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SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS INVOLVING GENERALIZED INTEGRAL OPERATOR

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Abstract. By making use of the generalized integral operator, we introduce and study subordination and superordination results for normalized analytic functions in the open unit disk. Relevant connections of the results are presented in this paper, with various other known results also pointed out.

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1. Introduction and Preliminaries

Let A denote the class of analytic functions of the form $f(z) = z+a_2z^2+a_3z^3+...$ or $f(z) = z+\sum_2^\infty a_nz^n$ in the open unit disk U normalized by $f(0) = f'(0)-1 = 0$. Let the functions f and g be analytic in U,then f is called subordinate to g and is denoted by $f(z) \prec g(z)$ or simply $f \prec g$ if there exist a Schwarz function w analytic in U such that $f(z) = g(w(z))$, $z \in U$.

Let $\phi : C^3 \times U \to C$ and let h analytic in U. Assume that p,ϕ are analytic and univalent in U and p satisfies the differential superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z). \quad (1)$$

An analytic function q is called a subordinant if $q \prec p$,for all p satisfying equation (1). A univalent function q such that $p \prec q$ for all subordinants p of equaction (1) is said to be the best subordinant.

Recently Miller and Mocanu [12] obtained conditions on h, q and ϕ for which the following implication holds

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q)(z) \prec p(z).$$

With the results of Miller and Mocanu [12], Bulboacă [18] investigated certain classes of first order differential superordinations as well as superordination-preserving integral operators [19]. Ali et al.[1] used the results obtained by Bulboacă [19] and gave the sufficient conditions for certain normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are given univalent functionsin U with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. obtained sufficient conditions for a normalized analytic functions f to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z)$$

and

$$q_1(z) \prec \frac{z^2 f'(z)}{(f(z))^2} \prec q_2(z)$$

where q_1 and q_2 are given univalent function in U with $q_1(0) = 1$ and $q_2(0) = 1$.

Let $f \in A$. Denote by $D^\lambda : A \longrightarrow A$ the operator defined by

$$D^\lambda = z/(1-z)^{\lambda+1} * f(z), \quad (\lambda > -1).$$

It is obvious that,

$$D^0 f(z) = f(z), \quad D^1 f(z) = zf'(z)$$

and,

$$D^\delta f(z) = z(z^{\delta-1} f(z))^\delta / \delta! \quad \delta \in N \setminus \{0\}.$$

The operator $D^\delta f$ is called the δth-order Ruscheweyh derivative of f.

Recently, K. I. Noor [2] and K. I. Noor and M. A. Noor [3] defined and studied an integral operator $I_n : \mathcal{A} \longrightarrow \mathcal{A}$, analogous to $D^\delta f$ as follows.

Let $f_n = z/(1-z)^{n+1}, (n \in \mathcal{N}_0$ and $f_n^{-1}(z)$ be defined such that

$$f_n(z) * f_n^{-1}(z) = z/(1-z)^2$$

Then,

$$f_n(z) = f_n^{-1}(z) * f(z) = (z/(1-z)^{n+1})^{-1} * f(z).$$

We note that $I_0 f(z) = f(z), I_1 f(z) = zf'(z)$. The operator I_n is called the Noor integral of nth order of f (see [4, 5]), which is an important tool in defining several classes of analytic functions. In recent years, it has been shown that Noor integral operator has fundamental and significant applications in the geometric function theory.

For real or complex numbers a, b, c other than $0, -1, -2, ...$, the hypergeometric series is defined by

$$_2F_1(a, b; c; z) = \sum_0^\infty \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k. \tag{2}$$

Where $(x)_n$ is the pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(n+x)}{\Gamma(x)} = \begin{cases} 1 & \text{if } n = 0 \\ x(x+1)(x+2)\cdots(x+n-1) & \text{if } n \in N \end{cases}$$

We note that the series (2) converges absolutely for all $z \in U$ so that it represents an analytic function in U. Also an incomplete beta function $\phi(a, c; z)$ is related to Gauss hypergeometric function $z_2F_1(a, b; c; z)$ as,

$$\phi(a, c; z) = z_2F_1(1, a; c; z)$$

and we note that $\phi(a, 1; z) = z/(1-z)^a$, where $\phi(a, 1; z)$ is Koebe function. Using $\phi(a, c; z)$, a convolution operator [6], was defined by Carlson and Shaferr. Furthermore, Hohlov [7] introduced a convolution operator using ${}_2F_1(a, b; c; z)$.

