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GROWTH OF SOLUTIONS OF COMPLEX LINEAR DIFFERENTIAL EQUATIONS WITH ENTIRE COEFFICIENTS OF FINITE ITERATED ORDER

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Abstract. In the present paper, we investigate the iterated order of solutions of higher order homogeneous linear differential equations with entire coefficients. We improve and extend some results of Belaïdi and Hamouda by using the concept of the iterated order. We also consider the non-homogeneous linear differential equations.

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1.Introduction and main results

In this paper, we shall use the fundamental results and the standard notation of the Nevanlinna value distribution theory of meromophic functions (see [10]). We also use the notations $\sigma(f)$ and $\mu(f)$ to denote respectively the order and the lower order of growth of a meromophic function $f(z)$.
We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t)dt$ and the logarithmic measure of a set $H \subset [1, +\infty)$ by $lm(H) = \int_1^{+\infty} \frac{\chi_H(t)}{t}dt$, where χ_F denote the characteristic function of a set F.
For the definition of the iterated order of a meromorphic function, we use the same definition as in [11], [4, p. 317], [12, p. 129]. For all $r \in \mathbb{R}$, we define $\exp_1 r := e^r$ and $\exp_{p+1} r := \exp\left(\exp_p r\right)$, $p \in \mathbb{N}$. We also define for all r sufficiently large $\log_1 r := \log r$ and $\log_{p+1} r := \log\left(\log_p r\right)$, $p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r := r$, $\log_0 r := r$, $\log_{-1} r := \exp_1 r$ and $\exp_{-1} r := \log_1 r$.
Definition 1.1 Let $p \geqslant 1$ be an integer. Then the iterated $p-$order $\sigma_p(f)$ of a meromorphic function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r}, \tag{1.1}$$

where $T(r,f)$ is the characteristic function of Nevanlinna. For $p=1$, this notation is called order and for $p=2$, hyper-order.
Remark 1.1 The iterated $p-$order $\sigma_p(f)$ of an entire function $f(z)$ is defined by

$$\sigma_p(f) = \limsup_{r\to+\infty} \frac{\log_p T(r,f)}{\log r} = \limsup_{r\to+\infty} \frac{\log_{p+1} M(r,f)}{\log r}, \tag{1.2}$$

where $M(r,f) = \max_{|z|=r} |f(z)|$.
Definition 1.2 The finiteness degree of the order of a meromorphic function f is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is rational,} \\ \min\{j \in \mathbb{N} : \sigma_j(f) < \infty\}, & \text{if } f \text{ is transcendental} \\ & \text{with } \sigma_j(f) < \infty \text{ for some } j \in \mathbb{N}, \\ \infty, & \text{if } \sigma_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.3}$$

Remark 1.2 Similarly, we can define the iterated lower $p-$order $\mu_p(f)$ of a meromorphic function $f(z)$ and the finiteness degree $i_\mu(f)$ of $\mu_p(f)$.
Definition 1.3 The iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to+\infty} \frac{\log_p N(r,1/f)}{\log r} \quad (p \geqslant 1 \text{ is an integer}), \tag{1.4}$$

where $N\left(r,\frac{1}{f}\right)$ is the counting function of zeros of $f(z)$ in $\{z : |z| < r\}$, and the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to+\infty} \frac{\log_p \overline{N}(r,1/f)}{\log r} \quad (p \geqslant 1 \text{ is an integer}), \tag{1.5}$$

where $\overline{N}\left(r,\frac{1}{f}\right)$ is the counting function of distinct zeros of $f(z)$ in $\{z : |z| < r\}$.
Definition 1.4 The finiteness degree of the iterated convergence exponent of the sequence of zeros of a meromorphic function $f(z)$ is defined by

$$i_\lambda(f) = \begin{cases} 0, & \text{if } n(r,1/f) = O(\log r), \\ \min\{j \in \mathbb{N} : \lambda_j(f) < \infty\}, & \text{if } \lambda_j(f) < \infty \\ & \text{for some } j \in \mathbb{N}, \\ \infty, & \text{if } \lambda_j(f) = \infty \text{ for all } j \in \mathbb{N}. \end{cases} \tag{1.6}$$

Remark 1.3 Similarly, we can define the finiteness degree $i_{\overline{\lambda}}(f)$ of $\overline{\lambda}_p(f)$.
Let $n \geqslant 2$ be an integer and let $A_0(z), ..., A_{n-1}(z), A_n(z)$ with $A_0(z) \not\equiv 0$ and

$A_n(z) \not\equiv 0$ be entire functions. It is well-known that if $A_n \equiv 1$, then all solutions of the linear differential equation

