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A NOTE ON TOTALLY UMBILICAL PSEUDO-SLANT SUBMANIFOLDS OF A NEARLY KAEHLER MANIFOLD

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Abstract. In this paper, we study pseudo-slant submanifolds of nearly Kaehler manifolds. A classification theorem on a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold is proved.

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1. Introduction

The study of the geometry of slant immersions was initiated by B.Y. Chen [2] as a natural generalization of both holomorphic and totally real immersions. Since then many researchers have studied slant immersions of almost Hermitian as well as almost contact manifolds. On the line of semi-slant submanifolds A. Carriazo defined and studied bi-slant submanifolds of almost Hermitian and almost contact metric manifolds and then he gave the notion of pseudo-slant submanifold [1]. Later on, V.A. Khan and M.A. Khan studied pseudo-slant submanifolds of a Sasakian manifold [4]. The purpose of the present paper is to study totally umbilical pseudo-slant submanifolds of a nearly Kaehler manifold. In this paper, we have obtained a classification theorem for a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold.

1. Preliminaries

Let $\bar{M}$ be an almost Hermitian manifold with almost complex structure J and a Riemannian metric g satisfying [7]:

$$(a)\quad J^2 = -I, \quad (b)\quad g(JX, JY) = g(X, Y)$$

for any $X, Y \in \Gamma(T\bar{M})$, where $\Gamma(T\bar{M})$ is the Lie algebra of the vector fields on $\bar{M}$. Further let $\bar{\nabla}$, the covariant differential operator on $\bar{M}$ with respect to g. If the almost complex structure J satisfies

$$(\bar{\nabla}_X J)Y + (\bar{\nabla}_Y J)X = 0 \tag{2.1}$$

for any $X, Y \in \Gamma(T\bar{M})$, then the manifold $\bar{M}$ is called a *nearly Kaehler manifold.* From the structure equation it is obvious that every Kaehler manifold is nearly Kaehler.

For an arbitrary submanifold M of a Riemannian manifold $\bar{M}$, the Gauss and Weingarten formulae are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{2.2}$$

and

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{2.3}$$

for all $X, Y \in \Gamma(TM)$, where ∇ is the induced Riemannian connection on M, V is a vector field normal to M, h is the second fundamental form of M, $\nabla^{\perp}$ is the normal connection in the normal bundle, $T^{\perp}M$ and A_V is the shape operator of the second fundamental form. Moreover,

$$g(A_V X, Y) = g(h(X, Y), V) \tag{2.4}$$

where g denotes the Riemannian metric on $\bar{M}$ as well as the metric induced on M. The mean curvature vector H of M is given by

$$H = \frac{1}{n}\sum_{i=1}^{n} h(e_i, e_i) \tag{2.5}$$

where n is the dimension of M and $\{e_1, e_2, \cdots, e_n\}$ is a local orthonormal frame of vector fields on M.

For any $X \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, the transformations JX and JV are decomposed into tangential and normal parts respectively as

$$JX = TX + NX \tag{2.6}$$

$$JV = tV + fV \tag{2.7}$$

A distribution D on a submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *slant distribution* if for each $U \in D_x$. The angle θ between JU and D_x is constant i.e., independent of $x \in M$ and $U \in D_x$. A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *slant* if the tangent bundle TM of M is slant.

A *pseudo-slant submanifold* M of a almost Hermitian manifold $\bar{M}$ is a submanifold which admits two orthogonal complementary distributions D_θ and $D^\perp$ such that D_θ is slant distribution with slant angle $\theta \in (0, \frac{\pi}{2})$ and $D^\perp$ is a totally real distribution i.e., $JD^\perp \subseteq T^\perp M$. In this case the tangent bundle TM is decomposed as

$$TM = D_\theta \oplus D^\perp.$$

Moreover, for a slant distribution D_θ, we have

$$T^2 X = -\cos^2\theta X \tag{2.8}$$

for any $X \in D_\theta$ [2]. Then from (2.8), we can calculate

$$g(TX, TY) = \cos^2\theta g(X, Y) \tag{2.9}$$

$$g(NX, NY) = \sin^2\theta g(X, Y) \tag{2.10}$$

for all $X, Y \in D_\theta$.

