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ON CONVEXITY OF CERTAIN INTEGRAL OPERATOR

D. O. Makinde and T. O. Opoola

Abstract. In this paper,we obtain the convexity property of the integral operator $\int_0^z \prod_{\imath=1}^k (\frac{f_\imath(s)}{s})^{\frac{1}{\alpha}} ds$.

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1. Introduction

Let $H(U)$ be the set of functions which are regular in the unit disc,

$$\begin{aligned} U &= \{z \in C : \mid z \mid < 1\} \\ A &= \{f \in H(u) : f(0) = f'(0) = 0\} \\ S &= \{f \in A : f \textit{ is univalent in } U, f(0) = f'(0) = 0\} \end{aligned}$$

where f is the function of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, n \in N \tag{1}$$

Furthermore, let

$$S^* = \left\{ f \in S : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in U \right\} \tag{2}$$

$$S^c = \left\{ f \in S : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in U \right\} \tag{3}$$

2. Preliminaries

Lemma 2.1 [1]: *Let M and N be analytic in U with $M(0) = N(0) = 0$. If $N(z)$ maps onto a many sheeted region which is starlike with respect to the origin and $Re\{\frac{M'(z)}{N'(z)}\} > 0$ in U, then $Re\{\frac{M(z)}{N(z)}\} > 0$ in U*

Lemma 2.2 [2]: *Let $f_\imath \in T_{n,\mu_\imath}(\imath = 1, 2, ..., k; k \in \ N^*)$ be defined by*

$$f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n \tag{4}$$

for all $\ i = 1, 2, ..., k; \alpha, \beta \in \mathcal{C}; R\{\beta\} \geq \gamma$ and $\gamma = \Sigma_{i=1}^{k} \frac{1+(1+\mu_i)M}{|\alpha|}(M \geq 1, 0 < \mu_i < 1, k \in N^*)$. If $|f_i(z)| \leq M(z \in U), i = 1, 2, ..., k$ then, the integral operator

$$F_{\alpha,\beta}(z) = \{\beta \int_0^z t^{\beta-1} \prod_{i=1}^{k} (\frac{f_i(t)}{t})^{\frac{1}{\alpha}} dt\}^{\frac{1}{\beta}} \tag{5}$$

is univalent
Lemma 2.3 [2]: *Let h be convex in $\ u$ and $\ Re\{\beta h(z) + \gamma\} > 0, z \in U$. If $\ p \in H(u)$ where$H(u)$ is the class of functions which are analytic in the unit disk, with $\ p(0) = h(0)$ and $p$satisfies the Briot-Bouquet differential subordinations:*
$p(z) + \frac{zp'(z)}{\beta p(z)+\gamma} \prec h(z), z \in U$. Then, $p(z) \prec h(z), z \in U$.
In [3],S. Kanas and F. Ronning introduced the following classes of functions for a fixed point $\ w$ in $\ U$:

$$\begin{aligned} A(w) &= \{f \in H(U) : f(w) = f'(w) - 1 = 0\} \\ S(w) &= \{f \in A(w) : f \text{ is univalent in } U\} \\ S^* &= \{f \in S : Re(\frac{zf'(z)}{f(z)}) > 0, z \in U\} \end{aligned}$$

The class of $\ S$ and S^* are called univalent and and starlike functions respectively. Let $\alpha \in R$ and $w\ $ be a fixed point in $\ U\ $. For $f \in S(w)$, we define $J(\alpha, f, w; z) = \{(1 - \alpha)\frac{(z-w)f'(z)}{f(z)} + (1 + \frac{(z-w)f''(Z)}{f'(Z)})$, f is $w - \alpha - convex$ function if $\frac{f(z)f'(n)}{z-w}, z \in U$ and $ReJ(\alpha, f, w; z) > 0, z \in U$. Let this class of functions be denoted by $M_\alpha(w)$.Let $D(w) = \{z \in U : Re(\frac{w}{z}) < 1,$ and $Re(\frac{z(1+z)}{(z-w)(1-z)}) > 0\}$ with $D(0) = U$ and $s(w) = \{f : D(w) \to C\} \cap S(w), w$ is a fixed point in U

3.The Main Results

We now give the proof of the following results:
Theorem 3.1: *Let $F_\alpha(z)$ be the function in U defined by:*

$$F_\alpha(z) = \int_0^z \prod_{i=1}^{k} (\frac{f_i(s)}{s})^{\frac{1}{\alpha}} ds, \alpha \in C. \tag{6}$$

If $f_i \in S^*$ then, $F(z) \in S^*$ wheref_i is as equation (5) above.

