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THE SOLUTION OF A SECOND-ORDER NONLINEAR DIFFERENTIAL EQUATION WITH NEUMANN BOUNDARY CONDITIONS USING TRIGONOMETRIC SCALING FUNCTIONS FOR HERMITE INTERPOLATION

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Abstract. A numerical technique for solving a second-order nonlinear Neumann problem is presented. The authors approach is based on trigonometric scaling function on $[0, 2\pi]$ which is constructed for Hermite interpolation. Two test problems are presented and errors plots show the efficiency of the proposed technique for the studied problem.

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1. Introduction

In this paper we solve the second-order nonlinear Neumann problem of the form

$$-\ddot{x}(t) = f(t, x(t)), \qquad t \in [0, 2\pi], \tag{1}$$

$$\dot{x}(0) = \alpha, \quad \dot{x}(2\pi) = \beta. \tag{2}$$

Here f, is a known function, α, and β are given real numbers and x is the unknown function to be found. The existence of a solution to equation (1) with Neumann boundary conditions is studied in [1] using the quasi-linearization method. In the present paper we apply Hermite interpolation by trigonometric scaling functions, to solve the nonlinear second-order Neumann problem of the form (1). The literature of numerical analysis contains little on the solution of second-order nonlinear Neumann problem of the above form. Equations (1)-(2) are investigated by few authors. A method using semi-orthogonal B-spline wavelets is employed in [2] to solve equations (1)-(2) on $[0, 1]$.

The outline of this paper is as follows. In Section 2, we describe the trigonometric scaling function on $[0, 2\pi]$ and its properties. In Section 3, we show the method of expanding a function using trigonometric scaling function on $[0, 2\pi]$ In section

4, we construct the operational matrix of derivative for these functions. In Section 5, the proposed method is used to approximate the solution of the problem. As a result a set of algebraic equations is formed and a solution of the considered problem is introduced. In Section 6, we report our computational results and demonstrate the accuracy of the proposed numerical scheme by presenting numerical examples. In section 7, we give the results of applying the presented method to the problem.

2. Trigonometric scaling function on $[0, 2\pi]$

In this section, we will give a brief introduction of Quak's work on the construction of Hermite interpolatory trigonometric scaling functions and their basic properties (see [3] , [4]).
For all $n \in \mathtt{N}$, the Dirichlet kernel $D_n(t)$ and its conjugate kernel $\tilde{D}_n(t)$ are defined as

$$D_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \cos kt = \begin{cases} \frac{\sin(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}, & t \notin 2\pi\mathtt{Z}, \\ n+\frac{1}{2}, & t \in 2\pi\mathtt{Z}, \end{cases} \tag{3}$$

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt = \begin{cases} \frac{\cos(\frac{1}{2}t)-\cos(n+\frac{1}{2})t}{2\sin\frac{1}{2}t}, & t \notin 2\pi\mathtt{Z}, \\ 0, & t \in 2\pi\mathtt{Z}. \end{cases} \tag{4}$$

Obviously, $D_n(t), \tilde{D}_n(t) \in T_n$, where T_n is the linear space of trigonometric polynomials with digree not exeeding n. The equally spaced nodes on the interval $[0, 2\pi)$ with a dyadic step are denoted by

$$t_{j,n} = \frac{n\pi}{2^j}, \qquad j \in \mathtt{N}_0, \qquad n = 0, 1, ..., 2^{j+1} - 1,$$

where $\mathtt{N}_0 = \mathtt{N} \cup \{0\}$, i.e., $\mathtt{N}_0$ is the set of non-negative integers [3], [4].

