1,\ \alpha_1 = 2, \alpha_2 = 1$ *and* $\beta_1 = k + 1 (k > -1)$ in Theorems 1 and 2, respectively, we obtain the following corollary:

Corollary 10. *If* $f(z) \in AT(k; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{(1-\alpha)(k+1)}{2(3-2\alpha+\beta)} z^2.$$

Putting $q = 3,\ s = 2,\ \alpha_1 = c,\ \alpha_2 = \lambda + 1$ *and* $\beta_1 = a$ in Theorems 1 and 2, respectively, we obtain the following corollary:

Corollary 11. *If* $f(z) \in FT(c, \lambda; a; \alpha, \beta), \delta > 0$, *then the assertion (3.2) holds true, where*

$$f_2(z) = z - \frac{a(1-\alpha)}{c(\lambda+1)(3-2\alpha+\beta)} z^2.$$

References

[1] S.D. Bernardi, Convex and starlike univalent functions, Trans. Amer. Math. Soc., 135 (1969), 429–449.

[2] B. C. Carlson and D. B. Shaffer, Starlike and prestarlike hypergeometric functions, J. Math. Anal. Appl., 15 (1984), 737-745.

[3] N. E. Cho, O. S. Kwon and H. M. Srivastava, Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators, J. Math. Anal. Appl., 292(2004), 470-483.

[4] J. H. Choi, M. Saigo and H. M. Srivastava, Some inclusion properties of a certain family of integral operators, J. Math. Anal. Appl., 276(2002), 432-445.

[5] J. Dziok and H.M. Srivastava, Classes of analytic functions with the generalized hypergeometric function, Appl. Math. Comput., 103 (1999), 1-13.

[6] J. Dziok and H.M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transform. Spec. Funct., 14 (2003), 7-18.

[7] J. S. Kang, S. Owa and H. M. Srivastava, Quasi-convolution properties of certain subclasses of analytic functions, Bull. Belg. Math. Soc., 3 (1996), 603-608.

[8] R. J. Libera, Some classes of regular univalent functions, Proc. Amer. Math. Soc., 16 (1969), 755-758.

[9] J. L. Littlewood, On inequalities in the theory functions, Proc. London Math. Soc., 23 (1925), 481-519.

[10] A. E. Livingston, On the radius of univalence of certain analytic functions, Proc. Amer. Math. Soc., 17 (1966), 352-357.

[11] S. Najafzadeh and S. R. Kulkarni, Study of certain properties of univalent functions based on Ruscheweyh derivative, South. Asian Bull. Math., 32 (2008), 475-488.

[12] K. I. Noor, On new classes of integral operators, J. Natur. Geom., 16(1999), 71-80.

[13] S. Owa, On the distortion theorems. I, Kyungpook Math. J., 18 (1978), 53-59.

[14] S. Owa and H. M. Srivastava, Univalent and starlike generalized hypergeometric functions, Canad. J. Math., 39 (1987), 1057-1077.

[15] St. Ruscheweyh, New criteria for univalent functions, Proc. Amer. Math. Soc., 49 (1975), 109-115.

[16] H. Silverman, Univalent functions with negative coefficients, Proc. Amer. Math. Soc., 51 (1975), 109-116.

[17] H. Silverman, Integral means for univalent functions with negative coefficients, Houston J. Math., 23 (1997), 169-174.

[18] H.M. Srivastava and S. Owa (Editors), Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992.

[19] R. Yamakawa, Certain subclasses of p-valently starlike functions with negative coefficients, in: Current Topics in Analytic Function Theory, World Scientific Publishing Company, Singapore, New Jersey, London, Hong Kong, 1992

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