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = 0 \tag{1.7}$$

are entire functions but when A_n is a nonconstant entire function, equation (1.7) can possess meromorphic solutions. For instance the equation $z^2 f^{'''} + 6z f^{''} + 6f' - z^2 f = 0$ has a meromorphic solution $f(z) = \frac{e^z}{z^2}$. We also know that if some of coefficients $A_0(z), ..., A_{n-1}(z)$ are transcendental and $A_n \equiv 1$, then equation (1.7) has at least one solution of infinite order. Thus the question which arises is: What conditions on $A_0(z), ..., A_{n-1}(z), A_n(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.7) has an infinite order? For the above question and when $A_n \equiv 1$, there are many results for the second and higher order linear differential equations see for example ([1], [2], [3], [5], [6], [7], [9], [11], [12], [13]). In 2002, Belaïdi and Hamouda have considered the higher order linear differential equations with entire coefficients and obtained the following result.

Theorem A [2] *Let $A_0(z), ..., A_{n-1}(z)$ with $A_0(z) \not\equiv 0$ be entire functions. Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ with $\lim_{k\to+\infty} z_k = \infty$ and three real numbers α, β and μ satisfying $0 \leqslant \beta < \alpha$ and $\mu > 0$ such that*

$$|A_0(z_k)| \geqslant \exp\{\alpha |z_k|^\mu\} \tag{1.8}$$

and

$$|A_j(z_k)| \leqslant \exp\{\beta |z_k|^\mu\} \quad (j = 1, 2, ..., n-1) \tag{1.9}$$

as $k \to +\infty$. Then every solution $f \not\equiv 0$ of the equation

$$f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = 0 \tag{1.10}$$

has an infinite order.

The main purpose of this paper is to extend Theorem A for equations of the form (1.7) by using the concept of the iterated order and considering some coefficient A_s $(s = 0, 1, ..., n-1)$. We also consider the non-homogeneous linear differential equations. We shall obtain the following results.

Theorem 1.1 *Let $p \geqslant 1$ be an integer and let $A_0(z), ..., A_{n-1}(z), A_n(z)$ be entire functions with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$, $i_\lambda(A_n) \leqslant 1$ and $i(A_j) = p$ $(j = 0, 1, ..., n)$ such that there exists some integer s $(s = 0, 1, ..., n-1)$ satisfying*

$$\max\{\sigma_p(A_j) \quad (j \neq s)\} < \mu_p(A_s) \leqslant \sigma_p(A_s) = \sigma. \tag{1.11}$$

Suppose that there exist a sequence of complex numbers $(z_k)_{k\in\mathbb{N}}$ *with* $\lim\limits_{k\to+\infty} z_k = \infty$ *and two real numbers* α *and* β *satisfying* $0 \leqslant \beta < \alpha$ *such that*

$$|A_s(z_k)| \geqslant \exp_p\left\{\alpha\,|z_k|^{\sigma-\varepsilon}\right\} \tag{1.12}$$

and

$$|A_j(z_k)| \leqslant \exp_p\left\{\beta\,|z_k|^{\sigma-\varepsilon}\right\} \quad (j \neq s) \tag{1.13}$$

as $k \to +\infty$. *Then every transcendental meromorphic solution* $f \not\equiv 0$ *whose poles are of uniformly bounded multiplicity of equation* (1.7) *has an infinite iterated* $p-$*order and satisfies* $i(f) = p+1, \sigma_{p+1}(f) = \sigma$.

Let $A_0(z), ..., A_{n-1}(z), A_n(z)$ with $A_0(z) \not\equiv 0$, $A_n(z) \not\equiv 0$ and $F \not\equiv 0$ be entire functions. Considering the non-homogeneous linear differential equation

$$A_n(z) f^{(n)} + A_{n-1}(z) f^{(n-1)} + ... + A_1(z) f' + A_0(z) f = F, \tag{1.14}$$

we obtain the following result.

Theorem 1.2 *Let* $A_0(z), ..., A_{n-1}(z), A_n(z)$ *with* $A_0(z) \not\equiv 0$ *and* $A_n(z) \not\equiv 0$ *be entire functions satisfying hypotheses of Theorem 1.1 and let* $F \not\equiv 0$ *be an entire function of iterated order with* $i(F) = q$.

(i) If $q < p+1$ *or* $q = p+1$ *and* $\sigma_{p+1}(F) < \sigma_p(A_s)$, *then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of equation* (1.14) *satisfies* $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ *and* $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$, *with at most one exceptional solution* f_0 *with* $i(f_0) < p+1$ *or* $\sigma_{p+1}(f_0) < \sigma$.

(ii) If $q > p+1$ *or* $q = p+1$ *and* $\sigma_p(A_s) < \sigma_{p+1}(F)$, *then every transcendental meromorphic solution* f *whose poles are of uniformly bounded multiplicity of equation* (1.14) *satisfies* $i(f) = q$ *and* $\sigma_q(f) = \sigma_q(F)$.

