ON THE DECAY OF SOLUTIONS TO AN INITIAL BOUNDARY VALUE PROBLEM FOR A CLASS OF DAMPED NONLINEAR WAVE EQUATIONS

Erhan Pİşkİn and Necat POLAT

ABSTRACT. In this paper, we consider a class of strongly damped multidimensional nonlinear wave equations in a bounded domain. We show that we can always find initial data in the stable set for which the solution of the problem decays exponentially. The key tool in the proof is an idea of Zuazua [6], which is based on the construction of a suitable Lyapunov function.

2000 *Mathematics Subject Classification*: 35L05, 35G31.

1. INTRODUCTION

In this paper, we consider the following problem

$$u_{tt} - \triangle u + \triangle^2 u - \alpha \triangle u_t = f(u), \; x \in \Omega, \; t > 0, \tag{1}$$

$$u(x,0) = u_0(x), \; u_t(x,0) = u_1(x), \; x \in \Omega, \tag{2}$$

$$u(x,t) = \triangle u(x,t) = 0, \; x \in \partial\Omega, \; t \geq 0, \tag{3}$$

where $\Omega \subset R^n$ is a bounded domain with smooth boundary. $f(u)$ is the given nonlinear function, $u_0(x)$ and $u_1(x)$ are the given initial value functions, $\alpha > 0$ is a constant, the subscript t indicates the partial derivative with respect to t, n is the dimension of space variable x, and $\triangle$ denotes the Laplace operator in R^n.

The present problem (1)-(3) has been studied by Lin et al. [5]. In their work, the authors proved the existence of global weak solutions and global strong solutions by means of the potential well method.

The purpose of this paper is to obtain decay estimate of solutions to problem (1)-(3). More precisely we show that we can always find initial data in the stable set for which the solution of problem (1)-(3) decays exponentially, motivated by [2, 4, 6].

Throughout this paper, we will consider the standard spaces $L^p(\Omega)$ with norm $\|u\|_p$, and we denote by $(.,.)$ and $\|.\|$ the inner product and norm on $L^2(\Omega)$.

Next, we state the local existence theorem.

Theorem 1.1 [5]. Suppose that $p > 1$, such that

$$p \leq \frac{n+2}{n-2}, \ n \geq 3$$

and let $(u_0, u_1) \in \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times L^2(\Omega)$ be given. Then problem (1)-(3) has a unique solution

$$
\begin{aligned}
u &\in L^\infty\left(0, T; H^2(\Omega) \cap H_0^1(\Omega)\right) & (4)\\
u_t &\in L^\infty\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right) & (5)
\end{aligned}
$$

for some T small.

Lemma 1.1 (Sobolev-Poincare inequality) [1]. Let q be a number with $2 \leq q < \infty$ $(n = 1, 2)$ or $2 \leq q \leq \frac{2n}{n-2}$ $(n \geq 3)$, then there is a constant $C_* = C_*(\Omega, \ q)$ such that

$$\|u\|_q \leq C_* \|\nabla u\| \ \text{for} \ u \in H_0^1(\Omega). \tag{6}$$

Lemma 1.2 (Young's inequality with ϵ) [3]. Assume that $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, then for $a, b > 0$, $\epsilon > 0$, we have

$$ab \leq \epsilon a^p + C(\epsilon) b^q, \tag{7}$$

where $C(\epsilon) = (\epsilon p)^{-\frac{q}{p}} q^{-1}$.

This paper is organized as follows. In section 2, the global existence of the solution is given. In section 3, we show the exponential decay of the solution the problem (1)-(3).

2. Global existence

In this section, we discuss the global existence of the solution for problem (1)-(3). In order to state and prove our main result we first introduce the following [5].

We assume that $f(u) \in C$, $uf(u) \geq 0$ and

$$|f(u)| < a |u|^p, \tag{8}$$

where $1 < p < \infty$ if $n = 1, 2$ and $1 < p \leq \frac{n+2}{n-2}$ if $n \geq 3$.