N. Shukla and P. Shukla [8] studied the mapping properties of a function f_μ to be as given in

$$f_\mu(a, b, c, z) = (1-\mu)z_2F_1(a, b; c; z) + \mu z(z_2F_1(a, b; c; z))',$$

and investigated the geometric properties of an integral operator of the form

$$I(z) = \int_0^z \frac{f_\mu(t)}{t} dt.$$

Kim and Shon [9] considered linear operator $L_\mu : \mathcal{A} \longrightarrow \mathcal{A}$ defined by

$$L_\mu(a, b, c)f(z) = f_\mu(a, b, c)(z) * f(z).$$

We now introduce a function $(f_\mu)^{(-1)}$ given by

$$f_\mu(a, b, c)(z) * (f_\mu(a, b, c)(z))^{-1} = z/(1-z)^{\lambda+1}, \quad (\mu \geq 0, \lambda > -1),$$

and Al-Shaqsi and M. Darus obtain the following generalized linear operator:

$$I_\mu^\lambda(a, b, c)f(z) = (f_\mu(a, b, c)(z))^{-1} * f(z).$$

The operator I_μ^λ is known as the generalized integral operator. Therefore, the function $(f_\mu)^{-1}$ has the following form

$$(f_\mu(a, b, c)(z))^{-1} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k+1)(a)_k (b)_k} z^{k+1} \quad z \in U.$$

Therefore,

$$I_\mu^\lambda(a,b,c)f(z) = z + \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k+1)(a)_k (b)_k} a_{k+1} z^{k+1}.$$

Also it can easily be verified that

$$z(I_\mu^\lambda(a,b,c)f(z))' = (\lambda+1)I_\mu^{\lambda+1}(a,b,c)f(z) - \lambda(I_\mu^\lambda(a,b,c)f(z)),$$

$$z(I_\mu^\lambda(a+1,b,c)f(z))' = aI_\mu^\lambda(a,b,c)f(z) - (a-1)I_\mu^\lambda(a+1,b,c)f(z).$$

Definition 1. *Let $f \in A$ belongs to the family of functions S^* (starlike) if and only if*

$$\Re\left(\frac{z(I_\mu^\lambda f(z))'}{I_\mu^\lambda f(z)}\right) > o, \quad z \in U,\, n \in N_0.$$

Definition 2. *Let $f \in A$ belongs to the family of functions $V_n^\star, n \in N_0$ if and only if $(I_\mu^\lambda f(z)) \in S^*$, $z \in U$.*

Definition 3. *Let $f \in A$ belongs to the family of functions $G_n^\star, n \in N_0$ if and only if there exists $g \in V_n^\star$ such that*

$$\Re\left(\frac{z(I_\mu^\lambda f(z))'}{g_\mu^\lambda f(z)}\right) > o, z \in U.$$

In the present work, we apply a method based on the differential subordination in order to obtain subordination results involving Noor Integral operator for a normalized analytic function f

$$q_1(z) \prec \left(\frac{z(I_\mu^\lambda f(z))'}{I_\mu^\lambda f(z)}\right) \prec q_2(z)$$

and

$$q_1(z) \prec \left(\frac{z(I_\mu^\lambda f(z))'}{g_\mu^\lambda f(z)}\right) \prec q_2(z).$$

Note also similar work has been seen for different subclasses done by other authors (see for example [13-16]). In order to prove our subordination and superordination results, we need to the following lemmas in the sequel.

