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STABILITY RESULTS FOR SOME FUNCTIONAL EQUATIONS OF QUADRATIC-TYPE

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Abstract. We present some theorems of stability, in the Hyers-Ulam-Rassias sense, for functional equations of quadratic-type, extending the results from [2], [8], [16], [19] and [20]. There are used both the direct and the fixed point methods.

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1. Introduction

Although there are known different methods to obtain stability properties for functional equations, almost all proofs used the *direct method*, discovered by Hyers (see [13], [1], [22] and [9], where a question of S. M. Ulam concerning the stability of group homomorphisms is affirmatively answered, for Banach spaces). The reader is referred to the expository papers [10,23] and the books [14,15], for more details.

In [21], there was proposed a *fixed point method* to prove stability results for different types of functional equations, including the case of unbounded differences (see also [3], [4], [5] and [6]). It is worth noting that the fixed point method does suggest a metrical context and is seen to better clarify the ideas of stability.

In this paper, by both the direct method and the fixed point method, we obtain stability results for functional equations of quadratic-type. Our idea is to rewrite some bi-quadratic and additive-quadratic equations as equations with the unknown function *in a single variable*. We have been able to slightly extend the results from [8], [16], [19] and [20].

Let us consider the real linear spaces X_1, X_2, Y, Z, where $Z := X_1 \times X_2$, and the linear involution mappings $P_{X_1} : Z \to Z,\ P_{X_1}(u) = (u_1, 0),\ P_{X_2} : Z \to Z,\ P_{X_2}(u) = (0, u_2)$, and $S : Z \to Z,\ S = S_{X_1} := P_{X_1} - P_{X_2}$. A function $F : Z \to Y$ is called a *quadratic-type* or *$Q-$type mapping* iff it satisfies the following equation

$$F(u+v) + F(u-v) + F(u+S(v)) + F(u-S(v)) = \tag{1}$$

$$= 4\left(F(u) + F(v) + F\left(\frac{u+S(u)+v-S(v)}{2}\right) + F\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)$$

for all $u, v \in Z$.
Notice that, whenever Z is an inner product space, the function

$$Z \ni u \to F(u) = a \cdot ||P_{X_1}u||^2 \cdot ||P_{X_2}u||^2$$

is a solution of (1) for any real constant a and recall that a mapping $h : X \to Y$, between linear spaces, is called *quadratic* if it satisfies the following functional equation:

$$h(x+y) + h(x-y) = 2h(x) + 2h(y), \forall x, y \in X. \tag{2}$$

If F is a solution of (1) for $X_1 = X_2 = X$, then $u = (x, z) \to f(x, z) := F(u)$ is *a bi-quadratic mapping*, verifying the following equation [19]:

$$f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) = \tag{3}$$

$$= 4\left(f(x,z) + f(y,w) + f(x,w) + f(y,z)\right),\ \forall x, y, z, w \in X.$$

For $X = Y = \mathbb{R}$, $(x, y) \to f(x, y) = ax^2y^2$ is a solution for (3).

Remark 1.1. *Any solution F of (1) has the following properties:*
(i) $F(0) = 0$ and F is an even mapping;
(ii) $F(2^n \cdot u) = 2^{4n} \cdot F(u),\ \forall u \in Z$ and $\forall n \in \mathbb{N}$;
(iii) $F \circ S = F$;
(iv) $F \circ P_{X_1} = F \circ P_{X_2} = 0$;
(v) If $f(x, z) = F(u)$, then f is quadratic in each variable.

In fact, we shall make use of the following easily verified result:

PROPOSITION 1.2. *Suppose that* $F : Z \to Y$ *is of the form*

$$F(u) = f_2(z)f_1(x),\ \forall u = (x, z) \in Z = X_1 \times X_2,$$

with nonidentically zero mappings $f_1 : X_1 \to Y$ *and* $f_2 : X_2 \to \mathbb{R}$*. Then* F *is of Q-type if and only if* f_1 *and* f_2 *are quadratic.*

For convenience, let us fix the following notation, related to (1):

$$Q_F(u, v) := F(u + v) + F(u - v) + F(u + S(v)) + F(u - S(v)) - \tag{4}$$

$$-4\left(F(u) + F(v) + F\left(\frac{u + S(u) + v - S(v)}{2}\right) + F\left(\frac{u - S(u) + v + S(v)}{2}\right)\right).$$

2. The stability of quadratic-type equations: the direct method

Let us consider a control mapping $\Phi : Z \times Z \to [0, \infty)$ such that

$$\Psi(u, v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{16^{i+1}} < \infty, \forall u, v \in Z, \tag{5}$$

and suppose Y is a Banach space. We can prove the following *generalized stability* result for the functional equation (1).