2. Preliminary lemmas

Lemma 2.1 [8] *Let* $f(z)$ *be a transcendental meromorphic function. Let* $\alpha > 1$ *and* $\Gamma = \{(k_1, j_1), (k_2, j_2), ..., (k_m, j_m)\}$ *denote a set of distinct pairs of integers satisfying* $k_i > j_i \geqslant 0$. *Then there exist a set* $E_1 \subset (1, +\infty)$ *having finite logarithmic measure and a constant* $B > 0$ *that depends only on* α *and* Γ *such that for all* z *satisfying* $|z| = r \notin [0,1] \cup E_1$ *and all* $(k,j) \in \Gamma$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \leqslant B\left[\frac{T(\alpha r, f)}{r}\left(\log^\alpha r\right)\log T(\alpha r, f)\right]^{k-j}. \tag{2.1}$$

Lemma 2.2 [5, 6] *Let* $p, q \geqslant 1$ *be integers and let* $f(z)$ *be an entire function with* $i(f) = p+1$, $\sigma_{p+1}(f) = \sigma$, $i_\mu(f) = q+1$ *and* $\mu_{q+1}(f) = \mu$. *Let* $\nu_f(r)$ *be the central* *index of* $f(z)$. *Then*

$$\limsup_{r\to+\infty}\frac{\log_{p+1}\nu_f(r)}{\log r}=\sigma \tag{2.2}$$

and

$$\liminf_{r\to+\infty}\frac{\log_{q+1}\nu_f(r)}{\log r}=\mu. \tag{2.3}$$

Lemma 2.3 [8] *Let $f(z)$ be a meromorphic function, let j be a positive integer, and let $\alpha>1$ be a real constant. Then there exists a constant $R>0$ such that for all $r\geqslant R$, we have*

$$T\left(r,f^{(j)}\right)\leqslant (j+2)\,T(\alpha r,f). \tag{2.4}$$

Lemma 2.4 [13] *Let $p\geqslant 1$ be an integer and let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire function satisfying $\mu_p(g)=\mu_p(f)=\mu\leqslant\sigma_p(g)=\sigma_p(f)=+\infty$, $i(d)<p$ or $i(d)=p$ and $\sigma_p(d)=\rho<\mu$. Let $\nu_g(r)$ be the central index of g. Then there exists a set E_2 of finite logarithmic measure such that the estimation*

$$\frac{f^{(n)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^n(1+o(1))\quad (n\in\mathbb{N}) \tag{2.5}$$

holds for all $|z|=r\notin E_2$ and $|g(z)|=M(r,g)$.

Lemma 2.5 *Let $p\geqslant 1$ be an integer and let $f(z)=\frac{g(z)}{d(z)}$ be a meromorphic function, where $g(z)$ and $d(z)$ are entire function satisfying $\mu_p(g)=\mu_p(f)=\mu\leqslant\sigma_p(g)=\sigma_p(f)\leqslant+\infty$, $i(d)<p$ or $i(d)=p$ and $\sigma_p(d)=\rho<\mu$. Then there exists a set E_3 of finite logarithmic measure such that $|z|=r\notin E_3, |g(z)|=M(r,g)$ and for r sufficiently large, we have*

$$\left|\frac{f(z)}{f^{(s)}(z)}\right|\leqslant r^{2s}\quad (s\geqslant 1 \text{ is an integer}). \tag{2.6}$$

Proof. By Lemma 2.4, there exists a set E_2 of finite logarithmic measure such that the estimation

$$\frac{f^{(s)}(z)}{f(z)}=\left(\frac{\nu_g(r)}{z}\right)^s(1+o(1))\quad (s\geqslant 1 \text{ is an integer}) \tag{2.7}$$

holds for all $|z|=r\notin E_2$ and $|g(z)|=M(r,g)$, where $\nu_g(r)$ is the central index of g. On the other hand, for any given ε $(0<\varepsilon<1)$, there exists $R>1$ such that for all $r>R$, we have

$$\nu_g(r)>\exp_{p-1}\left\{r^{\mu-\varepsilon}\right\}. \tag{2.8}$$

If $\mu = +\infty$, then $\mu - \varepsilon$ can be replaced by a large enough real number M. Set $E_3 = [1, R] \cup E_2$, $lm(E_3) < +\infty$. Hence from (2.7) and (2.8), we obtain

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| = \left|\frac{z}{\nu_g(r)}\right|^s \frac{1}{|1+o(1)|} \leqslant \frac{r^s}{\left(\exp_{p-1}\{r^{\mu-\varepsilon}\}\right)^s} \leqslant r^{2s}, \tag{2.9}$$

where $|z| = r \notin E_3, r \to +\infty$ and $|g(z)| = M(r, g)$.

