AN EXISTENCE RESULT FOR A CLASS OF NONCONVEX FUNCTIONAL DIFFERENTIAL INCLUSIONS

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ABSTRACT. Let σ be a positive number and $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], R^m)$ the Banach space of continuous functions from $[-\sigma, 0]$ into R^m and let $T(t)$ be the operator from $\mathcal{C}([-\sigma, T], R^m)$ into $\mathcal{C}_\sigma$, defined by $(T(t)x)(s) := x(t+s)$, $s \in [-\sigma, 0]$. We prove the existence of solutions for functional differential inclusion(differential inclusions with memory) $x' \in F(T(t)x) + f(t, T(t)x)$ where F is upper semicontinuous, compact valued multifunction such that $F(T(t)x) \subset \partial V((x(t))$ on $[0, T]$, V is a proper convex and lower semicontinuous function and f is a Carathéodory single valued function.

1.Introduction

Let R^m be the m-dimensional euclidean space with norm $||.||$ and scalar product $\langle ., . \rangle$. If I is a segment in R then we denote by $\mathcal{C}(I, R^m)$ the Banach space of continuous functions from I into R^m with the norm given by $||x(.)||_\infty := \sup\{||x(t)||; t \in I\}$. If σ is a positive number then we put $\mathcal{C}_\sigma := \mathcal{C}([-\sigma, 0], R^m)$ and for any $t \in [0, T]$, $T > 0$, we define the operator $T(t)$ from $\mathcal{C}([-\sigma, T], R^m)$ into $\mathcal{C}_\sigma$ as follows: $(T(t)x)(s) := x(t+s)$, $s \in [-\sigma, 0]$.

Let Ω be a nonempty subset of $\mathcal{C}_\sigma$. For a given multifunction $F : \Omega \to 2^{R^m}$ and a given function $f : R \times \Omega \to R^m$ we consider the following functional differential inclusion (differential inclusion with memory):

$$x' \in F(T(t)x) + f(t, T(t)x). \tag{1}$$

The existence of solutions for functional differential inclusion (1) was proved by Haddad [7] in the case in which F is upper semicontinuous and with convex compact values and $f \equiv 0$. In paper [1], Ancona and Colombo have obtained an existence result for Cauchy problem $x' \in F(x) + f(t,x), x(0) = \xi$, where $F : R^m \to 2^{R^m}$ is an upper semicontinuous, cyclically monotone multifunction, whose compact values are contained in the subdifferential ∂V of a proper convex and lower semicontinuous function V and f is a Carathéodory single valued function.

In this paper we prove the existence of solutions for functional differential inclusion (1) in the case in which F is upper semicontinuous, compact valued multifunction such that $F(\psi) \subset \partial V(\psi(0))$ for every $\psi \in \Omega$ and V is a proper convex and lower semicontinuous function.

2.Preliminaries and statement of the main result

For $x \in R^m$ and $r > 0$ let $B(x,r) := \{y \in R^m; ||y - x|| < r\}$ be the open ball centered in x with radius r, and let $\overline{B}(x,r)$ be its closure. For $\varphi \in \mathcal{C}_\sigma$ let $B(\varphi, r) := \{\psi \in R^m; ||\psi - \varphi|| < r\}$ and $\overline{B}(\varphi, r) := \{\psi \in R^m; ||\psi - \varphi|| \leq r\}$. For $x \in R^m$ and for a closed subset $A \subset R^m$ we denote by $d(x, A)$ the distance from x to A given by $d(x, A) := \inf\{||y - x||; y \in A\}$.

Let $V : R^m \to R$ be a proper convex and lower semicontinuous function. The multifunction $\partial V : R^m \to 2^{R^m}$, defined by

$$\partial V(x) := \{\xi \in R^m; V(y) - V(x) \geq \langle \xi, y - x \rangle, (\forall) y \in R^m\}, \tag{2}$$

is called subdifferential (in the sense of convex analysis) of the function V.

We say that a multifunction $F : \Omega \to 2^{R^m}$ is upper semicontinuous if for every $\varphi \in \Omega$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $F(\psi) \subset F(\varphi) + B(0, \varepsilon), (\forall) \psi \in B(\varphi, \delta)$.

We consider the functional differential inclusion (1) under the following assumptions:

(h_1) $\Omega \subset \mathcal{C}_\sigma$ is an open set and $F : \Omega \to 2^{R^m}$ is upper semicontinuous with compact values;

(h_2) There exists a a proper convex and lower semicontinuous function $V : R^m \to R$ such that

$$F(\psi) \subset \partial V(\psi(0)) \tag{3}$$

for every $\psi \in \Omega$;

(h$_3$) $f : R \times \Omega \to R^m$ is Carathéodory function, i.e. for every $\psi \in \Omega$, $t \to f(t, \psi)$ is measurable, for a.e. $t \in R$, $\psi \to f(t, \psi)$ is continuous and there exists $m \in L^2(R)$ such that $\|f(t, \psi)\| \leq m(t)$ for a.e.$t \in R$ and all $\psi \in \Omega$.