THEOREM 2.1. *Let* $F : Z \to Y$ *be a mapping such that* $F \circ P_{X_1} = F \circ P_{X_2} = 0$ *and suppose that*

$$||Q_F(u, v)|| \le \Phi(u, v), \forall u, v \in Z. \tag{6}$$

Then there exists a unique Q-type mapping $B : Z \to Y$*, such that*

$$||F(u) - B(u)|| \le \Psi(u, u),\ \forall u \in Z. \tag{7}$$

Proof. We shall use the (direct) method of Hyers. Letting $u = v$ in (6), we obtain

$$\left|\left|\frac{F(2u)}{16} - F(u)\right|\right| \le \frac{\Phi(u, u)}{16},\ \forall u \in Z.$$

In the next step, one shows that

$$\left|\left|\frac{F(2^p u)}{16^p} - \frac{F(2^m u)}{16^m}\right|\right| \le \sum_{i=p}^{m-1} \frac{\Phi(2^i u, 2^i u)}{16^{i+1}}, \ \forall u \in Z, \tag{8}$$

for given integers p, m, with $0 \le p < m$. Using (5) and (8), $\{\frac{F(2^n u)}{16^n}\}_{n \ge 0}$ is a Cauchy sequence for any $u \in Z$. Since Y is complete, the sequence $\{\frac{F(2^n u)}{16^n}\}_{n \ge 0}$ is convergent for all $u \in Z$. So, we can define the mapping $B : Z \to Y$,

$$B(u) = \lim_{n \to \infty} \frac{F(2^n u)}{16^n},$$

for all $u \in Z$. By using (8) for $p = 0$ and $m \to \infty$ we obtain the estimation (7). By (6), we have

$$\left|\left|\frac{F(2^n(u+v))}{16^n} + \frac{F(2^n(u-v))}{16^n} + \frac{F(2^n(u+S(v)))}{16^n} + \frac{F(2^n(u-S(v)))}{16^n} - \right.\right.$$
$$-4\left(\frac{F(2^n(u))}{16^n} + \frac{F(2^n(v))}{16^n} + \frac{1}{16^n}F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right) + \right.$$
$$\left.\left.\left.\frac{1}{16^n}F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\right|\right| \le \frac{\Phi(2^n u, 2^n v)}{16^n},$$

for all $u, v \in Z$. Using (5) and letting $n \to \infty$, we immediately see that B is a Q-type mapping.

Let B_1 be a Q-type mapping, which satisfies (7). Then

$$||B(u) - B_1(u)|| \le$$
$$\le \left|\left|\frac{B(2^n u)}{16^n} - \frac{F(2^n u)}{16^n}\right|\right| + \left|\left|\frac{F(2^n u)}{16^n} - \frac{B_1(2^n u)}{16^n}\right|\right| \le$$
$$\le 2\sum_{i=n}^{\infty} \frac{\Phi(2^i u, 2^i u)}{16^{i+1}} \longrightarrow 0, \text{for } n \to \infty.$$

Hence the uniqueness claim for B holds true. $\square$

Remark 2.2. *In the above proof we actually used the following fact only:*

$$F \circ P_{X_1} + F \circ P_{X_2} = 0.$$

Let us consider a mapping $\varphi : X \times X \times X \times X \to [0, \infty)$ such that

$$\psi(x, z, y, w) := \sum_{i=0}^{\infty} \frac{\varphi(2^i x, 2^i z, 2^i y, 2^i w)}{16^{i+1}} < \infty, \forall x, y, z, w \in X.$$

As a direct consequence of Theorem 2.1, we obtain the following result (see [19], Theorem 7):