Lemma 1. *[10] Let $q(z)$ be univalent in the open unit disk U and θ, ϕ be analytic in a domain D containing $q(U)$ with $\phi(w) \neq 0$ when $w \in q(U)$,*

$$Q(z) = zq(z)\phi(q'(z)) \quad h(z) = \theta(q(z)) + Q(z),$$

suppose that $Q(z)$ is starlike univalent in U and $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$, $z \in U$, if,

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)),$$

then,

$$p(z) \prec q(z),$$

and q is the best dominant.

Lemma 2. [11] Let $q(z)$ be convex univalent in the open unit disk U and ϑ, φ be analytic in a domain D containing $q(U)$. Suppose that,

.$zq'(z)\varphi(q(z))$ is starlike univalent in U and

.$\Re\left(\frac{\vartheta' q(z)}{\vartheta q(z)}\right) > 0, \quad z \in U,$

if $p(z) \in \mathcal{H}[q(0), 1] \bigcap Q$, with $p(U) \subseteq D$ and $\vartheta(p(z)) + zp'(z)\varphi p(z)$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\varphi q(z) \prec \vartheta(p(z)) + zp'(z)\varphi p(z),$$

then,

$$q(z) \prec p(z),$$

and $q(z)$ is the best subordinant.

Definition 4. [12] Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U} - E(f)$ where $E(f) = \zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U - E(f)$.

Lemma 3. [17] Let $q(z)$ be convex univalent in the unit disk U and ψ and $\gamma \in$ C with

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} + \frac{\psi}{\gamma}\right) > 0.$$

If; $p(z)$ is analytic in U and

$$\psi p(z) + \gamma zp'(z) \prec \psi q(z) + \gamma zq'(z),$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 4. [12] Let q be convex univalent in the unit disk U and $\gamma \in$ c. Further, assume that $\Re(\overline{\gamma} > 0)$. If $p(z) \in \mathrm{H}[q(0), 1] \cap Q$ with $p(z) + \gamma z p'(z)$ is univalent in U then

$$q(z) + zq'(z) \prec p(z) + zp'(z)$$

Implies $q(z) \prec p(z)$ and q is the best subordinant.

2.Main results about sandwich theorems

By using Lemmas 1 and Lemma 2, we prove subordination and superordination results for analytic functions as follows.

Theorem 1. Let $q(z) \neq 0$ be univalent in U such that $zq'(z)/q(z)$ is starlike univalent in U and

$$\Re\left(1 + \frac{\alpha}{\gamma} q(z) + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > 0, \quad \alpha, \gamma \in \mathcal{C}, \quad \gamma \neq 0. \tag{3}$$

If $f \in A$ satisfies the following subordination

$$\alpha \left(\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right) + \gamma\left(1 + \frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'} - \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right) \prec \alpha q(z) + \gamma z q'(z)/q(z),$$

then,

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)} \prec q(z),$$

and $q(z)$ is the best dominant.

*Proof.*Let,

$$p(z) = \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)},$$

then after computation, we have

$$zp'(z)/p(z) = 1 + \frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'} - \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)},$$

which yields the following subordination

$$\alpha p(z) + \gamma z p'(z)/p(z) \prec \alpha q(z) + \gamma z q'(z)/q(z), \quad \alpha, \gamma \in \mathcal{C}.$$

By setting,

$$\theta(\omega) = \alpha\omega \quad \phi(\omega) = \gamma/\omega, \quad \gamma \neq 0,$$

it can be easily observed that $\theta(\omega)$ is analytic in C and $\phi(\omega)$ is analytic in $C-\{0\}$ and that $\phi(\omega)\neq 0$ when $\omega\in C-\{0\}$. Also, by letting

$$Q(z)=zq'(z)\phi(q(z))=\gamma zq'(z)/q(z),$$

and,

$$h(z)=\theta(q(z))+Q(z)=\alpha q(z)+\gamma zq'(z)/q(z),$$

we find that $Q(z)$ is starlike univalent in U and that

$$\Re\left(\frac{zh'(z)}{Q(z)}\right)=\Re\left(1+\frac{\alpha}{\gamma}q(z)+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}\right)>0.$$

So by Lemma 1.,we have $\frac{z(I_\mu^\lambda f(z))'}{I_\mu^\lambda f(z)}\prec q(z)$.