Lemma 2.6 [5, 6] *Let $p \geqslant 1$ be an integer. Suppose that $f(z)$ is a meromorphic function such that $i(f) = p$, $\sigma_p(f) = \sigma$ and $i_\lambda\left(\frac{1}{f}\right) \leqslant 1$. Then for any given $\varepsilon > 0$, there exists a set $E_4 \subset (1, +\infty)$ that has finite linear measure and finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_4, r \to +\infty$, we have*

$$|f(z)| \leqslant \exp_p\left\{r^{\sigma+\varepsilon}\right\}. \tag{2.10}$$

Lemma 2.7 [11] *Let $p \geqslant 1$ be an integer and let $f(z)$ be a meromorphic function with $i(f) = p$. Then $\sigma_p(f) = \sigma_p(f')$.*

Lemma 2.8 [5, 6] *Let $p \geqslant 1$ be an integer and let $f(z)$ be a meromorphic solution of the differential equation*

$$f^{(n)} + B_{n-1}(z) f^{(n-1)} + ... + B_1(z) f' + B_0(z) f = F, \tag{2.11}$$

where $B_0(z), ..., B_{n-1}(z)$ and $F \not\equiv 0$ are meromorphic functions such that
(i) $\max\{i(F), i(B_j) \ \ (j = 0, ..., n-1)\} < i(f) = p+1$ *or*
(ii) $\max\{\sigma_{p+1}(F), \sigma_{p+1}(B_j) \ \ (j = 0, ..., n-1)\} < \sigma_{p+1}(f)$.
Then $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f)$.

To avoid some problems caused by the exceptional set, we recall the following lemma.

Lemma 2.9 [9] *Let $g : [0, +\infty) \to \mathbb{R}$ and $h : [0, +\infty) \to \mathbb{R}$ be monotone non-decreasing functions such that $g(r) \leqslant h(r)$ for all $r \notin E_5 \cup [0, 1]$, where $E_5 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\alpha > 1$ be a given constant. Then there exists an $r_0 = r_0(\alpha) > 0$ such that $g(r) \leqslant h(\alpha r)$ for all $r > r_0$.*

3.Proof of Theorem 1.1

Set

$$\max\{\sigma_p(A_j) \ \ (j \neq s)\} = \lambda < \mu_p(A_s) \leqslant \sigma_p(A_s) = \sigma < +\infty. \tag{3.1}$$

Let $f\,(\not\equiv 0)$ be a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of (1.7). Since the poles of $f(z)$ can only occur at the zeros of $A_n(z)$, it follows that $i_\lambda\left(\frac{1}{f}\right) \leqslant p$ and $\lambda_p\left(\frac{1}{f}\right) \leqslant \lambda < \mu_p(A_s)$. By Hadamard factorization theorem, we can write f as $f(z) = g(z)/d(z)$, where $g(z)$ and $d(z)$

are entire functions satisfying $i(f) = i(g) = t \geqslant p+1$, $\sigma_t(f) = \sigma_t(g)$ and that $i(d) \leqslant p$, $\sigma_p(d) = \lambda_p(1/f) \leqslant \lambda < \mu_p(A_s)$. For $j = 0, ..., n-1$, since

$$T\left(r, f^{(j+1)}\right) \leqslant 2T\left(r, f^{(j)}\right) + m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right), \tag{3.2}$$

$$m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O\left\{\log rT\left(r, f^{(j)}\right)\right\}, \tag{3.3}$$

we can obtain by using Lemma 2.3,

$$\begin{aligned} T\left(r, f^{(j+1)}\right) &\leqslant 2T\left(r, f^{(j)}\right) + O\left\{\log rT\left(r, f^{(j)}\right)\right\} \\ &\leqslant 2(j+2)T(2r, f) + O\left\{\log rT\left(r, f^{(j)}\right)\right\}. \end{aligned} \tag{3.4}$$

We have also

$$O\left\{\log rT\left(r, f^{(j)}\right)\right\} = o\left\{T\left(r, f^{(j)}\right)\right\} \tag{3.5}$$

which yields

$$O\left\{\log rT\left(r, f^{(j)}\right)\right\} \leqslant \frac{1}{2}T\left(r, f^{(j)}\right). \tag{3.6}$$

We can rewrite (1.7) as

$$\begin{aligned} -A_s(z) = A_n(z)\frac{f^{(n)}}{f^{(s)}} + A_{n-1}(z)\frac{f^{(n-1)}}{f^{(s)}} + ... + A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}} \\ +A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}} + ... + A_1(z)\frac{f'}{f^{(s)}} + A_0(z)\frac{f}{f^{(s)}}. \end{aligned} \tag{3.7}$$