Corollary 2.3. *Suppose that X is a real linear space, Y is a Banach space and let $f : X \times X \to Y$ be a mapping such that*

$$||f(x+y, z+w) + f(x+y, z-w) + f(x-y, z+w) + f(x-y, z-w) -$$
$$-4\left(f(x,z) + f(y,w) + f(x,w) + f(y,z)\right)|| \le \varphi(x, z, y, w),$$

and $f(x,0) + f(0,z) = 0$, for all $x, y, z, w \in X$. Then there exists a unique bi-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x,z) - b(x,z)|| \le \psi(x, z, x, z), \ \forall x, z \in X. \tag{9}$$

Proof. Let consider $X_1 = X_2 = X$, $u, v \in X \times X, u = (x, z), v = (y, w)$, $F(u) = f(x, z)$, and $\Phi(u, v) = \varphi(x, z, y, w)$. Since $\Psi(u, v) = \psi(x, z, y, w) < \infty$, then we can apply Theorem 2.1. Clearly, the mapping b, defined by $b(x, z) = B(u)$ is bi-quadratic and verifies (9). □

For some particulary forms of the mapping Φ, verifying (5), we will obtain some interesting consequences.

In the following corollary we give a stability result of Hyers-Rassias type for the equation (1).

Let us consider a Banach space Y, the linear spaces X_1, X_2, and suppose that $Z := X_1 \times X_2$ is endowed with a norm $||u||_Z$.

Corollary 2.4. *Let $F : Z \to Y$ be a mapping such that*

$$||Q_F(u, v)||_Y \le \varepsilon(||u||_Z^p + ||v||_Z^p), \ \forall u, v \in Z,$$

where $p \in [0, 4)$ and $\varepsilon \ge 0$ are fixed. If $F \circ (I - S) = 0$ and $F \circ (I + S) = 0$, then there exists a unique Q-type mapping $B : Z \to Y$, such that

$$||F(u) - B(u)||_Y \le \frac{2\varepsilon}{2^4 - 2^p} \cdot ||u||_Z^p, \ \forall u \in Z.$$

Proof. Consider the mapping $\Phi : Z \times Z \to [0, \infty)$, $\Phi(u, v) = \varepsilon(||u||_Z^p + ||v||_Z^p)$. Then

$$\Psi(u, v) := \sum_{i=0}^{\infty} \frac{\Phi(2^i u, 2^i v)}{16^{i+1}} = \varepsilon \cdot \frac{||u||_Z^p + ||v||_Z^p}{2^4 - 2^p}, \ \forall u, v \in Z,$$

and the conclusion follows directly from Theorem 2.1. $\square$

Now, let us consider, as in Corollarry 2.4: $X_1 = X_2 = X$, where X is a normed space, $u, v \in X \times X, u = (x, z), v = (y, w)$, $F(u) = f(x, z)$, with $f : X \times X \to Y$ and $||u|| = ||u||_p := \sqrt[p]{||x||^p + ||z||^p}, p \geq 0$. Although the functions $|| \cdot ||_p$ is not a norm, the above proof works as well. Actually, we obtain the following

Corollary 2.5. *Let $f : X \times X \to Y$ be a mapping such that*

$$| \, |f(x + y, z + w) + f(x + y, z - w) + f(x - y, z + w) + f(x - y, z - w) -$$

$$-4 \left(f(x, z) + f(y, w) + f(x, w) + f(y, z) \right) || \leq \varepsilon \left(||x||^p + ||y||^p + ||z||^p + ||w||^p \right),$$

for all $x, y, z, w \in X$ and for some fixed ε, p, $0 \leq p < 4$, $\varepsilon \geq 0$. If $f(x, 0) = 0$ and $f(0, y) = 0$, for all $x, y \in X$, then there exists a unique bi-quadratic mapping $b : X \times X \to Y$, such that

$$||f(x, z) - b(x, z)|| \leq \frac{2\varepsilon}{2^4 - 2^p} \cdot \left(||x||^p + ||z||^p \right),$$

for all $x, z \in X$.

