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SOME INCLUSION PROPERTIES FOR SUBCLASSES OF P-VALENT FUNCTIONS DEFINED BY A MULTIPLIER TRANSFORMATION

A. O. Mostafa and M.K.Aouf

Abstract. Using the principle of subordination, we obtain some inclusion properties of subclasses of p-valent functions defined by multiplier transformation. Also inclusion properties of classes involving the generalized Libera integral operator are obtained.

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1. Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z)=z^p+\sum_{k=p+1}^{\infty}a_kz^k \ \ (p\in N=\{1,2,...\}), \tag{1}$$

which are analytic and p-valent in the unit open disc $U=\{z:|z|<1\}$. If f and g are analytic functions in U, we say that f is subordinate to g, written $f\prec g$ if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0)=0$ and $|w(z)|<1$ for all $z\in U$, such that $f(z)=g(w(z)),\ z\in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [14], [15] and [3]):

$$f(z)\prec g(z)\Leftrightarrow f(0)=g(0)\ and\ f(U)\subset g(U).$$

For two functions f given by (1) and $g(z)=z^p+\sum\limits_{k=p+1}^{\infty}b_kz^k$, the Hadamard product (or convolution) is given by

$$(f*g)(z)=z^p+\sum_{k=p+1}^{\infty}a_kb_kz^k=(g*f)(z).$$

For any real number σ, Kumar and Taneja [11] defined the multiplier transformations $I^{\sigma}_{p,\lambda}$ of functions $f \in A(p)$ by:

$$I^{\sigma}_{p,\lambda}f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{k+\lambda}{p+\lambda}\right)^{\sigma} a_k z^k \ \ (\lambda \geqslant 0). \tag{2}$$

Obviously, we observe that

$$I^{s}_{p,\lambda}(I^{t}_{p,\lambda}f(z)) = I^{s+t}_{p,\lambda}f(z),$$

Specializing the parameters $\lambda, \sigma$and p, we obtain the following special operators:
i) $I^{\sigma}_{p,0}f(z) \equiv D^{\sigma}_{p}f(z)$ $(\sigma \in N_0 = N \cup \{0\}, p \in N$, see [1] and [16]);
ii) $I^{\sigma}_{1,0}f(z) = D^{\sigma}f(z)$ $(\sigma \in N_0$, see [20]);
iii) $I^{\sigma}_{1,\lambda}f(z) = I(\sigma, \lambda)$ (the multiplier transformation see [4], [5] and [8]);
iv) $I^{\sigma}_{1,1}$ (see[22]);
v) $I^{-1}_{1,\lambda}$ (see [18]);
vi) $I^{\sigma}_{1,1}$ (σ is any negative real number, see [2] and [10]).

For $0 \leq \eta < p$, $p \in N$, we denote by $S^*_p(\eta)$, $K_p(\eta)$ and C_p, the subclasses of $A(p)$ consisting of p-valent analytic functions which are, respectively, p-valent starlike of order η, p-valent convex of order η and p-valent close-to-convex functions in U (see [17], [19] and [21]).
Let S be the class of functions ϕ which are analytic and univalent in U and for which $\phi(U)$ is convex with $\phi(0) = 1$ and $\text{Re}\{\phi(z)\} > 0$, $z \in U$.

Making use of the principle of subordination between analytic functions, we introduce the subclasses $S^*_p(\eta;\phi)$, $K_p(\eta;\phi)$ and $C_p(\eta,\gamma;\phi,\psi)$ of the class $A(p)$, $0 \leq \eta, \gamma < p$ $(p \in N)$ and $\phi, \psi \in S$, which are defined by:

$$S^*_p(\eta;\phi) = \left\{ f \in A(p) : \frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)} - \eta\right) \prec \phi(z), \quad z \in U \right\};$$

$$K_p(\eta;\phi) = \left\{ f \in A(p) : \frac{1}{p-\eta}\left(1 + \frac{zf''(z)}{f'(z)} - \eta\right) \prec \phi(z), \quad z \in U \right\}$$

and

$$C_p(\eta,\gamma;\phi,\psi) = \left\{ f \in A(p) : \exists\, g \in S^*_p(\eta;\phi)\ s.t.\ \frac{1}{p-\gamma}\left(\frac{zf'(z)}{g(z)} - \gamma\right) \prec \psi(z),\ z \in U \right\}.$$