Definition 1. *(scaling functions). For all $j \in \mathtt{N}_0$, the scaling functions $\phi^0_{j,0}(t)$, $\phi^1_{j,0}(t)$ are defined as*

$$\phi^0_{j,0}(t) = \frac{1}{2^{2j+1}} \sum_{k=0}^{2^{j+1}-1} D_k(t), \tag{5}$$

$$\phi^1_{j,0}(t) = \frac{1}{2^{2j+1}} \left(\tilde{D}_{2^{j+1}-1}(t) + \frac{1}{2}\sin(2^{j+1}t) \right), \tag{6}$$

let $\phi^s_{j,n}(t) = \phi^s_{j,0}(t - t_{j,n})$, $s = 0, 1$, and $n = 0, 1, ..., 2^{j+1} - 1$. Furthermore, let $\phi^s_{j,n}(t) = \phi^s_{j,n \ mod \ 2^{j+1}}(t)$, $s = 0, 1$, and any $n \in \mathtt{N}$.

Lemma 1. *For any $j \in \mathtt{N}_0$, we have*

$$\phi^0_{j,0}(t) = \begin{cases} \frac{1}{2^{2j+1}} \frac{\sin^2(2^j t)}{\sin^2(\frac{t}{2})}, & t \notin 2\pi\mathtt{Z}, \\ 1, & t \in 2\pi\mathtt{Z}, \end{cases} \tag{7}$$

$$\phi^1_{j,0}(t) = \begin{cases} \frac{1}{2^{2j+1}} \left(1 - \cos(2^{j+1}t)\right) \cot(\frac{t}{2}), & t \notin 2\pi\mathtt{Z}, \\ 0, & t \in 2\pi\mathtt{Z}, \end{cases} \tag{8}$$

and their derivations are given by

$$\left(\phi^0_{j,0}(t)\right)' = \begin{cases} \frac{1}{2^{j+2}} \frac{\sin(2^{j+1}t)}{\sin^2(\frac{t}{2})} - \frac{1}{2^{2j+2}} \frac{\sin^2(2^j t)\cot(\frac{t}{2})}{\sin^2(\frac{t}{2})}, & t \notin 2\pi\mathtt{Z}, \\ 0, & t \in 2\pi\mathtt{Z}, \end{cases} \tag{9}$$

$$\left(\phi^1_{j,0}(t)\right)' = \begin{cases} \frac{1}{2^{2j+3}} \frac{\cos(2^{j+1}t)-1}{\sin(2^{j+1}t)} + \frac{1}{2^{j+1}} \sin(2^{j+1}t)\cot(\frac{t}{2}), & t \notin 2\pi\mathtt{Z}, \\ 0, & t \in 2\pi\mathtt{Z}. \end{cases} \tag{10}$$

Proof. see [3].

Theorem 2. *(interplatory properties of the scaling functions). For all $j \in \mathtt{N}_0$, the following interplatory properties hold for each $k, n = 0, 1, ..., 2^{j+1} - 1$*

$$\phi^0_{j,n}(t_{j,k}) = \delta^n_k, \qquad \left(\phi^0_{j,n}(t_{j,k})\right)' = 0, \tag{11}$$

$$\phi^1_{j,n}(t_{j,k}) = 0, \qquad \left(\phi^1_{j,n}(t_{j,k})\right)' = \delta^n_k. \tag{12}$$

Proof. see [3].

Theorem 3. *For any $j \in \mathtt{N}_0$, we have*

$$V_j = span\left\{1, \cos t, ..., \cos(2^{j+1} - 1)t, \sin t, ..., \sin 2^{j+1}t\right\}$$

consequently,

$$dimV_j = 2^{j+2}.$$

Proof. see [3], [4].

Lemma 4. *For $j \in \mathtt{N}_0$, $n = 0, 1, ..., 2^{j+1} - 1$, we have*

$$\sum_{k=0}^{2^{j+1}-1} k^2 \cos kt_{j,n} = \begin{cases} -2^{2j+1} + 2^j \sin^{-2} t_{j+1,n}, & n \neq 0, \\ \frac{1}{3} 2^j (2^{j+1}-1)(2^{j+2}-1), & n = 0, \end{cases} \tag{13}$$

$$\sum_{k=0}^{2^{j+1}-1} k^2 \sin kt_{j,n} = \begin{cases} -2^{2j+1} \cot t_{j+1,n}, & n \neq 0, \\ 0, & n = 0. \end{cases} \tag{14}$$

Proof. see [4].