3. A relationship between the stability of equations (1) and (2)

We recall the following stability result of type Hyers-Ulam-Rassias for the quadratic functional equation (2)(see [16], Theorem 2.2):

Proposition 3.1. *Let X be a real normed vector space and Y a real Banach space. If $\tilde{\varphi}$ and $\tilde{\phi} : X \times X \to [0, \infty)$ verify the condition*

$$\tilde{\phi}(x, y) := \sum_{i=0}^{\infty} \frac{\tilde{\varphi}(2^i x, 2^i y)}{4^{i+1}} < \infty, \ \textit{for all } x, y \in X \tag{10}$$

and the mapping $\tilde{f} : X \to Y$, with $\tilde{f}(0) = 0$, satisfies the relation

$$\left\|\tilde{f}(x+y) + \tilde{f}(x-y) - 2\tilde{f}(x) - 2\tilde{f}(y)\right\| \leq \tilde{\varphi}(x,y), \text{ for all } x, y \in X, \quad (11)$$

then there exists a unique quadratic mapping $\tilde{q} : X \to Y$ which satisfies the inequality

$$\left\|\tilde{f}(x) - \tilde{q}(x)\right\| \leq \tilde{\phi}(x,x), \text{ for all } x \in X.$$

We can show that Proposition 3.1 is a consequence of our Theorem 2.1. Namely, we have

THEOREM 3.2. *The stability of the Q-type equation (1) implies the Hyers-Ulam-Rassias stability of the quadratic equation (2).*

Proof. Let X, Y, $\tilde{\varphi} : X \times X \to [0, \infty)$ and $\tilde{f} : X \to Y$ as in Proposition 3.1. We take $X_1 = X$ and consider a linear space X_2 such that there exist a quadratic function $\tilde{h} : X_2 \to \mathbb{R}$, with $\tilde{h}(0) = 0$ and an element $z_0 \in X_2$, such that $\tilde{h}(z_0) \neq 0$ (In Hilbert spaces such a mapping is, for example, $z \to ||z||^2$).

If we set, for $u = (x, z), v = (y, w) \in X \times X_2$,

$$\Phi(u,v) = \Phi(x,z,y,w) = 2|\tilde{h}(z) + \tilde{h}(w)| \cdot \tilde{\varphi}(x,y)$$

and

$$F(u) = F(x,z) = \tilde{h}(z) \cdot \tilde{f}(x), \quad (14)$$

then we easily get (using (10) and the properties of the quadratic mapping):

$$\Psi(u,v) = \frac{1}{2}|\tilde{h}(z) + \tilde{h}(w)| \sum_{i=0}^{\infty} \frac{\tilde{\varphi}(2^i x, 2^i y)}{4^{i+1}} < \infty,$$

for all $u, v \in X \times X_2$. At the same time, by (11),

$$||Q_F(u,v)|| = 2|\tilde{h}(z) + \tilde{h}(w)| \cdot (\tilde{f}(x+y) + \tilde{f}(x-y) - 2\tilde{f}(x) - 2\tilde{f}(y)) \leq$$

$$\leq 2|\tilde{h}(z) + \tilde{h}(w)| \cdot \tilde{\varphi}(x,y) = \Phi(u,v), \ \forall u, v \in X \times X_2.$$

Now, by Theorem 2.1, there exists a unique Q-type mapping $B : X \times X_2 \to Y$, such that

$$||F(u) - B(u)|| \leq \Psi(u, u),$$

and

$$B(u) = \lim_{n\to\infty} \frac{F(2^n u)}{16^n} = \lim_{n\to\infty} \tilde{h}(z) \cdot \frac{\tilde{f}(2^n x)}{4^n}, \forall u = (x, z) \in X \times X_2.$$

Since $\tilde{h}(z_0) \neq 0$ for at least a $z_0 \in X_2$, then the following limit exists

$$\tilde{q}(x) = \lim_{n\to\infty} \frac{\tilde{f}(2^n x)}{4^n} \text{ for all } x \in X,$$

and

$$||\tilde{h}(z)\tilde{f}(x) - \tilde{h}(z)\tilde{q}(x)|| \leq \tilde{h}(z) \cdot \tilde{\phi}(x, x), \ \forall (x, z) \in X \times X_2.$$

Since

$$B(u) = \tilde{h}(z) \cdot \tilde{q}(x), \forall u = (x, z) \in Z,$$

then the estimation (11) holds .