For the problem (1)-(3), we define

$$J(t) = J(u(t)) = \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\triangle u\|^2 - \frac{a}{p+1} \|u\|_{p+1}^{p+1}, \tag{9}$$

$$I(t) = I(u(t)) = \|\nabla u\|^2 + \|\triangle u\|^2 - a \|u\|_{p+1}^{p+1}, \tag{10}$$

and

$$E(t) = E(u(t), u_t(t)) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|\nabla u\|^2 + \frac{1}{2} \|\triangle u\|^2 - \int_\Omega F(u)\,dx, \tag{11}$$

where $F(u) = \int_0^u f(s)\,ds.$

We also define

$$W = \left\{ w \in H^2(\Omega) \cap H_0^1(\Omega) : I(w) > 0 \right\} \cup \{0\}, \tag{12}$$

where we are using $w(t)$ instead of $w(.,t).$

Remark 2.1. By multiplying the equation (1) by u_t, integrating over Ω, and using integrating by parts, we get

$$E'(t) = -\alpha \|\nabla u_t\|^2 \leq 0, \tag{13}$$

for almost each t in $[0, T).$

Lemma 2.1. Suppose that

$$\begin{cases} 1 < p < \infty, \ n = 1, 2, \\ 1 < p \leq \frac{n+2}{n-2}, \ \ n \geq 3 \end{cases} \tag{14}$$

holds. If $u_0 \in W$ and $u_1 \in L^2(\Omega)$ such that

$$\beta = aC_*^{p+1} \left(\frac{2(p+1)}{p-1} E(u_0, u_1) \right)^{\frac{p-1}{2}} < 1 \tag{15}$$

then $u(t) \in W$, for each $t \in [0, T).$

Proof. Since $I(u_0) > 0$, it follows the continuity of $u(t)$ that

$$I(t) > 0,$$

for some interval near $t = 0$. Let $T_m > 0$ be a maximal time, when (10) holds on $[0, T_m].$

From (9) and (10), we have

$$
\begin{aligned}
J(t) &= \frac{1}{2}\|\nabla u\|^2 + \frac{1}{2}\|\triangle u\|^2 - \frac{a}{p+1}\|u\|_{p+1}^{p+1} \\
&= \frac{p-1}{2(p+1)}\left(\|\nabla u\|^2 + \|\triangle u\|^2\right) + \frac{1}{p+1}I(t) \\
&\geq \frac{p-1}{2(p+1)}\|\nabla u\|^2, \ \forall t \in [0, T_m).
\end{aligned}
\tag{16}
$$

Hence, we have

$$
\|\nabla u\|^2 \leq \frac{2(p+1)}{p-1}J(t).
$$

From (9) and (11), we have $\forall t \in [0, T_m)$, $J(t) \leq E(t)$. Thus we obtain;

$$
\begin{aligned}
\|\nabla u\|^2 &\leq \frac{2(p+1)}{p-1}E(t) \\
&\leq \frac{2(p+1)}{p-1}E(u_0, u_1), \ \forall t \in [0, T_m).
\end{aligned}
\tag{17}
$$

By exploiting (6), (15) and (17), we easily arrive at

$$
\begin{aligned}
a\|u(t)\|_{p+1}^{p+1} &\leq aC_*^{p+1}\|\nabla u(t)\|^{p+1} = aC_*^{p+1}\|\nabla u(t)\|^{p-1}\|\nabla u(t)\|^2 \\
&\leq aC_*^{p+1}\left(\frac{2(p+1)}{p-1}E(u_0, u_1)\right)^{\frac{p-1}{2}}\|\nabla u(t)\|^2 \\
&= \beta\|\nabla u(t)\|^2 \\
&< \|\nabla u(t)\|^2, \ \forall t \in [0, T_m);
\end{aligned}
\tag{18}
$$

hence $\|\nabla u(t)\|^2 - a\|u(t)\|_{p+1}^{p+1} > 0 \implies \|\nabla u(t)\|^2 + \|\triangle u(t)\|^2 - a\|u(t)\|_{p+1}^{p+1} > 0$, $\forall t \in [0, T_m)$. This shows that $u(t) \in W$, $\forall t \in [0, T_m)$. By noting that

$$
aC_*^{p+1}\left(\frac{2(p+1)}{p-1}E(u_0, u_1)\right)^{\frac{p-1}{2}} < 1
$$

we easily repeat the steps (16)-(18) to extend T_m to T_{2m}. We continue this procedure until $u(t) \in W$, $\forall t \in [0, T)$.

Theorem 2.1. Suppose that (14) holds. If $u_0 \in W$ and $u_1 \in L^2(\Omega)$ satisfying (18). Then the solution is global.