We recall that (see [7]) a continuous function $x(.) : [-\sigma, T] \to R^m$ is said to be a solution of (1) if $x(.)$ is absolutely continuous on $[0, T]$, $T(t)x \in \Omega$ for all $t \in [0, T]$ and $x'(t) \in F(T(t)x) + f(t, T(t)x)$ for almost all $t \in [0, T]$.

Our main result is the following:

Theorem 2.1. *If $F : \Omega \to 2^{R^m}$, $f : R \times \Omega \to R^m$ and $V : R^m \to R$ satisfy assumptions (h$_1$), (h$_2$) and (h$_3$) then for every $\varphi \in \Omega$ there exists $T > 0$ and $x(.) : [-\sigma, T] \to R^m$ a solution of the functional differential inclusion (1) such that $T(0)x = \varphi$* on $[-\sigma, 0]$.

3.Proof of main theorem

Let $\varphi \in \Omega$ be arbitrarily fixed. Since the multifunction $x \to \partial V(x)$ is locally bounded (see [3], Proposition 2.9) there exists $r > 0$ and $M > 0$ such that V is Lipschitz continuous with constant M on $B(\varphi(0), r)$. Since Ω is an open set we can choose r such that $\overline{B}(\varphi, r) \subset \Omega$. Moreover, by Proposition 1.1.3 in [2], F is also locally bounded; therefore, we can assume that

$$\|y\| < M, \tag{4}$$

$\forall y \in F(\psi)$and$\psi \in B(\varphi, r)$.

Since φ is continuous on $[-\sigma, 0]$ we can choose $T' > 0$ small enough such that for a fixed $r_1 \in (0, r/2)$ we have

$$\|\varphi(t) - \varphi(s)\| < r_1 \tag{5}$$

for all $t, s \in [-\sigma, 0]$ with $|t - s| < T'$.

By (h$_3$) there exists $T'' > 0$ such that

$$\int_0^{T''} (m(t) + M)dt + r_1 < r. \tag{6}$$

Let $0 < T \leq \min\{\sigma, T', T'', r_1/M\}$. We shall prove the existence of a solution of (1) defined on the interval $[-\sigma, T]$. For this, we define a family of approximate solutions and prove that a subsequence converges to a solution of (1).

First, we put

$$x_n(t) = \varphi(t), t \in [-\sigma, 0]. \tag{7}$$

Further on, for $n \geq 1$ we partition $[0,T]$ by points $t_n^j := \frac{jT}{n}$, $j = 0, 1, ..., n$, and, for every $t \in [t_n^j, t_n^{j+1}]$, we define

$$x_n(t) := x_n^j + (t - t_n^j)y_n^j + \int_{t_n^j}^t f(s, T(t_n^j)x_n)ds, \tag{8}$$

where $x_n^0 = x_n(0) := \varphi(0)$ and

$$x_n^{j+1} = x_n^j + \frac{T}{n}y_n^j, \tag{9}$$

$$y_n^j \in F(T(t_n^j)x_n) \tag{10}$$

for every $j \in \{0, 1, ..., n-1\}$.

It is easy to see that for every $j \in \{0, 1, ..., n\}$ we have

$$x_n^j = \varphi(0) + \frac{T}{n}(y_n^0 + y_n^1 + ... + y_n^{j-1}). \tag{11}$$

If for $t \in [0,T]$ and $n \geq 1$ we define $\theta_n(t) = t$ for all $t \in [t_n^j, t_n^{j+1}]$ then, by (9) and (10), we have

$$x_n(t) = x_n(\theta_n(t)) + (t - \theta_n(t))y_n + \int_{\theta_n(t)}^t f_n(s)ds \tag{12}$$

for every $t \in [0,T]$, where $f_n(t) := f(t, T(\theta_n(t)))$, $y_n \in F(T(\theta_n(t))x_n)$, and

$$x_n'(t) \in F(T(\theta_n(t))x_n) + f_n(t) \tag{13}$$

a.e. on $[0,T]$.

Moreover, since $|\theta_n(t) - t| \leq \frac{T}{n}$ for every $t \in [0,T]$, then $\theta_n(t) \to t$ uniformly on$[0,T]$.