By Proposition 1.2, $\tilde{q}$ is quadratic. If a quadratic mapping $\tilde{q}_1$ satisfies (12), then $B(u) = \tilde{h}(z)\tilde{q}_1(x) = \tilde{h}(z)\tilde{q}(x)$, for all $u = (x, z) \in X \times X_2$. Since $\tilde{h}$ is nonidentically 0, then $\tilde{q}(x) = \tilde{q}_1(x)$, for all $x \in X$. Hence $\tilde{q}$ is unique. $\square$

As a very particular case, we obtain the classical result of Hyers-Ulam-Rassias stability (see, e.g., [8]) for the quadratic equation (2):

Corollary 3.3. *Let $\tilde{f} : X \to Y$ be a mapping such that*

$$\left\|\tilde{f}(x+y) + \tilde{f}(x-y) - 2\tilde{f}(x) - 2\tilde{f}(y)\right\| \leq \varepsilon\left(||x||^p + ||y||^p\right), \text{ for all } x, y \in X,$$

and for any fixed ε, p, with $0 \leq p < 2$, $\varepsilon \geq 0$. If $\tilde{f}(0) = 0$, then there exists a unique quadratic mapping $\tilde{q} : X \to Y$ which satisfies the estimation

$$\left\|\tilde{f}(x) - \tilde{q}(x)\right\| \leq \frac{2\varepsilon}{2^2 - 2^p} \cdot \left(||x||^p + ||y||^p\right), \text{ for all } x \in X.$$

Indeed, let $\tilde{h} : \mathbb{R} \to \mathbb{R}, \tilde{h}(z) = z^2$ and $\tilde{f} : X \to Y$, where X is a normed space and Y a Banach space. We apply Theorem 2.1 for $X_1 = X, X_2 = \mathbb{R}$, $u, v \in X \times \mathbb{R}$, with $u = (x, z), v = (y, w)$ and the mappings

$$F(u) = F(x, z) = z^2 \cdot \tilde{f}(x),$$

$$\Phi(u,v) = \Phi(x,z,y,w) = 2(z^2+w^2)\cdot\varepsilon(||x||^p + ||y||^p),$$

to obtain the existence of a unique quadratic mapping $\tilde{q}$ and the required estimation.

Remark 3.4. *As it is wellknown (see [8,14]), Czervick showed that the quadratic equation (2) is* **not stable** *for $\tilde{\varphi}(x,y)$ of the form $\varepsilon(x^2+y^2)$, ε being a given positive constant. In fact, he proved that there exists a mapping $\tilde{f}:\mathbb{R}\to\mathbb{R}$ such that (11) holds with the above $\tilde{\varphi}$, and there exists* **no** *quadratic mapping $\tilde{q}$ to verify*

$$\left\|\tilde{f}(x) - \tilde{q}(x)\right\| \le c(\varepsilon)||x||^2, \text{ for all } x\in\mathbb{R}.$$

This suggests the following:

Example 3.5. Let us consider a real normed space X_1, a Banach space Y, and a quadratic function $\tilde{h}: X_2 = \mathbb{R}\to\mathbb{R}$, with $\tilde{h}(0)=0, \tilde{h}(1)=1$.

Then the equation (1) is **not stable** for

$$\Phi(u,v) = \Phi(x,z,y,w) = 2\varepsilon\cdot(||x||^2+||y||^2)(h(z)+h(w)). \tag{15}$$

In fact, we can show that there exists an F such that the relation (6) holds and there exists **no** Q-type mapping $B: X_1\times X_2\to Y$ which verifies

$$||F(u) - B(u)|| \le c(\varepsilon)\tilde{h}(z)||x||^2,\ \forall u=(x,z)\in X_1\times X_2. \tag{16}$$

Indeed, let

$$F(u) = F(x,z) = \tilde{h}(z)\cdot\tilde{f}(x),$$

and Φ as in (14), such that (6) holds. Therefore

$$\left\|\tilde{f}(x+y)+\tilde{f}(x-y)-2\tilde{f}(x)-2\tilde{f}(y)\right\| \le \varepsilon\left(||x||^2+||y||^2\right), \text{ for all } x,y\in X.$$

Let us suppose, for contradiction, that there exists a Q-type mapping B which verifies (15). By Remark 1.1, the mapping $\tilde{q}: X_1\to Y, \tilde{q}(x) := B(x,1)$ is a solution for quadratic equation (2). The estimation (15) gives us

$$||\tilde{f}(x) - \tilde{q}(x)|| \le c(\varepsilon)||x||^2,\ \forall x\in X_1.$$

This says that the quadratic equation (2) is stable for $\tilde{\varphi}(x,y) = \varepsilon(||x||^2+||y||^2)$, in contradiction with the assertions from Remark 3.4.