By $(3.4), (3.6), (3.7)$ and Lemma 2.3, we obtain

$$T(r, A_s) \leqslant cT(2r, f) + \sum_{j \neq s} T(r, A_j), \tag{3.8}$$

where $c\ (>0)$ is a constant. By (3.1) and (3.8), we conclude that $\mu_p(f) \geqslant \mu_p(A_s)$. By the fact that $\sigma_p(d) = \lambda_p(d) = \lambda_p\left(\frac{1}{f}\right) \leqslant \lambda < \mu_p(A_s)$ and the inequality $T(r,f) \leqslant T(r,g) + T(r,d) + O(1)$, it follows that $\sigma_p(d) \leqslant \lambda < \mu_p(g) = \mu_p(f)$ and $\sigma_p(g) = \sigma_p(f) \leqslant +\infty$. Hence by Lemma 2.5, there exists a set $E_3 \subset (1, +\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_3$ and $|g(z)| = M(r, g)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant r^{2s}. \tag{3.9}$$

From (3.7), it follows that

$$|A_s(z)| \leqslant |A_n(z)| \left|\frac{f^{(n)}}{f^{(s)}}\right| + |A_{n-1}(z)| \left|\frac{f^{(n-1)}}{f^{(s)}}\right| + ... + |A_{s+1}(z)| \left|\frac{f^{(s+1)}}{f^{(s)}}\right|$$

$$+ |A_{s-1}(z)| \left|\frac{f^{(s-1)}}{f}\right| \left|\frac{f}{f^{(s)}}\right| + ... + |A_1(z)| \left|\frac{f'}{f}\right| \left|\frac{f}{f^{(s)}}\right| + |A_0(z)| \left|\frac{f}{f^{(s)}}\right|. \tag{3.10}$$

By Lemma 2.1, there exist a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin E_1 \cup [0, 1]$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right| \leqslant Br\,[T(2r, f)]^{j-s+1} \quad (j = s+1, ..., n) \tag{3.11}$$

and

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant Br\,[T(2r, f)]^{j+1} \quad (j = 1, ..., s-1). \tag{3.12}$$

Hence from (1.12), (1.13), (3.10) − (3.12), we have

$$\exp_p\left\{\alpha |z_k|^{\sigma-\varepsilon}\right\} \leqslant Bnr_k^{2s+1}\,[T(2r_k, f)]^{n+1} \exp_p\left\{\beta |z_k|^{\sigma-\varepsilon}\right\} \tag{3.13}$$

as $k \to +\infty$, $|z_k| = r_k \notin [0, 1] \cup E_1 \cup E_3$ and $|g(z_k)| = M(r_k, g)$. Hence from (3.13) and Lemma 2.9, we obtain that $i(f) \geqslant p + 1$ and $\sigma_{p+1}(f) \geqslant \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we get $\sigma_{p+1}(f) \geqslant \sigma$.

Now we prove that $\sigma_{p+1}(f) \leqslant \sigma$. We can rewrite (1.7) as

$$-A_n(z)\frac{f^{(n)}}{f} = A_{n-1}(z)\frac{f^{(n-1)}}{f} + ... + A_{s+1}(z)\frac{f^{(s+1)}}{f}$$

$$+A_s(z)\frac{f^{(s)}}{f} + A_{s-1}(z)\frac{f^{(s-1)}}{f} + ... + A_1(z)\frac{f'}{f} + A_0(z). \tag{3.14}$$

By Lemma 2.4, there exist a set $E_2 \subset (1, +\infty)$ of finite logarithmic measure such that for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r, g)$, we have

$$\frac{f^{(k)}(z)}{f(z)} = \left(\frac{\nu_g(r)}{z}\right)^k (1+o(1)) \quad (k=1,2,...,n). \tag{3.15}$$

From Remark 1.1, we have for a sufficiently large r

$$|A_j(z)| \leqslant \exp_p\left\{r^{\sigma+\varepsilon}\right\} \quad (j=0,1,...,n-1). \tag{3.16}$$

By Lemma 2.6, there exists a set $E_4 \subset (1,+\infty)$ that has finite logarithmic measure such that for all z satisfying $|z| = r \notin [0,1] \cup E_4$, $r \to +\infty$, we have

$$\frac{1}{|A_n(z)|} \leqslant \exp_p\left\{r^{\sigma+\varepsilon}\right\}. \tag{3.17}$$

Substituting (3.15) into (3.14), for all z satisfying $|z| = r \notin E_2$ and $|g(z)| = M(r,g)$, we have

$$-A_n(z)\left(\frac{\nu_g(r)}{z}\right)^n (1+o(1)) = A_{n-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{n-1} (1+o(1))$$

$$+...+A_{s+1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s+1} (1+o(1)) + A_s(z)\left(\frac{\nu_g(r)}{z}\right)^{s} (1+o(1))$$

$$+A_{s-1}(z)\left(\frac{\nu_g(r)}{z}\right)^{s-1} (1+o(1)) + ... + A_1(z)\left(\frac{\nu_g(r)}{z}\right) (1+o(1)) + A_0(z). \tag{3.18}$$