3. Function approximation

For $j \in \mathbb{N}_0$, a function $f(t)$ defined on $[0, 2\pi]$ may be represented by trigonometric scaling functions as

$$f(t) \simeq \sum_{k=0}^{2^{j+1}-1} \left[a_k \phi^0_{j,k}(t) + b_k \phi^1_{j,k}(t) \right] = C^T \Phi, \tag{15}$$

where

$$C = [a_0, ..., a_{2^{j+1}-1}, b_0, ..., b_{2^{j+1}-1}]^T, \tag{16}$$

$$\Phi = \left[\phi^0_{j,0}, ..., \phi^0_{j,2^{j+1}-1}, \phi^1_{j,0}, ..., \phi^1_{j,2^{j+1}-1} \right]^T, \tag{17}$$

are vectors with dimension $2^{j+1} \times 1$.
Using (11) and (12) we get

$$a_k = f(t_{j,k}), \quad b_k = f'(t_{j,k}), \quad k = 0, 1, ..., 2^{j+1} - 1. \tag{18}$$

4. The operational matrix of derivative

The differentiation of vector Φ in (17) can be expressed as

$$\Phi' = D\Phi, \tag{19}$$

where D is $2^{j+2} \times 2^{j+2}$ operational matrix of derivative for trigonometric scaling functions. Suppose

$$\left(\phi^s_{j,n}(t) \right)' = \sum_{k=0}^{2^{j+1}-1} \left[a^s_{k,n} \phi^0_{j,k}(t) + b^s_{k,n} \phi^1_{j,k}(t) \right], \quad s = 0, 1, \tag{20}$$

where $n = 0, 1, ..., 2^{j+1} - 1$.
So the matrix D can be represented as a block matrix as

$$D = \begin{bmatrix} A^0 & B^0 \\ A^1 & B^1 \end{bmatrix}, \qquad k,n = 0,1,...,2^{j+1}-1, \tag{21}$$

where A^s and B^s, s=0,1 are $2^{j+1} \times 2^{j+1}$ matrices.
The entries of matrices A^s and B^s may be find by usig (18) as follows

$$A^s = \left(a^s_{k,n}\right) = \left(\phi^s_{j,k}(t_{j,n})\right)', \qquad B^s = \left(b^s_{k,n}\right) = \left(\phi^s_{j,k}(t_{j,n})\right)'', \quad s = 0,1, \tag{22}$$

and $k,n = 0,1,...,2^{j+1}-1$. Using (11) and (12) we get

$$A^0 = \left(a^0_{k,n}\right) = \left(\phi^0_{j,k}(t_{j,n})\right)' = 0, \qquad k,n = 0,1,...,2^{j+1}-1, \tag{23}$$

$$A^1 = \left(a^1_{k,n}\right) = \left(\phi^1_{j,k}(t_{j,n})\right)' = \delta^n_k, \qquad k,n = 0,1,...,2^{j+1}-1, \tag{24}$$

where δ^n_k is the Kronecker delta.
Regarding that using definition 1 we get

$$\phi^s_{j,n}(t) = \phi^s_{j,0}(t - t_{j,n}),$$

since

$$\phi^s_{j,k}(t_{j,n}) = \phi^s_{j,k}(\frac{n\pi}{2^j}) = \phi^s_{j,0}(\frac{n\pi}{2^j} - \frac{k\pi}{2^j}) = \phi^s_{j,0}(\frac{(n-k)\pi}{2^j}) = \phi^s_{j,0}(t_{j,n-k}),$$

so we have

$$\phi^s_{j,k}(t_{j,n}) = \phi^s_{j,0}(t_{j,n-k}). \tag{25}$$

Using (22) and (25) we get

$$B^s = \left(b^s_{k,n}\right) = \left(\phi^s_{j,k}(t_{j,n})\right)'' = \left(\phi^s_{j,0}(t_{j,n-k})\right)''. \tag{26}$$