If μ is the invariant subspace of the normal bundle $T^\perp M$ then, in the case of pseudo-slant submanifold the normal bundle $T^\perp M$ can be decomposed as

$$T^\perp M = ND_\theta \oplus ND^\perp \oplus \mu. \tag{2.11}$$

Now, denoting by $\mathcal{P}_X Y$ and $\mathcal{Q}_X Y$ the tangential and normal parts of $(\bar{\nabla}_X J)Y$, for any $X, Y \in \Gamma(TM)$, then making use of (2.6), (2.7) and the Gauss-Weingarten formulae, the following equations may be obtained easily

$$\mathcal{P}_X Y = (\bar{\nabla}_X T)Y - A_{NY}X - th(X, Y) \tag{2.12}$$

$$\mathcal{Q}_X Y = (\bar{\nabla}_X N)Y + h(X, TY) - fh(X, Y) \tag{2.13}$$

Similarly, for any $V \in \Gamma(T^\perp M)$, denote the tangential and normal parts of $(\bar{\nabla}_X J)V$ by $P_X V$ and $Q_X V$ respectively, then we find that

$$\mathcal{P}_X V = (\bar{\nabla}_X t)V + TA_V X - A_{fV}X \tag{2.14}$$

$$\mathcal{Q}_X V = (\bar{\nabla}_X f)V + h(tV, X) + NA_V X \tag{2.15}$$

where the covariant derivatives of $T,\ N,\ t$ and f are respectively defined by

$$(\bar{\nabla}_X T)Y = \nabla_X TY - T\nabla_X Y \tag{2.16}$$

$$(\bar{\nabla}_X N)V = \nabla^\perp_X NY - N\nabla_X Y \tag{2.17}$$

$$(\bar{\nabla}_X t)V = \nabla_X tV - t\nabla^\perp_X V \tag{2.18}$$

$$(\bar{\nabla}_X f)V = \nabla^{\perp}_X fV - f\nabla^{\perp}_X V \tag{2.19}$$

for all $X,\ Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$.

A submanifold M of an almost Hermitian manifold $\bar{M}$ is said to be *totally umbilical* if the second fundamental form h satisfies

$$h(X,Y) = g(X,Y)H. \tag{2.20}$$

The submanifold M is *totally geodesic* if $h(X,Y) = 0$, for all $X, Y \in \Gamma(TM)$ and minimal if $H = 0$.

For the integrability of the distributions involved in the definition of pseudo-slant submanifolds we have the following preparatory results:

Proposition 2.1. *Let M be a pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$, then the slant distribution D_θ is integrable if and only if*

$$h(X,TY) - h(Y,TX) + \nabla^{\perp}_X NY - \nabla^{\perp}_Y NX - 2\mathcal{Q}_X Y \in \Gamma(ND_\theta).$$

for any $X, Y \in \Gamma(D_\theta)$.

Proof. For any $X, Y \in \Gamma(D_\theta)$ and $Z \in \Gamma(D^{\perp})$, then from (2.2), (2.3), (2.6) and the normal components of $(\bar{\nabla}_X J)Y$, we have

$$\begin{aligned} g(N[X,Y], NZ) &= g(h(X,TY) - h(Y,TX), NZ) \\ &\quad + g(\nabla^{\perp}_X NY - \nabla^{\perp}_Y NX - 2\mathcal{Q}_X Y, NZ). \end{aligned}$$

The result follows from the above equation and the fact that ND_θ and $ND^{\perp}$ are orthogonal.

Proposition 2.2. *Let M be a pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$, then the totally real distribution $D^{\perp}$ is integrable if and only if*

$$A_{JW}Z - A_{JZ}W + 2\mathcal{P}_Z W \in \Gamma(D^{\perp})$$

for all $Z, W \in \Gamma(D^{\perp})$.

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D_\theta)$, by (2.6), we have

$$g([Z,W], TX) = g(\bar{\nabla}_W JZ - (\bar{\nabla}_W J)Z - \bar{\nabla}_Z JW + (\bar{\nabla}_Z J)W, X)$$

Using (2.2) and (2.3), the above equation reduced to

$$g([Z,W], TX) = g(A_{JW}Z - A_{JZ}W + 2\mathcal{P}_Z W, X).$$

Thus the assertion follows from the above equation.

3. Totally umbilical pseudo-slant submanifolds

Through out the section we consider M as a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. We have the following results for later use.

Theorem 3.1. *Let M be a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then the following conditions are equivalent*

(i) The submanifold M has a nearly Kaehler structure (T, g)

(ii) $H \in \Gamma(\mu)$

where H is the mean curvature vector on M.