Hence from (3.16) − (3.18), for all z satisfying $|z| = r \notin [0,1] \cup E_2 \cup E_4$, $r \to +\infty$ and $|g(z)| = M(r,g)$, we have

$$\left(1/\exp_p\left\{r^{\sigma+\varepsilon}\right\}\right)\left|\frac{\nu_g(r)}{z}\right|^n |1+o(1)| \leqslant \exp_p\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^{n-1} |1+o(1)|$$

$$+...+\exp_p\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right| |1+o(1)| + \exp_p\left\{r^{\sigma+\varepsilon}\right\}$$

$$\leqslant n\exp_p\left\{r^{\sigma+\varepsilon}\right\}\left|\frac{\nu_g(r)}{z}\right|^{n-1} |1+o(1)|. \tag{3.19}$$

By (3.19) and Lemma 2.9, we get

$$\limsup_{r\to+\infty} \frac{\log_{p+1} \nu_g(r)}{\log r} \leqslant \sigma + \varepsilon. \tag{3.20}$$

Since $\varepsilon > 0$ is arbitrary, by (3.20) and Lemma 2.2, we obtain $\sigma_{p+1}(g) \leqslant \sigma$. Hence $\sigma_{p+1}(f) \leqslant \sigma$. This and the fact that $\sigma_{p+1}(f) \geqslant \sigma$ yield $\sigma_{p+1}(f) = \sigma$.

4.Proof of Theorem 1.2

Assume that f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.14) and $f_1, f_2, ..., f_n$ is a solution base of the corresponding homogeneous equation (1.7) of (1.14). Then f can be expressed in the form

$$f(z) = B_1(z) f_1(z) + B_2(z) f_2(z) + ... + B_n(z) f_n(z), \tag{4.1}$$

where $B_1(z), ..., B_n(z)$ are suitable meromorphic functions determined by

$$\begin{array}{c} B_1'(z) f_1(z) + B_2'(z) f_2(z) + ... + B_n'(z) f_n(z) = 0 \\ B_1'(z) f_1'(z) + B_2'(z) f_2'(z) + ... + B_n'(z) f_n'(z) = 0 \\ .... \\ B_1'(z) f_1^{(n-1)}(z) + B_2'(z) f_2^{(n-1)}(z) + ... + B_n'(z) f_n^{(n-1)}(z) = F(z). \end{array} \tag{4.2}$$

Since the Wronskian $W(f_1, f_2, ..., f_n)$ is a differential polynomial in $f_1, f_2, ..., f_n$ with constant coefficients, it is easy by using Theorem 1.1 to deduce that

$$\sigma_{p+1}(W) \leqslant \max\{\sigma_{p+1}(f_j) : j = 1, ..., n\} = \sigma_p(A_s) = \sigma. \tag{4.3}$$

From (4.2), we get

$$B_j' = F.G_j(f_1, f_2, ..., f_n).W(f_1, f_2, ..., f_n)^{-1} (j = 1, 2, ..., n), \tag{4.4}$$

where $G_j(f_1, f_2, ..., f_n)$ are differential polynomials in $f_1, f_2, ..., f_n$ with constant coefficients. Thus

$$\sigma_{p+1}(G_j) \leqslant \max\{\sigma_{p+1}(f_j) : j = 1, 2, ..., n\}$$

$$= \sigma_p(A_s) = \sigma \ \ (j = 1, 2, ..., n). \tag{4.5}$$

(i) Suppose that $q < p+1$ or $q = p+1$ and $\sigma_{p+1}(F) < \sigma_p(A_s)$. First we show that (1.14) can possess at most one exceptional meromorphic solution f_0 satisfying $i(f_0) < p+1$ or $\sigma_{p+1}(f_0) < \sigma$. In fact, if f^* is another solution with $i(f^*) < p+1$ or $\sigma_{p+1}(f^*) < \sigma$ of equation (1.14), then $i(f_0 - f^*) < p+1$ or $\sigma_{p+1}(f_0 - f^*) < \sigma$. But $f_0 - f^*$ is a solution of the corresponding homogeneous equation (1.7) of (1.14). This contradicts Theorem 1.1. We assume that f is a transcendental meromorphic solution whose poles are of uniformly bounded multiplicity of equation (1.14) with $i(f) \geqslant p+1$. By Lemma 2.7, (4.3), (4.4) and (4.5), for $j = 1, 2, ..., n$, we have

$$\sigma_{p+1}(B_j) = \sigma_{p+1}\left(B_j'\right) \leqslant \max\{\sigma_{p+1}(F), \sigma_p(A_s)\} = \sigma_p(A_s). \tag{4.6}$$