To compute the entries of B^s if $n-k \neq 0$ then

$$t_{j,n-k} = \frac{(n-k)\pi}{2^j} = \frac{(n-k)}{2^{j+1}}.2\pi \notin 2\pi\mathrm{Z}, \qquad k,n = 0,1,...,2^{j+1}-1.$$

So for $n-k \neq 0$, from (9) we get

$$\left(\phi^0_{j,0}(t)\right)' = \frac{1}{2^{j+2}} \frac{\sin(2^{j+1}t)}{\sin^2(\frac{t}{2})} - \frac{1}{2^{2j+2}} \frac{\sin^2(2^j t)\cot(\frac{t}{2})}{\sin^2(\frac{t}{2})},$$

namely

$$2^{j+2}\sin^2(\frac{t}{2})\left(\phi^0_{j,0}(t)\right)' = \sin(2^{j+1}t) - \frac{1}{2^j}\sin^2(2^j t)\cot(\frac{t}{2}).$$

Taking the first derivative yields

$$2^{j+1}\sin(t)\left(\phi^0_{j,0}(t)\right)' + 2^{j+2}\sin^2(\frac{t}{2})\left(\phi^0_{j,0}(t)\right)'' =$$

$$2^{j+1}\cos(2^{j+1}t) - \sin(2^{j+1}t)\cot(\frac{t}{2}) + \frac{1}{2^j}\sin^2(2^j t)\left(1+\cot^2(\frac{t}{2})\right). \tag{27}$$

The result of an evaluation of (27) at the knots $t_{j,k-n}$ can be rewritten to produce

$$2^{j+2}\sin^2(\frac{t_{j,n-k}}{2})\left(\phi^0_{j,0}(t_{j,n-k})\right)'' = 2^{j+1}\cos(2^{j+1}t_{j,n-k}),$$

noting that the value of $\sin(kt_{j,k-n})$, when $k \geq 2^j$ are zero. Thus for $n-k \neq 0$

$$b^0_{k,n} = \left(\phi^0_{j,0}(t_{j,n-k})\right)'' = \frac{1}{2}\frac{\cos(2^{j+1}t_{j,n-k})}{\sin^2(\frac{t_{j,n-k}}{2})}. \tag{28}$$

When $n-k=0$, from (5) we get

$$\phi^0_{j,0}(t) = \frac{1}{2^{2j+1}}\sum_{p=0}^{2^{j+1}-1}\left(\frac{1}{2} + \sum_{k=1}^{p}\cos kt\right).$$

Taking the second derivative yields

$$\left(\phi^0_{j,0}(t)\right)'' = \frac{-1}{2^{2j+1}}\sum_{p=0}^{2^{j+1}-1}\sum_{k=1}^{p}k^2\cos kt. \tag{29}$$

So that the result of an evaluation of (29) at the knots $t_{j,n-k}$ can be rewritten to produce

$$\left(\phi^0_{j,0}(t_{j,n-k})\right)'' = \frac{-1}{2^{2j+1}}\sum_{p=0}^{2^{j+1}-1}\sum_{k=1}^{p}k^2\cos kt_{j,n-k}.$$

Using (13), when $n-k=0$, by replacing $p = 2^{j+1}-1$ we get

$$\sum_{k=0}^{p}k^2\cos kt_{j,n-k} = \frac{p(p+1)(2p+1)}{6}.$$

So

$$b^0_{k,n} = \left(\phi^0_{j,0}(t_{j,n-k})\right)'' = \frac{-1}{2^{2j+1}} \sum_{p=0}^{2^{j+1}-1} \frac{p(p+1)(2p+1)}{6}. \tag{30}$$

Now to find B^1, taking the second derivative of (6) we get

$$\left(\phi^1_{j,0}(t)\right)'' = \frac{-1}{2^{2j+1}} \left(\sum_{k=0}^{2^{j+1}-1} k^2 \sin kt + 2^{2j+1} \sin(2^{j+1}t) \right). \tag{31}$$