4. The stability of functional equations of quadratic-type: the fixed points method

As we will see, Corollary 2.4 can be extended using *the alternative of fixed point*, that is recalled in the next lemma:

Lemma 4.1. ([17,24]) *Suppose we are given a complete generalized metric space (X, d) and a strictly contractive mapping $J : X \to X$, that is*

$$(\mathbf{B_1}) \qquad d(Jx, Jy) \leq Ld\,(x, y)\,, \forall x, y \in X,$$

for some (Lipschitz constant) $L < 1$. Then, for each fixed element $x \in X$, ***either***

$$(\mathbf{A_1}) \qquad d(J^n x, J^{n+1}x) = +\infty\ ,\ \forall n \geq 0,$$

or

$$(\mathbf{A_2}) \qquad d(J^n x, J^{n+1}x) < +\infty, \forall n \geq n_0,$$

for some natural number n_0.

Actually, if $(\mathbf{A_2})$ holds then:
$(\mathbf{A_{21}})$ *The sequence $(J^n x)$ is convergent to a fixed point y^* of J;*
$(\mathbf{A_{22}})$ *y^* is the unique fixed point of J in the set*

$$Y = \{y \in X, d\,(J^{n_0}x, y) < +\infty\}\,;$$

$(\mathbf{A_{23}})$ $d\,(y, y^*) \leq \dfrac{1}{1-L} d\,(y, Jy)\,, \forall y \in Y.$

Let X_1, X_2 be linear spaces, $Z := X_1 \times X_2, Y$ a Banach space, and consider the mappings $S : Z \to Z, S(u) = (u_1, -u_2)$, Q_F defined by (4) and $\Phi : Z \times Z \to [0, \infty)$ an arbitrary given function. We can prove the following stability result:

Theorem 4.2. *Let $F : Z \to Y$ be a mapping such that $F \circ (I - S) = 0$ and $F \circ (I + S) = 0$. Suppose that*

$$||Q_F(u, v)|| \leq \Phi(u, v),\ \forall u \in Z \tag{17}$$

If, moreover, there exists $L < 1$ such that the mapping

$$x \to \Omega(u) = \Phi\left(\frac{u}{2}, \frac{u}{2}\right)$$

has the property

$$(\mathbf{H}) \qquad \Omega(u) \leq L \cdot 2^4 \cdot \Omega\left(\frac{u}{2}\right), \forall u \in Z,$$

and the mapping Φ *has the property*

$$(\mathbf{H}^*) \qquad \lim_{n\to\infty} \frac{\Phi\left(2^n u, 2^n v\right)}{2^{4n}} = 0, \forall u, v \in Z,$$

then there exists a unique Q-type mapping $B : Z \to Y$*, such that*

$$(\mathbf{Est}) \qquad ||F(u) - B(u)|| \leq \frac{L}{1-L}\Omega(u),$$

for all $u \in Z$*.*

Proof. We consider the set

$$\mathcal{F} := \{G : Z \to Y, G(0) = 0\}$$

and introduce a *generalized metric* on $\mathcal{F}$:

$$d\left(G, H\right) = d_\Omega\left(G, H\right) = \inf\left\{K \in \mathbb{R}_+, \|G\left(u\right) - H\left(u\right)\| \leq K\Omega(u), \forall u \in Z\right\}$$

Obviously, $(\mathcal{F}, d)$ *is complete.* Now, we consider the (linear) mapping

$$J : \mathcal{F} \to \mathcal{F}, JG\left(u\right) := \frac{G\left(2u\right)}{2^4}.$$

We have, for any $G, H \in \mathcal{F}$:

$$d(G, H) < K \Longrightarrow \|G\left(u\right) - H\left(u\right)\| \leq K\Omega(u), \forall u \in Z \Longrightarrow$$

$$\left\|\frac{1}{2^4}G\left(2u\right) - \frac{1}{2^4}H\left(2u\right)\right\| \leq LK\Omega(u), \forall u \in Z \Longrightarrow$$

$$d\left(JG, JH\right) \leq LK.$$

Therefore we see that

$$d\left(JG, JH\right) \leq L \cdot d\left(G, H\right), \forall G, H \in \mathcal{F},$$

that is J is a *strictly contractive* self-mapping of $\mathcal{F}$, with the Lipschitz constant L. If we set $u = v = t$ in the relation (16), then we see that

$$\|F(2t) - 16F(t)\| \leq \Omega(2t), \forall t \in Z.$$

Using the hypothesis (**H**) we obtain

$$\left\| \frac{F(2t)}{16} - F(t) \right\| \leq \frac{\Omega(2t)}{16} \leq L\Omega(t), \forall t \in Z,$$

that is $d(F, JF) \leq L < \infty$.