Then from (4.1) and (4.6), we get

$$\sigma_{p+1}(f) \leqslant \max\{\sigma_{p+1}(f_j), \sigma_{p+1}(B_j) : j = 1, 2, ..., n\} = \sigma_p(A_s) = \sigma < +\infty. \tag{4.7}$$

Since $i(f) \geqslant p+1$, it follows from (4.7) that $i(f) = p+1$. Set

$$\max\{\sigma_p(A_j) \;\; (j \neq s), \; \sigma_p(F)\} = \gamma < \mu_p(A_s) \leqslant \sigma_p(A_s) = \sigma < +\infty. \tag{4.8}$$

By the fact that the poles of $f(z)$ can only occur at the zeros of $A_n(z)$, it follows that $i_\lambda\left(\frac{1}{f}\right) \leqslant p$ and $\lambda_p\left(\frac{1}{f}\right) \leqslant \lambda < \mu_p(A_s)$. By Hadamard factorization theorem, we can write f as $f(z) = \frac{g(z)}{d(z)}$, where $g(z)$ and $d(z)$ are entire functions satisfying $i(f) = i(g) = t \geqslant p+1$, $\sigma_t(f) = \sigma_t(g)$ and that $i(d) \leqslant p$, $\sigma_p(d) = \lambda_p(1/f) \leqslant \lambda < \mu_p(A_s)$. For $j = 0, ..., n-1$, since

$$T\left(r, f^{(j+1)}\right) \leqslant 2T\left(r, f^{(j)}\right) + m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right), \tag{4.9}$$

$$m\left(r, \frac{f^{(j+1)}}{f^{(j)}}\right) = O\left\{\log rT\left(r, f^{(j)}\right)\right\}, \tag{4.10}$$

we can obtain by using Lemma 2.3,

$$\begin{aligned} T\left(r, f^{(j+1)}\right) &\leqslant 2T\left(r, f^{(j)}\right) + O\left\{\log rT\left(r, f^{(j)}\right)\right\} \\ &\leqslant 2(j+2)T(2r, f) + O\left\{\log rT\left(r, f^{(j)}\right)\right\}. \end{aligned} \tag{4.11}$$

We have also

$$O\left\{\log rT\left(r, f^{(j)}\right)\right\} = o\left\{T\left(r, f^{(j)}\right)\right\} \tag{4.12}$$

which yields

$$O\left\{\log rT\left(r, f^{(j)}\right)\right\} \leqslant \frac{1}{2}T\left(r, f^{(j)}\right). \tag{4.13}$$

We can rewrite (1.14) as

$$-A_s(z) = A_n(z)\frac{f^{(n)}}{f^{(s)}} + A_{n-1}(z)\frac{f^{(n-1)}}{f^{(s)}} + ... + A_{s+1}(z)\frac{f^{(s+1)}}{f^{(s)}}$$

$$+A_{s-1}(z)\frac{f^{(s-1)}}{f^{(s)}}+...+A_1(z)\frac{f'}{f^{(s)}}+A_0(z)\frac{f}{f^{(s)}}-\frac{F}{f^{(s)}}. \tag{4.14}$$

By (4.11), (4.13) and Lemma 2.3, we can obtain from (4.14) that

$$T(r,A_s)\leqslant T(r,F)+cT(2r,f)+\sum_{j\neq s}T(r,A_j), \tag{4.15}$$

where $c\,(>0)$ is a constant. By (4.8) and (4.15), we conclude $\mu_p(f)\geqslant\mu_p(A_s)$. By the fact that $\sigma_p(d)=\lambda_p(d)=\lambda_p\left(\frac{1}{f}\right)\leqslant\gamma<\mu_p(A_s)$ and the inequality $T(r,f)\leqslant T(r,g)+T(r,d)+O(1)$, it follows that $\sigma(d)<\mu_p(f)=\mu_p(g)$ and $\sigma_p(g)=\sigma_p(f)=+\infty$. Hence by Lemma 2.5, there exists a set $E_3\subset(1,+\infty)$ that has finite logarithmic measure such that for all z satisfying $|z|=r\notin[0,1]\cup E_3$, $r\to+\infty$ and $|g(z)|=M(r,g)$, we have

$$\left|\frac{f(z)}{f^{(s)}(z)}\right|\leqslant r^{2s}. \tag{4.16}$$

By Lemma 2.1, there exist a constant $B>0$ and a set $E_1\subset(1,+\infty)$ of finite logarithmic measure such that for all z satisfying $|z|=r\notin E_1\cup[0,1]$, we have

$$\left|\frac{f^{(j)}(z)}{f^{(s)}(z)}\right|\leqslant Br\left[T(2r,f)\right]^{j-s+1}\quad(j=s+1,...,n), \tag{4.17}$$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right|\leqslant Br\left[T(2r,f)\right]^{j+1}\quad(j=1,...,s-1). \tag{4.18}$$