The result of an evaluation of (31) at the knots $t_{j,n-k}$ can be rewritten to produce

$$\left(\phi^1_{j,0}(t_{j,n-k})\right)'' = \frac{-1}{2^{2j+1}} \left(\sum_{k=0}^{2^{j+1}-1} k^2 \sin kt_{j,n-k} + 2^{2j+1} \sin(2^{j+1}t_{j,n-k}) \right).$$

Using (14) and after simplification we have

$$B^1 = \left(b^1_{k,n}\right) = \left(\phi^1_{j,0}(t_{j,n-k})\right)'' = \begin{cases} -\cot t_{j+1,n-k}, & k \neq n, \\ 0, & k = n, \end{cases} \tag{32}$$

noting that the value of $\sin(2^{j+1}t)$ in $t_{j,n-k}$ are zero.

So the matrix D can be found using (21), where A^0 is a $2^{j+1} \times 2^{j+1}$ zero matrix, A^1 is a $2^{j+1} \times 2^{j+1}$ identity matrix and matrix $B^0 = \left(b^0_{k,n}\right)$ as

$$B^0 = \left(b^0_{k,n}\right) = \begin{cases} \frac{1}{2} \frac{\cos((n-k)2\pi)}{\sin^2(\frac{(n-k)}{2^{j+1}}\pi)}, & k \neq n, \\ \frac{-1}{2^{2j+1}} \sum_{p=0}^{2^{j+1}-1} \frac{p(p+1)(2p+1)}{6}, & k = n, \end{cases}$$

and

$$B^1 = \left(b^1_{k,n}\right) = \begin{cases} -\cot \frac{(n-k)\pi}{2^{j+1}}, & k \neq n, \\ 0, & k = n, \end{cases}$$

for $k, n = 0, 1, ..., 2^{j+1} - 1$.

5. The second-order nonlinear differential equation

In this section we solve a second-order nonlinear Neumann problem of the form (1) by using trigonometric scaling functions. For this purpose, we use equation (15) to approximation $x(t)$ and $f(t, x(t))$ as

$$x(t) = C^T \Phi(t), \tag{33}$$

$$f(t, x(t)) = F^T \Phi(t), \tag{34}$$

where $\Phi(t)$ is defined in (17) and C and F are $2^{j+2} \times 1$ unknown vectors defined in (16). Using (19) and (33) gives

$$\dot{x}(t) = C^T \Phi'(t) = C^T D \Phi(t), \tag{35}$$

$$\ddot{x}(t) = C^T D^2 \Phi(t). \tag{36}$$

Using (1) ,(34) and (36) we have

$$-C^T D^2 \Phi(t) = F^T \Phi(t), \tag{37}$$

The entries of vector $\Phi(t)$ are independent, so using (37) we get

$$-C^T D^2 = F^T. \tag{38}$$

Equations (33) and (34) yield

$$f(t, C^T \Phi(t)) = F^T \Phi(t), \tag{39}$$

also using boundary values in (2) we have

$$C^T D \Phi(0) = \alpha, \tag{40}$$

$$C^T D \Phi(2\pi) = \beta. \tag{41}$$

To find the solution in (1) we first collocate (39) in $t_{j,n}$, $n = 1, 2, ..., 2^{j+1} - 2$. The resulting equation generates $2^{j+1} - 2$ algebraic equations. Also expression (38) gives 2^{j+1} algebraic equations. The total unknowns for vectors C in (33) and F in (34) are 2^{j+2}. These unknowns can be obtained by using expressions (38)-(41).

6. Numerical examples

Example 1. Consider the second-order nonlinear Neumann problem

$$-\ddot{x}(t) = e^{-\sin t}(\sin t - \cos^2 t)x^2(t), \qquad t \in [0, 2\pi],$$
$$\dot{x}(0) = \dot{x}(2\pi) = 1.$$

The exact solution of this problem is e^{sint} . Figure 1 shows the plot of error using the method proposed in this paper.