Proof. It is sufficient to show that $\|u_t\|^2 + \|\nabla u\|^2 + \|\triangle u\|^2$ is bounded independently of t. To achieve this we use (12) and (13), so

$$
\begin{aligned}
E\left(u_0, u_1\right) \quad \geq \quad & E\left(t\right) = \frac{1}{2}\left\|u_t\right\|^2 + \frac{1}{2}\left\|\nabla u\right\|^2 + \frac{1}{2}\left\|\triangle u\right\|^2 - \int\limits_\Omega F\left(u\right) dx \\
\geq \quad & \frac{1}{2}\left\|u_t\right\|^2 + \frac{1}{2}\left\|\nabla u\right\|^2 + \frac{1}{2}\left\|\triangle u\right\|^2 - \frac{a}{p+1}\left\|u\right\|_{p+1}^{p+1} \\
= \quad & \frac{1}{p+1}I\left(u\right) + \frac{1}{2}\left\|u_t\right\|^2 + \frac{p-1}{2\left(p+1\right)}\left(\left\|\nabla u\right\|^2 + \left\|\triangle u\right\|^2\right) \\
\geq \quad & \frac{1}{2}\left\|u_t\right\|^2 + \frac{p-1}{2\left(p+1\right)}\left(\left\|\nabla u\right\|^2 + \left\|\triangle u\right\|^2\right)
\end{aligned}
$$

since $I\left(u\right) \geq 0$. Therefore

$$
\|u_t\|^2 + \|\nabla u\|^2 + \|\triangle u\|^2 \leq CE\left(u_0, u_1\right)
$$

for $C = \max\left\{2, \frac{2(p+1)}{p-1}\right\}$. This completes the proof.

3. EXPONENTIAL DECAY

In this section we consider the energy decay of the solution to (1)-(3).

Theorem 3.1. Suppose that (14) and (15) hold. Let $u_0 \in W$ and $u_1 \in L^2\left(\Omega\right)$. Then there exist positive constants K and k such that the global solution of (1)-(3) satisfies

$$
E\left(t\right) \leq Ke^{-kt}, \ \forall t \geq 0. \tag{19}
$$

Proof. We define

$$
F\left(t\right) = E\left(t\right) + \varepsilon \int\limits_\Omega \left(uu_t + \frac{1}{2}u^2\right) dx, \tag{20}
$$

for $\varepsilon > 0$, to be chosen later. It is straightforward to see that $F\left(t\right)$ and $E\left(t\right)$ are equivalent in the sense that there exist two positive constants α_1, $\alpha_2 > 0$ depending on ε such that for $t \geq 0$

$$
\alpha_1 F\left(t\right) \leq E\left(t\right) \leq \alpha_2 F\left(t\right). \tag{21}
$$

By taking the time derivative of the function $F\left(t\right)$ defined above in equation (20),

using equation (1), and performing several integration by parts, we get:

$$
\begin{aligned}
F'(t) &= E'(t) + \varepsilon \int_{\Omega} \left(u_t^2 + u u_{tt} + u u_t \right) dx \\
&= -\alpha \|\nabla u_t\|^2 + \varepsilon \int_{\Omega} \left(u_t^2 + \left(u, \triangle u - \triangle^2 u + \alpha \triangle u_t + f(u) \right) + u u_t \right) dx \\
&= -\alpha \|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 - \varepsilon \|\nabla u\|^2 - \varepsilon \|\triangle u\|^2 \\
&\quad + \varepsilon \int_{\Omega} u f(u) \, dx - \varepsilon \alpha \int_{\Omega} \nabla u \nabla u_t dx + \varepsilon \int_{\Omega} u u_t dx.
\end{aligned} \tag{22}
$$

Now, we estimate the fifth, sixth terms and last term in the right hand side of (22) as follows.

$$
\int_{\Omega} u f(u) \, dx < a \|u\|_{p+1}^{p+1} \le \beta \|\nabla u\|^2, \tag{23}
$$

where, used (8) and (18). By using Young's inequality (7), we obtain, for any $\delta > 0$

$$
\int_{\Omega} u u_t dx \le \frac{1}{4\delta} \|u_t\|^2 + \delta \|u\|^2, \tag{24}
$$

$$
-\varepsilon \alpha \int_{\Omega} \nabla u \nabla u_t dx \le \varepsilon \alpha \left(\frac{1}{4\delta} \|\nabla u_t\|^2 + \delta \|\nabla u\|^2 \right). \tag{25}
$$