From (4.14), it follows that

$$\begin{aligned}|A_s(z)|\leqslant{}&|A_n(z)|\left|\frac{f^{(n)}}{f^{(s)}}\right|+|A_{n-1}(z)|\left|\frac{f^{(n-1)}}{f^{(s)}}\right|+...+|A_{s+1}(z)|\left|\frac{f^{(s+1)}}{f^{(s)}}\right|\\&+|A_{s-1}(z)|\left|\frac{f^{(s-1)}}{f}\right|\left|\frac{f}{f^{(s)}}\right|+...+|A_1(z)|\left|\frac{f'}{f}\right|\left|\frac{f}{f^{(s)}}\right|\\&+|A_0(z)|\left|\frac{f}{f^{(s)}}\right|+\left|\frac{F}{f}\right|\left|\frac{f}{f^{(s)}}\right|.\end{aligned} \tag{4.19}$$

On the other hand, for any given ε $(0<2\varepsilon<\sigma-\gamma)$, we have for a sufficiently large r

$$|F(z)|\leqslant\exp_{p-1}\left\{r^{\gamma+\varepsilon}\right\}\ \text{and}\ |d(z)|\leqslant\exp_{p-1}\left\{r^{\gamma+\varepsilon}\right\}. \tag{4.20}$$

Since $M(r,g) \geqslant 1$, it follows from (4.16) and (4.20) that

$$\left|\frac{F(z)}{f(z)}\right|\left|\frac{f(z)}{f^{(s)}(z)}\right| = \frac{|F(z)||d(z)|}{|g(z)|}\left|\frac{f(z)}{f^{(s)}(z)}\right| \leqslant r^{2s}\left(\exp_{p-1}\left\{r^{\gamma+\varepsilon}\right\}\right)^2 \tag{4.21}$$

as $|z| = r \to +\infty$ and $|g(z)| = M(r,g)$. From (1.12), (1.13), (4.16) $-$ (4.19) and (4.21), it follows that

$$\exp_p\left\{\alpha|z_k|^{\sigma-\varepsilon}\right\} \leqslant Bn|z_k|^{2s+1}[T(2r_k,f)]^{n+1}\exp_p\left\{\beta|z_k|^{\sigma-\varepsilon}\right\} + |z_k|^{2s}\left(\exp_{p-1}\left\{|z_k|^{\gamma+\varepsilon}\right\}\right)^2 \tag{4.22}$$

as $k \to +\infty$, $|z_k| = r_k \notin [0,1] \cup E_1 \cup E_3$ and $|g(z_k)| = M(r_k,g)$. From (4.22) and Lemma 2.9, we get $\sigma_{p+1}(f) \geqslant \sigma - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, it follows that $\sigma_{p+1}(f) \geqslant \sigma$. This and the fact that $\sigma_{p+1}(f) \leqslant \sigma$ yield $\sigma_{p+1}(f) = \sigma$. Hence by Lemma 2.8, we deduce that $i_{\overline{\lambda}}(f) = i_\lambda(f) = i(f) = p+1$ and $\overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \sigma_{p+1}(f) = \sigma$.

(ii) Suppose that $q > p+1$ or $q = p+1$ and $\sigma_p(A_s) < \sigma_{p+1}(F)$. If $q = p+1$ and $\sigma_p(A_s) < \sigma_{p+1}(F)$, then by Lemma 2.7, (4.3), (4.4) and (4.5), for $j = 1,2,...,n$, we have

$$\sigma_{p+1}(B_j) = \sigma_{p+1}\left(B_j'\right) \leqslant \max\left\{\sigma_{p+1}(F), \sigma_p(A_s)\right\} = \sigma_{p+1}(F). \tag{4.23}$$

Then from (4.1) and (4.23), we get

$$\sigma_{p+1}(f) \leqslant \max\left\{\sigma_{p+1}(f_j), \sigma_{p+1}(B_j)\right\} = \sigma_{p+1}(F). \tag{4.24}$$

If $q > p+1$, we have

$$\sigma_q(B_j) = \sigma_q\left(B_j'\right) \leqslant \max\left\{\sigma_q(F), \sigma_q(G_j)\right\} = \sigma_q(F) \quad (j = 1,2,...,n). \tag{4.25}$$

Then from (4.1) and (4.25), we get

$$\sigma_q(f) \leqslant \max\left\{\sigma_q(f_j), \sigma_q(B_j)\right\} = \sigma_q(F). \tag{4.26}$$

On the other hand, if $q > p+1$ or $q = p+1$ and $\sigma_p(A_s) < \sigma_{p+1}(F)$, it follows from (1.14) that a simple consideration of order implies $\sigma_q(f) \geqslant \sigma_q(F)$. Hence $\sigma_q(f) = \sigma_q(F)$.

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