Proof. As M is a totally umbilical pseudo-slant, then for any $X \in \Gamma(TM)$, we have

$$h(X, TX) = g(X, TX)H = 0.$$

Using (2.2), we obtain

$$\bar{\nabla}_X TX - \nabla_X TX = 0.$$

Then from (2.6), we get

$$\bar{\nabla}_X JX - \bar{\nabla}_X NX = \nabla_X TX.$$

Thus by the covariant derivative property of J, we arrive at

$$(\bar{\nabla}_X J)X + J\bar{\nabla}_X X - \bar{\nabla}_X NX = \nabla_X TX.$$

Using (2.1), (2.2) and (2.3), we get

$$J(\nabla_X X + h(X, X)) + A_{NX}X - \nabla^{\perp}_X NX = \nabla_X TX.$$

Then from (2.6) and (2.7), we obtain

$$T\nabla_X X + N\nabla_X X + th(X, X) + fh(X, X) + A_{NX}X - \nabla^{\perp}_X NX = \nabla_X TX.$$

Equating the tangential components of the above equation, we get

$$T\nabla_X X + th(X, X) + A_{NX}X = \nabla_X TX.$$

As M is totally umbilical, the above equation takes the form

$$(\bar{\nabla}_X T)X = g(X, X)tH + g(H, NX)X.$$

The above equation has a solution if $H \in \Gamma(\mu)$, then $(\bar{\nabla}_X T)X = 0$ and vice-versa. This completes the proof of the theorem.

Note. *The above result is not only true for a totally pseudo-slant submanifold but also true for all totally umbilical submanifolds of a nearly Kaehler manifold.*

Theorem 3.2 [6]. *Let M be a totally umbilical proper slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then M is totally geodesic with nearly Kaehler structure (T, g).*

Now, we consider M as a pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$ and D_θ and $D^\perp$ are integrable distributions corresponding to the slant and totally real submanifolds of $\bar{M}$, respectively. Now, for any $U \in \Gamma(TM)$, we have $(\bar{\nabla}_U J)U = 0$, using this fact, we get

$$(\bar{\nabla}_Z J)Z = 0 \tag{3.1}$$

for any $Z \in \Gamma(D^\perp)$. Therefore the tangential and normal parts of the above equation are $\mathcal{P}_Z Z = 0$ and $\mathcal{Q}_Z Z = 0$, respectively. From (2.12) and the tangential component of (3.1), we obtain

$$\mathcal{P}_Z Z = 0 = (\bar{\nabla}_Z T)Z - A_{NZ} Z - th(Z, Z).$$

The above can be written as

$$(\bar{\nabla}_Z T)Z = A_{NZ} Z + th(Z, Z). \tag{3.2}$$

As M is totally umbilical and for any $Z \in \Gamma(D^\perp)$, $TZ = 0$, using these two facts and (2.16), the above equation takes the form

$$T\nabla_Z Z = -g(H, NZ)Z - \|Z\|^2 tH. \tag{3.3}$$

Taking the inner product in (3.3) with $W \in \Gamma(D^\perp)$, we obtain

$$g(H, NZ)g(Z, W) - \|Z\|^2 g(tH, W) = 0. \tag{3.4}$$

Thus the equation (3.4) has a solution if one of the following holds:

(a) $dim D^\perp = 1$

(b) $H \in \Gamma(\mu)$.

Now, we are in the position to prove our main result.

Theorem 3.3. *Let M be a totally umbilical pseudo-slant submanifold of a nearly Kaehler manifold $\bar{M}$. Then at least one of the following statements is true*

(i) M has a nearly Kaehler structure (T, g),

(ii) M is totally geodesic submanifold with nearly Kaehler structure (T, g),

(iii) $dimD^{\perp} = 1$.

Proof. We consider two cases: either $H \in \Gamma(\mu)$ or $H \notin \Gamma(\mu)$. If $H \in \Gamma(\mu)$, then by Theorem 3.1, there is a nearly Kaehler structure (T, g) on M, this is the part (i) of the theorem. Moreover, if $H \in \Gamma(\mu)$ and M has a nearly Kaehler structure then by Theorem 3.2, M is totally geodesic, which proves part (ii). Finally, if $H \notin \Gamma(\mu)$, then the equation (3.4) has a solution if $dimD^{\perp} = 1$, which is part (iii). This proves the theorem completely.

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