We can apply the fixed point alternative, and we obtain the existence of a mapping $B : \mathcal{F} \to \mathcal{F}$ such that:

- B is a fixed point of J, that is

$$B(2u) = 16B(u), \forall u \in Z. \tag{18}$$

The mapping B is the unique fixed point of J in the set

$$\{G \in \mathcal{F},\ d(F, G) < \infty\}.$$

This says that B is the unique mapping with *both* the properties (17) and (18), where

$$\exists K \in (0, \infty) \text{ such that } \|B(u) - F(u)\| \leq K\Omega(u), \forall u \in Z. \tag{19}$$

Moreover,

- $d(J^n F, B) \longrightarrow 0$, for $n \to \infty$, which implies the equality

$$\lim_{n\to\infty} \frac{F(2^n u)}{2^{4n}} = B(u), \forall u \in Z. \tag{20}$$

- $d(F, B) \leq \dfrac{1}{1-L} d(F, JF)$, which implies the inequality

$$d(F, B) \leq \frac{L}{1-L},$$

that is (**Est**) is seen to be true.

The statement that B is a Q-type mapping follows immediately: replacing u by $2^n u$ and v by $2^n v$ in (16), then we obtain

$$\left\| \frac{F(2^n(u+v))}{16^n} + \frac{F(2^n(u-v))}{16^n} + \frac{F(2^n(u+S(v)))}{16^n} + \frac{F(2^n(u-c(v)))}{16^n} - \right.$$

$$-4\left(\frac{F(2^n(u))}{16^n}+\frac{F(2^n(v))}{16^n}+\frac{1}{16^n}F\left(2^n\left(\frac{u+S(u)+v-S(v)}{2}\right)\right)+\right.$$
$$\left.\left.\frac{1}{16^n}F\left(2^n\left(\frac{u-S(u)+v+S(v)}{2}\right)\right)\right)\right\|\leq\frac{\Phi(2^n u,2^n v)}{16^n},$$

for all $u,v\in Z$. Using (19) and ($\mathbf{H}^*$) and letting $n\to\infty$, we see that B satisfies equation (1). $\square$

REMARK 4.3. *In fact, in Theorem 2.1. it suffices to suppose, as in [7]-Theorem 3.1, that the series*

$$\Psi(u,u):=\sum_{i=0}^{\infty}\frac{\Phi(2^i u,2^i u)}{16^{i+1}}$$

is convergent for each $u\in Z$, *and that*

$$\lim_{n\to\infty}\frac{\Phi\left(2^n u,2^n v\right)}{2^{4n}}=0,\forall u,v\in Z.$$

EXAMPLE 4.4. If we apply Theorem 4.2 with the mapping

$$\Phi:Z\times Z\to[0,\infty),\ \Phi(u,v)=\varepsilon(||u||_Z^p+||v||_Z^p),$$

where $\varepsilon\geq 0$ and $p\in[0,4)$ are fixed, then we obtain Corollary 2.4.

5. Stability properties of functional equations of additive-quadratic-type

The following equation was discussed in [20]:

$$f(x+y,z+w)+f(x+y,z-w)= \tag{21}$$

$$=2\left(f(x,z)+f(y,w)+f(x,w)+f(y,z)\right),\ \forall x,y,z,w\in X.$$

with $f:X\times X\to Y$. A solution of this equation is called an *additive-quadratic mapping*. Obviously, for $X=Y=\mathbb{R}$, $(x,y)\to f(x,y)=axy^2$ verifies (20).