Therefore a combination of (22)-(25) gives

$$
\begin{aligned}
F'(t) &< -\alpha \left(1 - \frac{\varepsilon}{4\delta} \right) \|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 + \varepsilon (\alpha \delta - 1) \|\nabla u\|^2 \\
&\quad -\varepsilon \|\triangle u\|^2 + \varepsilon \beta \|\nabla u\|^2 + \frac{\varepsilon}{4\delta} \|u_t\|^2 + \varepsilon \delta \|u\|^2 \\
&\le \varepsilon \left(\alpha \delta + \underbrace{a C_*^{p+1} \left(\frac{2(p+1)}{p-1} E(u_0, u_1) \right)^{\frac{p-1}{2}} - 1}_{<0} \right) \|\nabla u\|^2 \\
&\quad -\alpha \left(1 - \frac{\varepsilon}{4\delta} \right) \|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 - \varepsilon \|\triangle u\|^2 + \frac{\varepsilon}{4\delta} \|u_t\|^2 + \varepsilon \delta \|u\|^2 \\
&\le \varepsilon \left(C_* \delta + \alpha \delta + a C_*^{p+1} \left(\frac{2(p+1)}{p-1} E(u_0, u_1) \right)^{\frac{p-1}{2}} - 1 \right) \|\nabla u\|^2 \\
&\quad -\alpha \left(1 - \frac{\varepsilon}{4\delta} \right) \|\nabla u_t\|^2 + \varepsilon \|u_t\|^2 - \varepsilon \|\triangle u\|^2 + \frac{\varepsilon}{4\delta} \|u_t\|^2.
\end{aligned} \tag{26}
$$

From (15), we have

$$aC_*^{p+1} \left(\frac{2\left(p+1\right)}{p-1} E\left(u_0, u_1\right) \right)^{\frac{p-1}{2}} - 1 < 0. \tag{27}$$

Now, let us choose δ small enough such that:

$$C_* \delta + \alpha \delta + aC_*^{p+1} \left(\frac{2\left(p+1\right)}{p-1} E\left(u_0, u_1\right) \right)^{\frac{p-1}{2}} - 1 < 0. \tag{28}$$

From (26) we may find $\eta > 0$, which depends only on δ, such that:

$$F'\left(t\right) < -\alpha \left(1 - \frac{\varepsilon}{4\delta}\right) \|\nabla u_t\|^2 + \varepsilon \left(1 + \frac{1}{4\delta}\right) \|u_t\|^2 - \varepsilon\eta \|\nabla u\|^2 - \varepsilon \|\triangle u\|^2. \tag{29}$$

Consequently, using the definition of the energy (11), for any positive constant M, we obtain:

$$\begin{aligned} F'\left(t\right) \quad < \quad & -\varepsilon M E\left(t\right) + \left(\varepsilon C_* \left(\frac{M}{2} + 1 + \frac{1}{4\delta} \right) - \alpha \left(1 - \frac{\varepsilon}{4\delta} \right) \right) \|\nabla u_t\|^2 \\ & + \varepsilon \left(\frac{M}{2} - \eta \right) \|\nabla u\|^2 + \varepsilon \left(\frac{M}{2} - 1 \right) \|\triangle u\|^2. \end{aligned} \tag{30}$$

Now, choosing $M \leq \min\{2, 2\eta\}$, and α such that

$$\varepsilon C_* \left(\frac{M}{2} + 1 + \frac{1}{4\delta} \right) - \alpha \left(1 - \frac{\varepsilon}{4\delta} \right) < 0,$$

inequality (30) becomes

$$\begin{aligned} F'\left(t\right) \quad & < \quad -\varepsilon M E\left(t\right) \\ & \leq \quad -\varepsilon M \alpha_1 F\left(t\right) \end{aligned} \tag{31}$$

by virtue of (21). A simple integrating of (31) then leads to

$$F\left(t\right) \leq F\left(0\right) e^{-kt}, \ \forall t \geq 0,$$

where $k = \varepsilon M \alpha_1$. Consequently, by using (21) once again, we conclude

$$E\left(t\right) \leq K e^{-kt}, \ \forall t \geq 0.$$

where $K = \alpha_2 F\left(0\right)$. This completes the proof.

References

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces,* Academic Press, 2003.

[2] A. Benaissa and S.A. Messaoudi, *Exponential decay of solutions of nonlinearly damped wave equation,* Nonlinear Differ. Equ. Appl., 12 (2005) 391-399.

[3] L.C. Evans, *Partial Differential Equations,* Graduate Studies in Mathematics, vol.19, 1998.

[4] S. Gerbi and B.S. Houari, *Exponential decay for solutions to semilinear damped wave equations,* Discrete Cont Dyn-S, 5(3) (2012) 559-566.

[5] Q. Lin, Y.H. Wu and S. Lai, *On global solution of an initial boundary value problem for a class of damped nonlinear equations,* Nonlinear Analysis, 69 (2008) 4340-4351.

[6] E. Zuazua, *Exponential decay for the semilinear wave equation with locally distributed damping,* Comm. Partial. Diff. Eq., 15(2) (1990) 205–235.

Erhan Pişkin
Department of Mathematics
University of Dicle
21280 Diyarbakır, Turkey
E-mail: *episkin@dicle.edu.tr*

Necat Polat
Department of Mathematics
University of Dicle
21280 Diyarbakır, Turkey
E-mail: *npolat@dicle.edu.tr*