From these definitions, we can obtain some well-known subclasses of $A(p)$ by special choices of the functions ϕ and ψ. For example, we have

$$S^*_p\left(\eta; \frac{1+z}{1-z}\right) = S^*_p(\eta), \quad K_p\left(\eta; \frac{1+z}{1-z}\right) = K_p(\eta)$$

and

$$C_p\left(0,0;\frac{1+z}{1-z},\frac{1+z}{1-z}\right)=C_p.$$

For real or complex numbers a, b, c other than $0, -1, -2, ...$, the hypergeometric series is defined by

$$_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k(1)_k}z^k, \tag{3}$$

where

$$(d)_k=\begin{cases}1 & (k=0; d\in C\backslash\{0\})\\ d(d+1)...(d+k-1) & (k\in N; d\in C).\end{cases}$$

We note that, the series (3) converges absolutely for all $z\in U$, so that it represents an analytic function in U.

Setting

$$h^{\sigma}_{p,\lambda}(z)=z^p+\sum_{k=p+1}^{\infty}\left(\frac{k+\lambda}{p+\lambda}\right)^{\sigma}z^k \quad (\sigma\in R; \lambda\geqslant 0). \tag{4}$$

With the aid of the function $h^{\sigma}_{p,\lambda}(z)$ given by (4), we define the function $h^{\sigma *}_{p,\lambda}(z)$ in terms of the Hadamard product (or convolution) by:

$$\left(h^{\sigma}_{p,\lambda}*h^{\sigma *}_{p,\lambda}\right)(z)=z^p\ _2F_1(a,b;c;z)=z^p+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}(1)_{k-p}}z^k\ (z\in U).$$

This function yields the following family of linear operators $I^{\sigma}_{p,\lambda}(a,b;c):A(p)\to A(p)$ which are given by:

$$I^{\sigma}_{p,\lambda}(a,b;c,z)f(z)=h^{\sigma *}_{p,\lambda}(z)*f(z)$$

$$=z^p+\sum_{k=p+1}^{\infty}\frac{(a)_{k-p}(b)_{k-p}}{(c)_{k-p}(1)_{k-p}}\left(\frac{p+\lambda}{k+\lambda}\right)^{\sigma}a_kz^k\ (z\in U; \lambda\geqslant 0; \sigma\in R). \tag{5}$$

Specializing the parameters a, b, c and σ, we have:

i) $I^{0}_{p,\lambda}(p,1;p)f(z)=I^{0}_{p,\lambda}(1,1;1)f(z)=f(z)$;

ii) $I^{0}_{p,\lambda}(p+1,1;p)f(z)=\frac{zf'(z)}{p}$;

iii) $I^{0}_{p,\lambda}(\delta+p,1;\delta+p+1)f(z)=F_{p,\delta}(f)(z)=\frac{\delta+p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)dt,\ \delta>-p; p\in N$ (the generalized Libera operator defined by (18), see [2], [12] and [18]);

iv) $I^{0}_{p,\lambda}(a,n+p;a)f(z)=D^{n+p-1}f(z)\,(n>-p)$ (the Ruscheweyh derivative of $th-(n+p-1)$ order, see [9]);

v) $I^{0}_{p,\lambda}(p+1,1;n+p)f(z)=I_{n+p-1}f(z)\,(n>-p)$ (Noor integral operator, see [13]);

vi) $I^{0}_{1,\lambda}(\mu, 1; \tau + 1)f(z) = I_{\mu,\tau}f(z)$ $(\tau > -1)$ (Choi-Saigo-Srivastava operator, see [6]).

