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SOME RESULTS ASSOCIATED WITH FRACTIONAL INTEGRALS INVOLVING THE EXTENDED CHEBYSHEV FUNCTIONAL

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Abstract. In this paper, the Riemann-Liouville fractional integral is used to establish a new class of inequalities for the extended Chebyshev functional.

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1. Introduction

Let us consider the functional [3]

$$T(f,g,p,q) := \int_a^b q(x)dx \int_a^b p(x)f(x)g(x)\,dx + \int_a^b p(x)dx \int_a^b q(x)f(x)\,g(x)\,dx$$
$$- \left(\int_a^b q(x)f(x)\,dx\right)\left(\int_a^b p(x)g(x)\,dx\right) - \left(\int_a^b p(x)f(x)\,dx\right)\left(\int_a^b q(x)g(x)\,dx\right), \tag{1}$$

where f and g are two integrable functions on $[a,b]$ and p,q are two positive integrable functions on $[a,b]$.
In the case of $f', g' \in L_\infty(a,b)$, S.S. Dragmir [9] proved that

$$|S(f,g,p)| \le ||f'||_\infty ||g'||_\infty \Big[\int_a^b p(x)dx \int_a^b x^2 p(x)dx - \Big(\int_a^b xp(x)dx\Big)^2\Big], \tag{2}$$

where

$$S(f,g,p) := \frac{1}{2}T(f,g,p,p) = \int_a^b p(x) \int_a^b p(x)f(x)\,g(x)\,dx - \int_a^b p(x)f(x)\,dx \int_a^b p(x)g(x)\,dx. \tag{3}$$

If f is M-g-Lipschitzian on $[a,b]$: i.e.

$$|f(x)-f(y)| \le M|g(x)-g(y)|;\, M>0, x,y \in [a,b], \tag{4}$$

Dragomir proved that

$$|S(f,g,p)| \leq M\Big[\int_a^b p(x)dx \int_a^b g^2(x)p(x)dx - \Big(\int_a^b g(x)p(x)dx\Big)^2\Big]. \tag{5}$$

Many researchers have given considerable attention to (1) and (3) and several inequalities related to these functionals have appeared in the literature, to mention a few, see [1,2,4-12,14-18]and the references cited therein.
The main aim of this paper is to establish some new generalizations for the extended Chebyshev functional (1) by using the Riemann-Liouville fractional integrals. We give our results in the case of differentiable functions whose derivative belong to $L_\infty([0,\infty[$. Then, under the condition (4), we give another class of inequalities.

2. Basic Definitions of the Fractional Integrals

Definition 1. *A real valued function $f(t), t \geq 0$ is said to be in the space $C_\mu, \mu \in IR$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C([0,\infty[)$.*
Definition 2. *A function $f(t), t \geq 0$ is said to be in the space $C_\mu^n, \mu \in IR$, if $f^{(n)} \in C_\mu$.*
Definition 3. *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$, for a function $f \in C_\mu, (\mu \geq -1)$ is defined as*

$$\begin{gathered} J^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau; \quad \alpha > 0, t > 0, \\ J^0 f(t) = f(t), \end{gathered} \tag{6}$$

where $\Gamma(\alpha) := \int_0^\infty e^{-u}u^{\alpha-1}du$.
For the convenience of establishing the results, we give the semigroup property:

$$J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t), \alpha \geq 0, \beta \geq 0, \tag{7}$$

which implies the commutative property

$$J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t). \tag{8}$$

More details, one can consult [13].

3. Main Results

Theorem 3.1. *Let p, q be two positive integrable functions on $[0,\infty[$ and let f and g be two differentiable functions on $[0,\infty[$. If $f', g' \in L_\infty([0,\infty[)$, then for all $t > 0, \alpha > 0$, we have:*

$$\begin{aligned}&\Big|J^{\alpha}q(t)J^{\alpha}pfg(t)+J^{\alpha}p(t)J^{\alpha}qfg(t)-J^{\alpha}qf(t)J^{\alpha}pg(t)-J^{\alpha}pf(t)J^{\alpha}qg(t)\Big|\\&\leq ||f'||_{\infty}||g'||_{\infty}\Big[J^{\alpha}q(t)J^{\alpha}t^{2}p(t)+J^{\alpha}p(t)J^{\alpha}t^{2}q(t)-2(J^{\alpha}tq(t))(J^{\alpha}tp(t))\Big].\end{aligned} \tag{9}$$

Proof. Let f and g be two functions satisfying the conditions of Theorem 3.1 and let p,q be two positive integrable functions on $[0,\infty[$.
Define

$$H(\tau,\rho):=(f(\tau)-f(\rho))(g(\tau)-g(\rho));\ \tau,\rho\in(0,t),t>0. \tag{10}$$