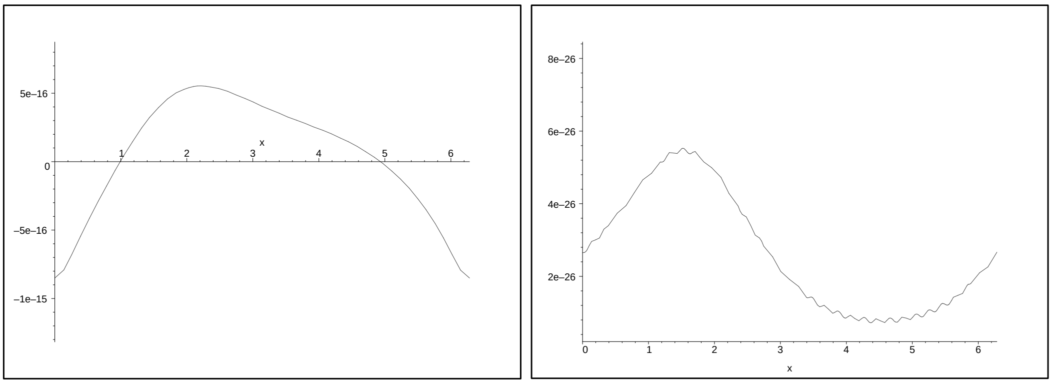

Figure 1: plot of error for J=3 (left) and J=4 (right)

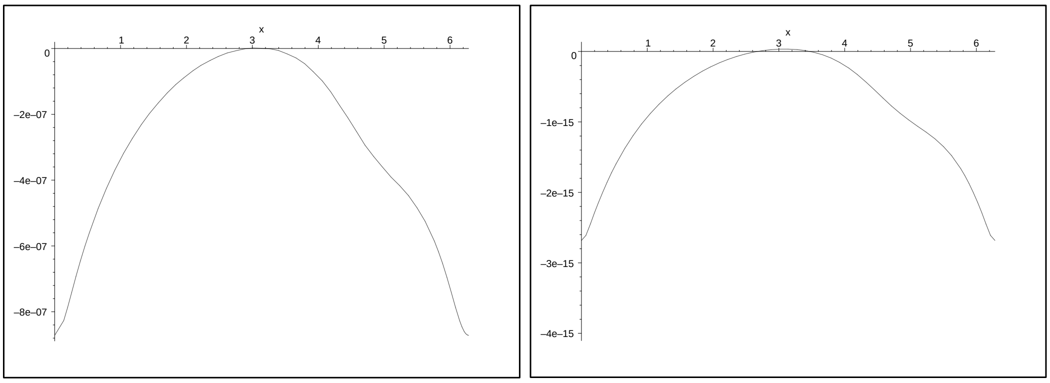

Figure 2: plot of error for J=3 (left) and J=4 (right)

Example 2. As the second test problem, consider the nonlinear Neumann problem

$$-\ddot{x}(t) = (\sin^2 t - 2\sin t - 2)x^3(t), \qquad t \in [0, 2\pi]\,,$$

$$\dot{x}(0) = \dot{x}(2\pi) = -\frac{1}{2}.$$

The exact solution of this problem is $1/(2 + \sin^2 t)$. Figure 2 shows the plot of error using the method proposed in this paper.

7. Main Results

In this paper a second-order differential equation with Neumann boundary conditions was investigated. The trigonometric scaling functions was proposed for solving this problem. The proposed technique was tested on some examples. The results obtained show the efficiency of the method presented.

References

[1] R. A. Khan, *Existence and approximation of solutions of second order nonlinear Neumann problems*, Electronic Journal of Differential Equations, 3, (2005), 1-10.

[2] M. Lakestani and M. Dehghan, *The solution of a second-order nonlinear differential equation with Neumann boundary conditions using semi-orthogonal B-spline wavelets*, Int. J. Computer Mathematics, 83, 8-9, (2006), 685-694.

[3] E. Quak,*Trigonometric wavelets for hermite interpolation*, J. Mathematic of computation, 65, (1996), 683-722.

[4] Z. Shan, Q. Du, *Trigonometric wavelet method for some elliptic boundary value problems*,J. Math. Anal. Appl., 344, (2008), 1105-1119.

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