Proof: By differentiating (6), we obtain: F'(z)= $\prod\limits_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}$ Thus,

$$\frac{zF'(z)}{F(z)} = \frac{\prod\limits_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}}{\int_0^z \prod\limits_{i=1}^{k}(\frac{f_i(s)}{s})^{\frac{1}{\alpha}}ds} \tag{7}$$

Let

$$M = zF'(z), N(z) = F(z) \tag{8}$$

From (7) and (8) we have: $\frac{M'(z)}{N'(z)} = 1 + \frac{zF"(z}{F'(z)}$

$$\frac{M'(z)}{N'(z)} = 1 + \frac{\sum_{i=1}^{k}\frac{1}{\alpha}(\frac{zf_i(z)}{f(z)} - 1)}{\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}}$$

$$|\frac{M'(z)}{N'(z)} - 1| = \frac{|\sum_{i=1}^{k}\frac{1}{\alpha}(\frac{zf_i(z)}{f(z)} - 1)|}{|\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}|} \leq \frac{\sum_{i=1}^{k}|\frac{1}{\alpha}||\frac{zf_i(z)}{f(z)} - 1|}{|\prod_{i=1}^{k}(\frac{f_i(z)}{z})^{\frac{1}{\alpha}}|}$$

By hypothesis $f_i \in S^*$, this means that:$|\frac{zf_i(z)}{f(z)} - 1| < 1$ which implies that: $|\frac{M'(z)}{N'(z)} - 1| < 1$. Thus, $Re\{\frac{M'(z)}{N'(z)}\} > 0$ and by lemma 2.1 $Re\{\frac{M(z)}{N(z)}\} > 0$. This implies that $Re\{\frac{zF'(z)}{F(z)}\} > 0$. Hence $F \in S^*$

Remark: *The integral in (6) is equivalent to that in (5) of section 2 with $\beta = 1$.*
Let $s = \{f : U \to C\} \cap S$. Let $F(z) \in U$ be defined by

$$F(z) = \int_0^z \prod_{i=1}^{k}(\frac{f_i(s)}{s})^{\frac{1}{\alpha}}ds \tag{9}$$

Theorem 3.2: *Let $z \in U, \alpha \in C, Re\alpha > 0$ and $m_\alpha = M_\alpha \cap s$. If $F \in m_\alpha$, then $F \in S^*$that is $m_\alpha \subset S^*$*

Proof: From (7)above, we have: $\frac{F(z)F'(z)}{z} \neq 0$ and for $F \in m_\alpha$, we have:

$$ReJ(\alpha, f; z) = Re\{(1-\alpha)\frac{zF'(z)}{F(z)} + \alpha(1 + \frac{zF'(z)}{F(z)})\} \tag{10}$$

for $p(z) = \frac{zF'(z)}{F(z)}, \frac{zp'(z)}{p(z)} = 1 + \frac{zF"(z)}{F'(z)} - p(z)$ This implies that:

$$1 + \frac{zF''(z)}{F'(z)} = \frac{zp'(z)}{p(z)} + p(z) \tag{11}$$

using (9) and (11) in (10), we obtain:

$$ReJ(\alpha, f; z) = Re\{(1-\alpha)p(z) + \alpha(\frac{zp'(z)}{p(z)} + p(z))\} \quad (12)$$

Simplifying (12), we obtain:

$$ReJ(\alpha, f; z) = Re\{p(z) + \alpha(\frac{zp'(z)}{p(z)})\} \quad (13)$$

$p(0)+\frac{\alpha zp'(0)}{p(0)} = 1$ and $p(0) = h(0) = 1$. Thus, using lemma 2.3 with $\beta = 1$ and $\gamma = 0$, we have $p(z) + \frac{\alpha zp'(z)}{p(z)} < h(z) = \frac{1+z}{1-z}$. This implies that $p(z) \prec h(z)$
That is $Re\{p(z)\} > 0$. Thus, $Re\{\frac{zF'(z)}{F(z)}\} > 0$. Hence, $F \in S^*$.

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Deborah O. Makinde
Department of Mathematics,
Obafemi Awolowo University,
Ile Ife 220005, Nigeria.
email:*dmakinde@oauife.edu.ng, makindemyiuv@yahoo.com*

Timothy O. Opoola
Department of Mathematics,
University of Ilorin
ILORIN,Nigeria.
email:*opoolato@unilorin.edu.ng*