In case $\Phi(\omega)=\omega$ in theorem 1, then we get the following result:

Corollary 1. *If $f\in\mathcal{A}$ and assume that (3) holds then*

$$1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}\prec\frac{1+Az}{1+Bz}+\frac{(A-B)z}{(1+Az)(1+Bz)},$$

implies,

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\prec\frac{1+Az}{1+Bz},\quad -1\leq B\leq A\leq 1,$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Proof. By setting $\alpha=\gamma=1$ and $q(z)=1+Az/1+Bz$ where $-1\leq B\leq A\leq 1$.

Corollary 2. *If $f\in\mathcal{A}$ and assume that (3) holds then*

$$1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}\prec\frac{1+z}{1-z}+\frac{2z}{(1+z)(1-z)},$$

implies,

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\prec\frac{1+z}{1-z},$$

and $\frac{1+z}{1-z}$ is the best dominant.

Proof. By setting $\alpha=\gamma=1$ and $q(z)=1+z/1-z$

Corollary 3. *If $f \in \mathcal{A}$ and assume that (3) holds then*

$$1+\frac{z\left(I_{\mu}^{\lambda}f(z)\right)''}{\left(I_{\mu}^{\lambda}f(z)\right)'} \prec e^{Az}+Az,$$

implies,

$$\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)} \prec e^{Az},$$

and e^{Az} is the best dominant.

Proof. By setting $\alpha=\gamma=1$ and $q(z)=e^{Az}$, $|A|<\Pi$.

Theorem 2. *Let $q(z)\neq 0$ be convex univalent in the unit disk U. Suppose that*

$$\Re\left(\frac{\alpha}{\gamma}q(z)\right)>0, \quad \alpha,\gamma\in C\, for\, z\in U, \tag{4}$$

and $zq'(z)/q(z)$ is starlike univalent in U. if,

$$\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)} \in \mathcal{H}[q(0),1]\bigcap Q, \quad f\in A,$$

$$\alpha\left(\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)}\right)+\gamma\left(1+\frac{z\left(I_{\mu}^{\lambda}f(z)\right)''}{\left(I_{\mu}^{\lambda}f(z)\right)'}-\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)}\right),$$

is univalent is U and the subordination

$$q(z)+\gamma zq'(z)/q(z) \prec \alpha\left(\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)}\right)+\gamma\left(1+\frac{z\left(I_{\mu}^{\lambda}f(z)\right)''}{\left(I_{\mu}^{\lambda}f(z)\right)'}-\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)}\right),$$

holds, then

$$q(z) \prec \frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)},$$

and q is the best subordinant.

Proof. Let

$$p(z)=\frac{z\left(I_{\mu}^{\lambda}f(z)\right)'}{I_{\mu}^{\lambda}f(z)},$$

then after doing some calculation, we get

$$zp'(z)/p(z) = 1 + \frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'} - \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)},$$

this implies that,

$$\alpha q(z) + \gamma zq'(z)/q(z) \prec \alpha p(z) + \gamma zp'(z)/p(z), \quad \alpha, \gamma \in \mathcal{C},$$

By setting

$$\vartheta(\omega) = \alpha\omega \quad \varphi(\omega) = \gamma/\omega, \quad \gamma \neq 0.$$

It can be easily observed that $\vartheta(\omega)$ is analytic in C and $\varphi(\omega)$ is analytic in $C - \{0\}$ and that $\varphi(\omega) \neq 0$ when $\omega \in C - \{0\}$. Also, we obtain

$$\Re\left(\frac{\vartheta'(q(z))}{\varphi(q(z))}\right) = \Re\left(\frac{\alpha}{\gamma}q(z)\right) > 0.$$