For simplicity, we write $I^{\sigma}_{p,\lambda}(a, b; c; z) = I^{\sigma}_{p,\lambda}(a)$.
From equation (5), it can be eaily to verify that:

$$z(I^{\sigma+1}_{p,\lambda}(a)f(z))' = (p + \lambda)I^{\sigma}_{p,\lambda}(a)f(z) - \lambda I^{\sigma+1}_{p,\lambda}(a)f(z) \tag{6}$$

and

$$z(I^{\sigma}_{p,\lambda}(a)f(z))' = aI^{\sigma}_{p,\lambda}(a + 1)f(z) - (a - p)I^{\sigma}_{p,\lambda}(a)f(z). \tag{7}$$

Using the operator $I^{\sigma}_{p,\lambda}(a)$, we introduce the following classes of p-valent analytic functions for $\phi, \psi \in S; \sigma \in R, \lambda \geqslant 0$ and $0 \leq \eta, \delta < p, p \in N$:

$$S^{\sigma}_{p,\lambda}(a; \eta; \phi) = \left\{f \in A(p) : I^{\sigma}_{p,\lambda}(a)f(z) \in S^{*}_{p}(\eta; \phi), \quad z \in U\right\};$$

$$K^{\sigma}_{p,\lambda}(a; \eta; \phi) = \left\{f \in A(p) : I^{\sigma}_{p,\lambda}(a)f(z) \in K_{p}(\eta; \phi), \quad z \in U\right\}$$

and

$$C^{\sigma}_{p,\lambda}(a; \eta, \gamma; \phi, \psi) = \left\{f \in A(p) : I^{\sigma}_{p,\lambda}(a)f(z) \in C_{p}(\eta, \delta; \phi, \psi),\ z \in U\right\}.$$

We note that

$$f(z) \in K^{\sigma}_{p,\lambda}(a; \eta; \phi) \Leftrightarrow \frac{zf'(z)}{p} \in S^{\sigma}_{p,\lambda}(a; \eta; \phi). \tag{8}$$

In particular, we set

$$S^{\sigma}_{p,\lambda}\left(a; \eta; \left(\frac{1 + Az}{1 + Bz}\right)^{\alpha}\right) = S^{\sigma}_{p,\lambda}(a; \eta; A, B; \alpha)\ \ (0 < \alpha \leq 1; -1 \leq B < A \leq 1)$$

$$K^{\sigma}_{p,\lambda}\left(a; \eta; \left(\frac{1 + Az}{1 + Bz}\right)^{\alpha}\right) = K^{\sigma}_{p,\lambda}(a; \eta; A, B; \alpha)\ \ (0 < \alpha \leq 1; -1 \leq B < A \leq 1).$$

In this paper, we obtain several inclusion properties of the classes $S^{\sigma}_{p,\lambda}(a; \eta; \phi)$, $K^{\sigma}_{p,\lambda}(a; \eta; \phi)$ and $C^{\sigma}_{p,\lambda}(a; \eta, \gamma; \phi, \psi)$. We also obtain some applications involving of classes of integral operators.

2. Inclusion Properties Involving the Operator $I^{\sigma}_{p,\lambda}(a)$

In order to prove our results, we shall make use of the following known results.
Lemma 1.[7] *Let ϕ be a convex univalent function in U with $\phi(0) = 1$ and $Re\{\beta\phi(z) + \nu\} > 0$ $(\beta, \nu \in C)$. If p is analytic in U with $p(0) = 1$, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \nu} \prec \phi(z)\ \ (z \in U), \tag{9}$$

implies that

$$p(z) \prec \phi(z) \quad (z \in U).$$

Lemma 2.[15] *Let ϕ be a convex univalent function in U and w be analytic in U with $Re\{w(z)\} \geqslant 0$. If p is analytic in U and $p(0) = \phi(0)$, then*

$$p(z) + w(z)zp'(z) \prec \phi(z) \quad (z \in U), \tag{10}$$

implies that

$$p(z) \prec \phi(z) \quad (z \in U).$$

Unless otherwise mentioned we shall assume that $-1 \leq B < A \leq 1; \phi, \psi \in S; \sigma \in R, \lambda \geqslant 0$ and $0 \leq \eta, \delta < p, p \in N$.