We again rewrite (20) as an equation with the unknown function *in a single variable* to obtain, by the fixed point alternative, a generalized stability result. A function $F:Z=X_1\times X_2\to Y$, which verifies the equation

$$F(u+v)+F(u+S(v))=$$

$$= 2\left(F(u) + F(v) + F\left(\frac{u + S(u) + v - S(v)}{2}\right) + F\left(\frac{u - S(u) + v + S(v)}{2}\right)\right)$$

for all $u, v \in Z$, is called an *A-quadratic-type* or $A - Q-$*type mapping.* In the case of inner product spaces, the function given by

$$F(u) = a \cdot ||P_{X_1}u|| \cdot ||P_{X_2}u||^2, \forall u \in Z,$$

is a solution of (21).

It is easy to verify that any solution F of (21) has the following properties:

(i) $F(0) = 0$ and F is an odd mapping;

(ii) $F(2^n \cdot u) = 2^{3n} \cdot F(u),\ \forall u \in Z,\ \forall n \in \mathbb{N}$;

(iii) $F \circ S = F$;

(iv) $F \circ P_{X_1} = F \circ P_{X_2} = 0$;

(v) If $f(x, z) = F(u)$, then f is additive in the first variable and quadratic in the second variable.

Let us introduce the following notation, related to (21):

$$D_F(u, v) := F(u + v) + F(u + S(v)) -$$

$$-2\left(F(u) + F(v) + F\left(\frac{u + S(u) + v - S(v)}{2}\right) + F\left(\frac{u - S(u) + v + S(v)}{2}\right)\right).$$

THEOREM 5.1. *Let $\Theta : Z \times Z \to [0, \infty)$ be an arbitrary function and consider a mapping $F : Z \to Y$ such that $F \circ (I + S) = 0$. Suppose that*

$$(i) \qquad ||D_F(u, v)|| \leq \Theta(u, v),\ \forall u \in Z;$$

(ii) There exists $L < 1$ such that the mapping

$$x \to \Gamma(u) = \Theta\left(\frac{u}{2}, \frac{u}{2}\right)$$

has the property

$$\Gamma(u) \leq L \cdot 2^3 \cdot \Gamma\left(\frac{u}{2}\right), \forall u \in Z;$$

$$(iii) \qquad \lim_{n\to\infty} \frac{\Theta\left(2^n u, 2^n v\right)}{2^{3n}} = 0, \forall u, v \in Z.$$

Then there exists a unique A-Q-type mapping $A : Z \to Y$, such that

$$||F(u) - A(u)|| \leq \frac{L}{1-L}\Gamma(u),$$

for all $u \in Z$.

Except for obvious modifications, the *proof* coincides with that of Theorem 4.2. We note only that one uses the following formula for the (complete) generalized metric:

$$d\left(G, H\right) = d_{\Gamma}\left(G, H\right) = \inf\left\{K \in \mathbb{R}_{+}, \left\|G\left(u\right) - H\left(u\right)\right\| \leq K\Gamma(u), \forall u \in Z\right\}$$

and that the (linear) mapping

$$J : \mathcal{F} \to \mathcal{F}, JG\left(u\right) := \frac{G\left(2u\right)}{2^3},$$

is a *strictly contractive* self-mapping of $\mathcal{F} := \{G : Z \to Y, G(0) = 0\}$, with the Lipschitz constant $L < 1$:

$$d\left(JG, JH\right) \leq L \cdot d\left(G, H\right), \forall G, H \in \mathcal{F}. \ \square$$

As a direct consequence of Theorem 5.1, we obtain the following

COROLLARY 5.2. *Let $F : Z \to Y$ be a mapping such that*

$$||D_F(u, v)||_Y \leq \varepsilon(||u||_Z^p + ||v||_Z^p), \ \forall u, v \in Z,$$

for some fixed ε, p, with $0 \leq p < 3$, $\varepsilon \geq 0$. If $F \circ (I + S) = 0$, then there exists a unique A-Q-type mapping $A : Z \to Y$, such that

$$||F(u) - A(u)||_Y \leq \frac{2\varepsilon}{2^3 - 2^p} \cdot ||u||_Z^p, \ \forall u \in Z.$$

Proof. This follows immediately, by setting $\Theta(u, v) = \varepsilon(||u||_Z^p + ||v||_Z^p)$. $\square$

REMARK 5.3 *We emphasize that Theorem 5.1 can be used to obtain a counterpart of Theorem 3.2.*

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