By (11) we infer $||x_n^j - \varphi(0)|| \leq \frac{jT}{n}M < r_1$, proving that $x_n(t_n^j) = x_n^j \in B(\varphi(0), r_1)$ for every $j \in \{0, 1, ..., n\}$ and $n \geq 1$ and hence that

$$x_n(\theta_n(t)) \in B(\varphi(0), r_1) \tag{14}$$

for every $t \in [0,T]$and for every $n \geq 1$.

Now, by (h_3), (4), (6), (12), (14) and our choose of T we have

$$\begin{aligned} ||x_n(t) - \varphi(0)|| &\leq ||x_n(t) - x_n(\theta_n(t))|| + ||x_n(\theta_n(t)) - \varphi(0)|| \\ &\leq TM + \int_0^T ||f_n(s)||ds + r_1 \\ &= \int_0^T (m(s) + M) + r_1 < r \end{aligned}$$

and so $x_n(t) \in B(\varphi(0), r)$, for every $t \in [0,T]$ and for every $n \geq 1$.

Moreover, by (4) and (13) we have $||x_n'(t)|| \leq M + m(t)$ for every $t \in [0,T]$ and for every $n \geq 1$, hence $\int_0^T ||x_n'(t)||^2 dt \leq \int_0^T (M + m(t))^2 dt$ and therefore the sequence $(x_n')_n$ is bounded in $L^2([0,T], R^m)$.

For all $t, s \in [0,T]$, we have $\|x_n(t) - x_n(s)\| \leq \left|\int_s^t ||x_n'(\tau)|| d\tau\right| \leq |\int_0^T (M + m(\tau)) d\tau|$ so that the sequence $(x_n)_n$ is equiuniformly continuous.

Therefore, $(x_n')_n$ is bounded in $L^2([0,T], R^m)$ and $(x_n)_n$ is bounded in $\mathcal{C}([0,T], R^m)$ and equiuniformly continuous on $[0,T]$, hence, by Theorem 0.3.4 in [2], there exists a subsequence, still denoted by $(x_n)_n$, and an absolute continuous function $x : [0,T] \to R^m$ such that:

(i) $(x_n)_n$ converges uniformly on $[0,T]$ to x;

(ii) $(x_n')_n$ converges weakly in $L^2([0,T], R^m)$ to x'.

Moreover, since by (7) all functions x_n agree with φ on $[-\sigma, 0]$, we can obviously say that $x_n \to x$ on $[-\sigma, T]$, if we extend x in such a way that $x \equiv \varphi$ on $[-\sigma, 0]$. By the uniform convergence of x_n to x on $[0,T]$ and the uniform convergence of θ_n to t on $[0,T]$ we deduce that $x_n(\theta_n(t)) \to x(t)$ uniformly on $[0,T]$. Also, it is clearly that $T(0)x = \varphi$ on $[-\sigma, 0]$.

Further on, let us denote the modulus continuity of a function ψ defined on interval I of R by $\omega(\psi, I, \varepsilon) := \sup\{||\psi(t) - \psi(s)||; s, t \in I, |s - t| < \varepsilon\}, \varepsilon > 0$.

Then we have (see [6]):

$$\begin{aligned} ||T(\theta_n(t))x_n - T(t)x_n||_\infty &= -\sigma \leq s \leq 0 \sup ||x_n(\theta_n(t) + s) - x_n(t + s)|| \\ &\leq \omega(x_n, [-\sigma, T], \frac{T}{n}) \\ &\leq \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \omega(x_n, [0,T], \frac{T}{n}) \\ &\leq \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n} M, \end{aligned}$$

hence

$$||T(\theta_n(t))x_n - T(t)x_n||_\infty \leq \delta_n \tag{15}$$

for every $n \geq 1$, where $\delta_n := \omega(\varphi, [-\sigma, 0], \frac{T}{n}) + \frac{T}{n}M$.

Thus, by continuity of φ, we have $\delta_n \to 0$ as $n \to \infty$ and hence $\|T(\theta_n(t))x_n - T(t)x_n\|_\infty \to 0$ as $n \to \infty$ and since the uniform convergence of x_n to x on $[-\sigma, T]$ implies

$$T(t)x_n \to T(t)x \tag{16}$$

uniformly on $[-\sigma, 0]$,we deduce that

$$T(\theta_n(t))x_n \to T(t)x \tag{17}$$

in C_σ.

Now, we have to estimate $||(T(\theta_n(t))x_n)(s) - \varphi(s)||$ for each $s \in [-\sigma, 0]$. If $-\theta_n(t) \leq s \leq 0$, then $\theta_n(t) + s \geq 0$ and there exists $j \in \{0, 1, ..., n-1\}$ such that $\theta_n(t) + s \in [t_n^j, t_n^{j+1}]$.