Theorem 1. *For $f(z) \in A(p)$, we have*

$$S^{\sigma}_{p,\lambda}(a+1;\eta;\phi) \subset S^{\sigma}_{p,\lambda}(a;\eta;\phi) \subset S^{\sigma+1}_{p,\lambda}(a;\eta;\phi) \ \ (0 \leq \eta < p, p \in N; \phi \in S).$$

We will first of all, show that

$$S^{\sigma}_{p,\lambda}(a+1;\eta;\phi) \subset S^{\sigma}_{p,\lambda}(a;\eta;\phi).$$

Let $f \in S^{\sigma}_{p,\lambda}(a+1;\eta;\phi)$ and let

$$q(z) = \frac{1}{p-\eta}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)f(z))'}{I^{\sigma}_{p,\lambda}(a)f(z)} - \eta\right), \tag{11}$$

where $q(z)$ is analytic in U with $q(0) = 1$. Applying (7) in (11), we have

$$a\frac{I^{\sigma}_{p,\lambda}(a+1)f(z)}{I^{\sigma}_{p,\lambda}(a)f(z)} = (p-\eta)q(z) + \eta + (a-p). \tag{12}$$

Differentiating (12) logarithmically with respect to z, we have

$$\frac{1}{p-\eta}\left(\frac{z(I^{\sigma}_{p,\lambda}(a+1)f(z))'}{I^{\sigma}_{p,\lambda}(a+1)f(z)} - \eta\right) = q(z) + \frac{zq'(z)}{(p-\eta)q(z) + \eta + (a-p)} \quad (z \in U). \tag{13}$$

Applying Lemma 1 to (13), it follows that $q \prec \phi$, that is, that, $f \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$. The proof of the second part, will follow by using arguments similar to those used in the first part with $f \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$ and taking

$$h(z) = \frac{1}{p-\eta}\left(\frac{z(I^{\sigma+1}_{p,\lambda}(a)f(z))'}{I^{\sigma+1}_{p,\lambda}(a)f(z)} - \eta\right),$$

where h is analytic in U with $h(0) = 1$ and using (6). It follows that $h \prec \phi$ in U, which implies that $f \in S_{p,\lambda}^{\sigma+1}(a;\eta;\phi)$. This completes the proof of Theorem 1.

Theorem 2. *For $f(z) \in A(p)$, we have*

$$K_{p,\lambda}^{\sigma}(a+1;\eta;\phi) \subset K_{p,\lambda}^{\sigma}(a;\eta;\phi) \subset K_{p,\lambda}^{\sigma+1}(a;\eta;\phi) \ \ (0 \le \eta < p, p \in N; \phi \in S).$$

Applying (8) and using Theorem 1, we have

$$\begin{aligned} f(z) \in K_{p,\lambda}^{\sigma}(a+1;\eta;\phi) &\Leftrightarrow I_{p,\lambda}^{\sigma}(a+1)f(z) \in K_p(\eta;\phi) \\ &\Leftrightarrow \frac{z(I_{p,\lambda}^{\sigma}(a+1)f(z))'}{p} \in S_p^*(\eta;\phi) \\ &\Leftrightarrow I_{p,\lambda}^{\sigma}(a+1)(\frac{zf'(z)}{p}) \in S_p^*(\eta;\phi) \\ &\Leftrightarrow \frac{zf'(z)}{p} \in S_{p,\lambda}^{\sigma}(a+1;\eta;\phi) \\ &\Rightarrow \frac{zf'(z)}{p} \in S_{p,\lambda}^{\sigma}(a;\eta;\phi) \\ &\Leftrightarrow I_{p,\lambda}^{\sigma}(a)(\frac{zf'(z)}{p}) \in S_p^*(\eta;\phi) \\ &\Leftrightarrow \frac{z(I_{p,\lambda}^{\sigma}(a)f(z))'}{p} \in S_p^*(\eta;\phi) \\ &\Leftrightarrow I_{p,\lambda}^{\sigma}(a)f(z) \in K_p(\eta;\phi) \\ &\Leftrightarrow f(z) \in K_{p,\lambda}^{\sigma}(a;\eta;\phi). \end{aligned}$$