So by using Lemma 2.,we have

$$q(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}.$$

Theorem 3. *Let $q_1(z) \neq 0, q_2(z) \neq 0$ be convex univalent in the unit disk U satisfying* (3) *and* (4) *respectively. Suppose that and $zq_1'(z)/q_1(z)$, $zq_2'(z)/q_2(z)$ is starlike univalent in U. If,*

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)} \in \mathcal{H}[q(0), 1] \bigcap Q, \quad f \in A,$$

$$\alpha\left(\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right) + \gamma\left(1 + \frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'} - \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right),$$

is univalent is U and the subordination

$$q_1(z) + \gamma\frac{zq_1'(z)}{q_1(z)} \prec \alpha\left(\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right) + \gamma\left(1 + \frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'} - \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)}\right)$$

$$\prec q_2(z) + \gamma\frac{zq_2'(z)}{q_2(z)},$$

holds, then

$$q_1(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)} \prec q_2(z),$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Proof. To prove the result, we use Theorem 1 and Theorem 2 simultaneously.

Theorem 4. *Let $q(z)$ be convex univalent in the unit disk U and $\gamma \in \mathrm{C}$ satisfying that*

$$\Re\left(1+\frac{zq''(z)}{q'(z)}+\frac{1}{\gamma}\right)>0,\ \gamma \in \mathrm{C}. \tag{5}$$

If $f \in G_n^\star$ for $n \in N_0$ and exists $g \in V_n^\star$ such that $\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}$ is analytic in U and the subordination

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right)\right) \prec q(z)+\gamma z q'(z),\ \gamma \in \mathrm{C}$$

holds, then

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)} \prec q(z),$$

and q is the best dominant.

Proof. Let

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}.$$

Therefore,

$$zp'(z)=\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right),$$

which yields the following subordination

$$p(z)+\gamma z p'(z) \prec q(z)+\gamma z q'(z),\ \gamma \in \mathrm{C}$$

This implies that

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)} \prec q(z).$$

Theorem 5. Let be convex univalent in the unit disk U and $\gamma \in \mathrm{C}$. Further, assume that $\Re(\overline{\gamma}) > 0$. If

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)} \in \mathrm{H}[q(0),1] \cap Q,$$

with

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right)\right),$$

is univalent in U then

$$q(z)+\gamma z q'(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right)\right),$$

this implies that

$$q(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)},$$

and q is the best subordinant.

Proof. Let

$$p(z)=\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}.$$

Hence by using the same method as above, we get

$$zp'(z)=\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right),$$

which yields the following subordination

$$q(z)+\gamma z q'(z) \prec p(z)+\gamma z p'(z),\ \gamma \in \mathrm{C}.$$

Thus in view of Lemma 4, we have

$$q(z) \prec p(z) \Rightarrow q(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}.$$

Theorem 6. Let $q_1(z), q_2(z)$ be convex univalent in the unit disk U such that

$$\Re\left(1+\frac{zq_2''(z)}{q_1'(z)}+\frac{1}{\gamma}\right)>0,\ \gamma \in \mathrm{C},\ \Re\left(\overline{\gamma}\right)>0.$$

If $f \in G_n^\star$ for $n \in N_0$ and exists $g \in V_n^\star$ such that $\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)} \in \mathrm{H}[q_1(0), 1] \cap Q$, with

$$\frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right)\right),$$

is univalent in U then

$$q(z)+\gamma z q'(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda g(z)}\left(1+\left(1+\frac{z\left(I_\mu^\lambda f(z)\right)''}{\left(I_\mu^\lambda f(z)\right)'}-\frac{z\left(I_\mu^\lambda g(z)\right)'}{I_\mu^\lambda g(z)}\right)\right) \prec p(z)+\gamma z p'(z)$$

holds, then

$$q_1(z) \prec \frac{z\left(I_\mu^\lambda f(z)\right)'}{I_\mu^\lambda f(z)} \prec q_2(z),$$

and $q_1(z)$ is the best subordinant and $q_2(z)$ is the best dominant.

Proof. To prove the result, we use Theorem 4 and Theorem 5 simultaneously.

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