Multiplying (10) by $\frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)}p(\tau);\ \tau\in(0,t)$ and and integrating the resulting identity with respect to τ from 0 to t, we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_{0}^{t}(t-\tau)^{\alpha-1}p(\tau)H(\tau,\rho)d\tau\\&=J^{\alpha}pfg(t)-f(\rho)J^{\alpha}pg(t)-g(\rho)J^{\alpha}pf(t)+f(\rho)g(\rho)J^{\alpha}p(t).\end{aligned} \tag{11}$$

Multiplying (11) by $\frac{(t-\rho)^{\alpha-1}}{\Gamma(\alpha)}q(\rho);\ \rho\in(0,t)$ and integrating the resulting identity with respect to ρ over $(0,t)$, we can write

$$\begin{aligned}&\frac{1}{\Gamma^{2}(\alpha)}\int_{0}^{t}\int_{0}^{t}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)q(\rho)H(\tau,\rho)d\tau d\rho\\&=J^{\alpha}q(t)J^{\alpha}pfg(t)-J^{\alpha}qf(t)J^{\alpha}pg(t)-J^{\alpha}pf(t)J^{\alpha}qg(t)+J^{\alpha}p(t)J^{\alpha}qfg(t).\end{aligned} \tag{12}$$

On the other hand, we know that

$$H(\tau,\rho)=\int_{\tau}^{\rho}\int_{\tau}^{\rho}f'(y)g'(z)dydz, \tag{13}$$

From the hypothesis $f',g'\in L_{\infty}([0,\infty[)$, it follows

$$|H(\tau,\rho)|\leq\Big|\int_{\tau}^{\rho}f'(y)dy\Big|\Big|\int_{\tau}^{\rho}g'(z)dz\Big|\leq||f'||_{\infty}||g'||_{\infty}(\tau-\rho)^{2}. \tag{14}$$

Hence,

$$\begin{aligned}&\frac{1}{\Gamma^{2}(\alpha)}\int_{0}^{t}\int_{0}^{t}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho\\&\leq\tfrac{||f'||_{\infty}||g'||_{\infty}}{\Gamma^{2}(\alpha)}\textstyle\int_{0}^{t}\int_{0}^{t}(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}(\tau^{2}-2\tau\rho+\rho^{2})p(\tau)q(\rho)d\tau d\rho.\end{aligned} \tag{15}$$

Thus, we obtain the following inequality

$$\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho$$
$$\leq ||f'||_\infty||g'||_\infty\Big[J^\alpha q(t)J^\alpha t^2p(t)-2(J^\alpha tp(t))(J^\alpha tq(t))+J^\alpha p(t)J^\alpha t^2q(t)\Big]. \quad (16)$$

By the relations (12), (16) and using the properties of the modulus, we get the desired inequality (9).

Remark 3.2. Applying Theorem 3.1 for $\alpha=1, p(x)=q(x); x\in[0,\infty[$, we obtain the inequality (2) on $[0,t]$.

The second result is the following theorem.

Theorem 3.3. *Let p,q be two positive integrable functions on $[0,\infty[$ and let f,g be two differentiable functions on $[0,\infty[$. If $f',g'\in L_\infty([0,\infty[)$, then for all $t>0,\alpha>0,\beta>0$, we have*

$$\Big|J^\alpha p(t)J^\beta qfg(t)+J^\beta q(t)J^\alpha pfg(t)-J^\alpha pf(t)J^\beta qg(t)-J^\beta qf(t)J^\alpha pg(t)\Big|$$
$$\leq ||f'||_\infty||g'||_\infty\Big[J^\alpha p(t)J^\beta t^2q(t)-2(J^\alpha tp(t))(J^\beta tq(t))+J^\beta q(t)J^\alpha t^2p(t)\Big]. \quad (17)$$

Proof. The relation (11) implies that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)q(\rho)H(\tau,\rho)d\tau d\rho$$
$$=J^\beta q(t)J^\alpha pfg(t)+J^\alpha p(t)J^\beta qfg(t)-J^\beta qf(t)J^\alpha pg(t)-J^\alpha pf(t)J^\beta qg(t). \quad (18)$$

From the relation (13), it follows

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}p(\tau)|H(\tau,\rho)|d\tau$$
$$\leq ||f'||_\infty||g'||_\infty\Big[J^\alpha t^2p(t)-2\rho J^\alpha tp(t)+\rho^2J^\alpha p(t)\Big]. \quad (19)$$

The inequality (19) implies that

$$\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho$$
$$\leq ||f'||_\infty||g'||_\infty\Big[J^\beta q(t)J^\alpha t^2p(t)-2(J^\alpha tp(t))(J^\beta tq(t))+J^\alpha p(t)J^\beta t^2q(t)\Big]. \quad (20)$$

Using $(18),(20)$ and the properties of modulus, we finally obtain (17).