Also

$$\begin{aligned} f(z) \in K_{p,\lambda}^{\sigma}(a;\eta;\phi) &\Leftrightarrow \frac{zf'(z)}{p} \in S_{p,\lambda}^{\sigma}(a;\eta;\phi) \\ &\Rightarrow \frac{zf'(z)}{p} \in S_{p,\lambda}^{\sigma+1}(a;\eta;\phi) \\ &\Leftrightarrow \frac{z(I_{p,\lambda}^{\sigma+1}(a)f(z))'}{p} \in S_p^*(\eta;\phi) \\ &\Leftrightarrow I_{p,\lambda}^{\sigma+1}(a)f(z) \in K_p(\eta;\phi) \\ &\Leftrightarrow f(z) \in K_{p,\lambda}^{\sigma+1}(a;\eta;\phi). \end{aligned}$$

This completes the proof of Theorem 2.

Taking

$$\phi(z) = \left(\frac{1+Az}{1+Bz}\right)^{\alpha} \quad (-1 \leq B < A \leq 1; 0 < \alpha \leq 1; z \in U)$$

in Theorem 1 and Theorem 2, we have the following corollary.
Corollary 1. *For $f(z) \in A(p)$, we have*

$$S^{\sigma}_{p,\lambda}(a+1; A, B; \phi) \subset S^{\sigma}_{p,\lambda}(a; A, B; \phi) \subset S^{\sigma+1}_{p,\lambda}(a; A, B; \phi)$$

and

$$K^{\sigma}_{p,\lambda}(a+1; A, B; \phi) \subset K^{\sigma}_{p,\lambda}(a; A, B; \phi) \subset K^{\sigma+1}_{p,\lambda}(a; A, B; \phi).$$

Now, using Lemma 2, we obtain similar inclusion relations for the class $C^{\sigma}_{p,\lambda}(a; \eta, \gamma; \phi, \psi)$.
Theorem 3. *For $f(z) \in A(p), a \geqslant p, p \in N$, we have*

$$C^{\sigma}_{p,\lambda}(a+1; \eta, \gamma; \phi, \psi) \subset C^{\sigma}_{p,\lambda}(a; \eta, \gamma; \phi, \psi) \subset C^{\sigma+1}_{p,\lambda}(a; \eta, \gamma; \phi, \psi),$$

$$(0 \leq \eta, \gamma < p, p \in N; \phi, \psi \in S).$$

First, we will prove that

$$C^{\sigma}_{p,\lambda}(a+1; \eta, \gamma; \phi, \psi) \subset C^{\sigma}_{p,\lambda}(a; \eta, \gamma; \phi, \psi).$$

Let $f \in C^{\sigma}_{p,\lambda}(a+1; \eta, \gamma; \phi, \psi)$. Then, from the defintion of the class $C^{\sigma}_{p,\lambda}(a+1; \eta, \gamma; \phi, \psi)$, there exist a function $g \in S^{\sigma}_{p,\lambda}(a+1; \eta; \phi)$ such that

$$\frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a+1)f(z))'}{I^{\sigma}_{p,\lambda}(a+1)g(z)} - \gamma\right) \prec \psi(z).$$

Now, let

$$q(z) = \frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)f(z))'}{I^{\sigma}_{p,\lambda}(a)g(z)} - \gamma\right), \tag{14}$$

where $q(z)$ is analytic in U with $q(0) = 1$. Applying (7) in (14), we have

$$[(p-\gamma)q(z)+\gamma]I^{\sigma}_{p,\lambda}(a)g(z) + (a-p)I^{\sigma}_{p,\lambda}(a)f(z) = aI^{\sigma}_{p,\lambda}(a+1)f(z). \tag{15}$$