Thus, by (5), (14) and by the fact that $|\theta_n(t) - t| \leq T$ and $|s| \leq T$, we have

$$||(T(\theta_n(t))x_n)(s) - \varphi(s)|| = ||x_n(\theta_n(t) + s) - \varphi(s)||$$

$$\leq ||x_n(\theta_n(t) + s) - \varphi(0)|| + ||\varphi(s) - \varphi(0)||$$

$$\leq r_1 + r_1 < r.$$

If $-\sigma \leq s \leq -\theta_n(t)$ then $s + \theta_n(t) \leq 0$ and by (5) we have

$$\|(T(\theta_n(t))x_n)(s) - \varphi(s)\| = \|x_n(\theta_n(t) + s) - \varphi(s)\| \leq r_1 < r.$$

Therefore, $T(\theta_n(t))x_n \in B(\varphi, r)$, for every $t \in [0, T]$ and for every $n \geq 1$ and so that, by (16), $T(t)x \in \overline{B}(\varphi, r) \subset \Omega$ on $[-\sigma, 0]$.

Further on, by (13) and (15) we have

$$d((T(t)x_n, x_n'(t) - f_n(t)), graph(F)) \leq \delta_n \tag{18}$$

for every $n \geq 0$.

By (h_3) and (16) we have $f_n(.) := f(., T(.)x_n) \to f(., T(.)x)$ in $L^2([0,T], R^m)$ and hence by (ii), (18) and Theorem 1.4.1 in [2] we obtain that

$$x'(t) \in co(T(t)x) + f(t, T(t)x) \tag{19}$$

a.e. on $[0,T]$,
where co stands for the closed convex hull.

By (h_2) we have that

$$x'(t) - f(t, T(t)x) \in \partial V(x(t)) \tag{20}$$

a.e. on $[0,T]$.

Since the functions $t \to x(t)$ and $t \to \partial V(x(t))$ are absolutely continuous, we obtain from Lemma 3.3 in [4] and (19) that

$$\tfrac{d}{dt}V(x(t)) = \langle x'(t), x'(t) - f(t, T(t)x)\rangle$$

a.e. on $[0,T]$;
therefore,

$$V(x(T)) - V(x(0)) = \int_0^T ||x'(t)||^2 dt - \int_0^T \langle x'(t), f(t, T(t)x)\rangle dt. \tag{21}$$

On the other hand, since
$x'_n(t) - f_n(t) \in F(T(t_n^j)x_n) \subset \partial V(x_n(t_n^j)) \forall t \in [t_n^j, t_n^{j+1}]$,
it follows that

$$\begin{aligned} V(x_n(t_n^{j+1})) - V(x_n(t_n^j)) &\geq \langle x'_n(t) - f_n(t), x_n(t_n^{j+1}) - x_n(t_n^j)\rangle = \\ \langle x'_n(t) - f_n(t), \int_{t_n^j}^{t_n^{j+1}} x'_n(t)dt\rangle &= \int_{t_n^j}^{t_n^{j+1}} ||x'_n(t)||^2 dt - \int_{t_n^j}^{t_n^{j+1}} \langle f_n(t), x'_n(t)\rangle dt. \end{aligned}$$

By adding the n inequalities from above, we obtain

$$V(x_n(T)) - V(x(0)) \geq \int_0^T ||x'_n(t)||^2 dt - \int_0^T \langle f_n(t), x'_n(t)\rangle dt \tag{22}$$

Thus, the convergence of $(f_n)_n$ in L^2-norm and of and $(x'_n)_n$ in the weak topology of L^2 implies that

$$\lim_{n\to\infty} \int_0^T \langle f_n(t), x'_n(t)\rangle dt = \int_0^T \langle f(t), x'(t)\rangle dt.$$

By passing to the limit for $n \to \infty$ in (22) and using the continuity of V, a comparison with (21), we obtain

$$||x'||^2_{L^2} \geq \limsup_{n\to\infty} ||x'_n||^2_{L^2}.$$

Since, by the weak lower semicontinuity of the norm,$|||x'||^2_{L^2} \leq \liminf_{n\to\infty} ||x'_n||^2_{L^2}$, we have that $||x'||^2_{L^2} = \lim_{n\to\infty} ||x'_n||^2_{L^2}$ i.e. $(x'_n)_n$ converges strongly in $L^2([0,T], R^m)$(see [5], Proposition III.30). Hence there exists a subsequence (again denote by) $(x'_n)_n$ which converges pointwiesely a.e. to x'.

Since by (h_1) the graph of F is closed and, by (18), $\lim_{n\to\infty} d((T(t)x_n, x'_n(t) - f_n(t)), graph(F)) = 0$, we obtain that $x'(t) \in F(T(t)x) + f(t, T(t)x) a.e. on [0,T]$ and so functional differential inclusion (1) does have solutions.

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