Remark 3.4 (i) Applying Theorem 3.3 for $\alpha=\beta$ we obtain Theorem 3.1.
(ii) Applying Theorem 3.3 for $\alpha=\beta=1, p(x)=q(x), x\in[0,\infty[$, we obtain the inequality (2) on $[0,t]$.

Theorem 3.5. *Let p,q be two positive integrable functions on $[0,\infty[$ and let f,g be two integrable functions on $[0,\infty[$ satisfying the condition (4) on $[0,\infty[$. Then for all $t>0,\alpha>0$, we have:*

$$\Big|J^{\alpha}q(t)J^{\alpha}pfg(t)+J^{\alpha}p(t)J^{\alpha}qfg(t)-J^{\alpha}qf(t)J^{\alpha}pg(t)-J^{\alpha}qf(t)J^{\alpha}pg(t)\Big| \\ \leq M\Big[J^{\alpha}p(t)J^{\alpha}qg^2(t)+J^{\alpha}q(t)J^{\alpha}pg^2(t)-2J^{\alpha}pg(t)J^{\alpha}qg(t)\Big]. \tag{21}$$

Proof. Let f and g be two functions satisfying the condition (4) on $[0,\infty[$. Then for every $\tau,\rho\in[0,t];t>0$, we have

$$|f(\tau)-f(\rho)|\leq M|g(\tau)-g(\rho)|. \tag{22}$$

This implies that

$$\Big|H(\tau,\rho)\Big|\leq M\Big(g(\tau)-g(\rho)\Big)^2,\tau,\rho\in[0,t]. \tag{23}$$

Hence, it follows that

$$\frac{1}{\Gamma(\alpha)}\int_0^t(t-\tau)^{\alpha-1}p(\tau)|H(\tau,\rho)|d\tau \\ \leq M\Big(J^{\alpha}pg^2(t)-2g(\rho)J^{\alpha}pg(t)+g^2(\rho)J^{\alpha}p(t)\Big). \tag{24}$$

Consequently,

$$\frac{1}{\Gamma^2(\alpha)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\alpha-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho \\ \leq M\Big[J^{\alpha}q(t)J^{\alpha}pg^2(t)-2J^{\alpha}qg(t)J^{\alpha}pg(t)+J^{\alpha}p(t)J^{\alpha}qg^2(t)\Big]. \tag{25}$$

Using (12) and (25), we finally get the desired fractional inequality (21).
Remark 3.6 Applying Theorem 3.5 for $\alpha=1, p(x)=q(x); x\in[0,\infty[$, we obtain the inequality (5) on $[0,t]$.

Another estimation depending on two fractional parameters is the following theorem:

Theorem 3.7. *Let f and g be two integrable functions on $[0,\infty[$ satisfying the condition (4) on $[0,\infty[$ and let p,q be two positive integrable functions on $[0,\infty[$. Then the inequality*

$$\begin{aligned}&\Big|J^{\alpha}p(t)J^{\beta}qfg(t)+J^{\beta}q(t)J^{\alpha}pfg(t)-J^{\alpha}pf(t)J^{\beta}qg(t)-J^{\beta}qf(t)J^{\alpha}pg(t)\Big|\\&\leq M\Big[J^{\beta}q(t)J^{\alpha}pg^{2}(t)+J^{\alpha}p(t)J^{\beta}qg^{2}(t)-2J^{\alpha}pg(t)J^{\beta}qg(t)\Big].\end{aligned}\tag{26}$$

is valid for all all $t>0,\alpha>0,\beta>0$.

Proof. Using the relation (24), we obtain

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho\\&\leq \tfrac{M}{\Gamma(\beta)}\textstyle\int_0^t\Big((t-\rho)^{\beta-1}q(\rho)\Big[J^{\alpha}pg^{2}(t)-2g(\rho)J^{\alpha}pg(t)+g^{2}(\rho)J^{\alpha}p(t)\Big]\Big)d\rho.\end{aligned}\tag{27}$$

Consequently,

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t\int_0^t(t-\tau)^{\alpha-1}(t-\rho)^{\beta-1}p(\tau)q(\rho)|H(\tau,\rho)|d\tau d\rho\\&\leq M\Big[J^{\beta}q(t)J^{\alpha}pg^{2}(t)-2J^{\beta}qg(t)J^{\alpha}pg(t)+J^{\beta}qg^{2}(t)J^{\alpha}p(t)\Big].\end{aligned}\tag{28}$$

Theorem 3.7 is thus proved.

Remark 3.8. (i) Applying Theorem 3.7 for $\alpha=\beta$, we obtain Theorem 3.5.
(ii) Applying Theorem 3.7 for $\alpha=\beta=1, p(x)=q(x); x\in[0,\infty[$, we obtain the inequality (5) on $[0,t]$.

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