Differentiating (15) with respect to z and multiplying by z, we have

$$[(p-\gamma)q(z)+\gamma]z(I^{\sigma}_{p,\lambda}(a)g(z))' + (p-\gamma)zq'(z)I^{\sigma}_{p,\lambda}(a)g(z) + (a-p)z(I^{\sigma}_{p,\lambda}(a)f(z))'$$
$$= az(I^{\sigma}_{p,\lambda}(a+1)f(z))'. \tag{16}$$

Since $g \in S^{\sigma}_{p,\lambda}(a+1;\eta;\phi)$, then by Theorem 1, we have $g \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$. Let

$$h(z) = \frac{1}{p-\eta}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)g(z))'}{I^{\sigma}_{p,\lambda}(a)g(z)} - \eta\right).$$

Applying (7) again, we have

$$a\frac{I^{\sigma}_{p,\lambda}(a+1)g(z)}{I^{\sigma}_{p,\lambda}(a)g(z)} = (p-\eta)h(z) + \eta + (a-p). \tag{17}$$

From (16) and (17), we have

$$\frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a+1)f(z))'}{I^{\sigma}_{p,\lambda}(a+1)g(z)} - \gamma\right) = q(z) + \frac{zq'(z)}{(p-\eta)h(z)+\eta+(a-p)}.$$

Since $a \geqslant p, p \in N$ and $h \prec \phi$ in U, then

$$Re\{(p-\eta)h(z)+\eta+(a-p)\} > 0 \ \ (z \in U).$$

Hence applying Lemma 2, we can show that $p \prec \psi$, that is, that $f \in C^{\sigma}_{p,\lambda}(a;\eta,\gamma;\phi,\psi)$. The second part can be proved by using similar arguments and using (6). This completes the proof of Theorem 3.

3. Inclusion Properties Involving the Integral Operator $F_{p,\delta}$

Now, we consider the generalized Libera integral operator $F_{p,\delta}$ (see [2], [12] and [18]), defined by

$$\begin{aligned} F_{p,\delta}(f)(z) &= \frac{\delta+p}{z^{\delta}}\int_0^z t^{\delta-1}f(t)dt \\ &= z^p + \sum_{k=p+1}^{\infty}\frac{\delta+p}{\delta+k}a_k z^k \ \ (\delta > -p). \end{aligned} \tag{18}$$

From (18), we have

$$z\left(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z)\right)' = (\delta+p)I^{\sigma}_{p,\lambda}(a)f(z) - \delta I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z). \tag{19}$$

Theorem 4. Let $\delta > -p$. If $f \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$, then $F_{p,\delta}(f)(z) \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$.

Let $f \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$ and put

$$q(z) = \frac{1}{p-\eta}\left(\frac{z\left(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z)\right)'}{I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z)} - \eta)\right), \tag{20}$$

where q is analytic in U with $q(0) = 1$. then, by using (19) and (20), we have

$$(\delta + p)\frac{I^{\sigma}_{p,\lambda}(a)f(z)}{I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z)} = (p-\eta)q(z) + (\eta + \delta). \tag{21}$$

Differentiating (21) logarithmically with respect to z, we have

$$q(z) + \frac{zq'(z)}{(p-\eta)q(z) + (\eta+\delta)} = \frac{1}{p-\eta}\left(\frac{z\left(I^{\sigma}_{p,\lambda}(a)f(z)\right)'}{I^{\sigma}_{p,\lambda}(a)f(z)} - \eta\right).$$

Applying Lemma 1, we conclude that $p \prec \phi$ $(z \in U)$, which implies that $F_{p,\delta}(f)(z) \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$. This completes the proof of Theorem 4.

Theorem 5. Let $\delta > -p, p \in N$. If $f \in K^{\sigma}_{p,\lambda}(a;\eta;\phi)$, then $F_{p,\delta}(f)(z) \in K^{\sigma}_{p,\lambda}(a;\eta;\phi)$.

Applying Theorem 4 and (8), we have

$$\begin{aligned} f(z) \in K^{\sigma}_{p,\lambda}(a;\eta;\phi) &\Leftrightarrow \frac{zf'(z)}{p} \in S^{\sigma}_{p,\lambda}(a;\eta;\phi) \\ &\Rightarrow F_{p,\delta}(\frac{zf'}{p})(z) \in S^{\sigma}_{p,\lambda}(a;\eta;\phi) \\ &\Leftrightarrow \frac{z}{p}\left(F_{p,\delta}(f)(z)\right)' \in S^{\sigma}_{p,\lambda}(a;\eta;\phi) \\ &\Leftrightarrow F_{p,\delta}(f)(z) \in K^{\sigma}_{p,\lambda}(a;\eta;\phi). \end{aligned}$$

This completes the proof of Theorem 5.

From Theorem 4 and Theorem 5, we have the following corollary.

Corollary 2. Let $\delta > -p, p \in N$. If $f \in S^{\sigma}_{p,\lambda}(a;A,B;\phi)$ (or $K^{\sigma}_{p,\lambda}(a;A,B;\phi)$), then $F_{p,\delta}(f) \in S^{\sigma}_{p,\lambda}(a;A,B;\phi)$ (or $K^{\sigma}_{p,\lambda}(a;A,B;\phi)$).

Theorem 6. Let $\delta > -p, p \in N$. If $f \in C^{\sigma}_{p,\lambda}(a;\eta,\gamma;\phi,\psi)$, then $F_{p,\delta}(f)(z) \in K^{\sigma}_{p,\lambda}(a;\eta,\gamma;\phi,\psi)$.

Let $f \in C^{\sigma}_{p,\lambda}(a;\eta,\gamma;\phi,\psi)$. Then, from the defintion of the class $C^{\sigma}_{p,\lambda}(a;\eta,\gamma;\phi,\psi)$, there exist a function $g \in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$ such that

$$\frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)(f)(z))'}{I^{\sigma}_{p,\lambda}(a)g(z)} - \gamma\right) \prec \psi(z).$$

Now, let

$$q(z)=\frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z))'}{I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z)}-\gamma\right), \tag{22}$$

where $q(z)$ is analytic in U with $q(0)=1$. Applying (19) in (22), we have

$$[(p-\gamma)q(z)+\gamma]I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z)+\delta I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z)=(\delta+p)I^{\sigma}_{p,\lambda}(a)f(z). \tag{23}$$

Differentiating (23) with respect to z, we have

$$[(p-\gamma)q(z)+\gamma]z(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z))'+(p-\gamma)zq'(z)I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z)+\delta z(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(f)(z))'$$

$$=(\delta+p)z(I^{\sigma}_{p,\lambda}(a)f(z))'. \tag{24}$$

Since $g\in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$, then by Theorem 4, we have $F_{p,\delta}(g)(z)\in S^{\sigma}_{p,\lambda}(a;\eta;\phi)$. Let

$$h(z)=\frac{1}{p-\eta}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z))'}{I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z)}-\eta\right).$$

Applying (19) again, we have

$$(\delta+p)\frac{I^{\sigma}_{p,\lambda}(a)g(z)}{I^{\sigma}_{p,\lambda}(a)F_{p,\delta}(g)(z)}=(p-\eta)h(z)+\eta+\delta. \tag{25}$$

From (24) and (25), we have

$$\frac{1}{p-\gamma}\left(\frac{z(I^{\sigma}_{p,\lambda}(a)f(z))'}{I^{\sigma}_{p,\lambda}(a)g(z)}-\gamma\right)=p(z)+\frac{zp'(z)}{(p-\eta)h(z)+\eta+\delta}.$$

The remaining part of the proof is similar to that of Theorem 5 and so we omit it.
Remark. Putting $p=1$ in the above results, we obtain the results obtained by Cho and Kim [4].

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A. O. Mostafa and M.K.Aouf

Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt.
emails: *adelaeg254@yahoo.com,* *mkaouf127@yahoo.com*