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APPLICATIONS OF THE DIRAC SEQUENCES IN MECHANICS

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Abstract. In the paper the manner in which certain Dirac sequences appear to the representation of the concentrated forces and moment by distributions is shown. The generalized solutions of the problems from elastostatics regarding the elastic help-plane and half-space are given with the help of the certain Dirac sequences.

1. Introduction

The Dirac sequences have important applications in the representation of the physical quantities with punctual support as well as in solving of boundary value problems from mathematical-physics.

We shall exemplify these ideas by writing in the distributions space D'(**R**) the concentrated force and momentum in a point, as well as the solutions of the boundary problems from elastostatics regarding the half-plane and half-space. We shall denote with D(**R**^{n}) the Schwartz's space of indefinitely differentiable functions with compact support, and with D'(**R**^{n}) the linear continuous functionals defined on D(**R**^{n}), named as L. Schwartz distributions.

Definition 1. Let $f_\varepsilon : R^n \to R, \varepsilon > 0$ be a family of locally integrable functions $f_\varepsilon \in L_{loc}(R^n)$. We say that the functions f_ε form a representative Dirac family or "Dirac sequence" if in the sense of the convergence of D'(**R**^{n}) we have

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = \delta(x) \qquad (1.1)$$

This means that $(\forall)\varphi \in D(R^n)$ we have

$$\lim_{\varepsilon \to 0} f_\varepsilon((x), \varphi(x)) = (\delta(x), \varphi(x)) = \varphi(0)\,. \qquad (1.2)$$

If, $f_\varepsilon \in C^\infty(R^n),$ then from (1.1) we obtain

$$\lim_{\varepsilon \to 0} D^a f_\varepsilon(x) = D^a \delta(x), \qquad (1.3)$$

where $D^a f_\varepsilon(x) = \frac{\partial^{|a|} f_\varepsilon(x_1,...x_n)}{\partial x_1^{a_1} \partial x_2^{a_2} \partial x_n^{a_n}}$ represents the partial derivative of the order $|\alpha| = a_1 + ... + a_n$ of the function f_ε.

2. General results

Continuous functions with certain properties allow the construction of „the Dirac sequences". Thus, according to [1], p. 163 we can state:

Proposition 2.1. Let $f \in C^0(R^n), f : R^n \to R$ be with the property $\int_{R^n} (x)dx = 1$. Thus, the family of functions $f_\varepsilon, \varepsilon > 0$, having the expression

$$f_\varepsilon(x) = \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right), x \in R^n, \varepsilon > 0, \tag{2.1}$$

form a "Dirac sequences", hence $\lim_{\varepsilon \to 0} f_\varepsilon(x) = \delta(x)$.

We consider the function, *f*: **R**$\to$**R,**

$$f(x) = \frac{1}{\pi} \frac{1}{1+x^2} \tag{2.2}$$

Obviously $f \in C^0(R)$ and we have

$$\int_{\mathbf{R}} f(x)dx = 1, \quad \left(\int_{\mathbf{R}} f dx = \frac{1}{\pi} \arctan x \Big|_{-\infty}^{\infty} = \frac{1}{\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = 1 \right)$$

Because the conditions of the proposition 2.1 are satisfied (n=1) we obtain the "Dirac
sequences"

$$f_\varepsilon(x) = \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}, \varepsilon > 0, x \in \mathbf{R}, \tag{2.3}$$

and thus we can write

$$\lim_{\varepsilon \to 0} f_\varepsilon(x) = \delta(x). \tag{2.4}$$

We observe that the family of functions f_ε, *>0* is of the class $C^\infty(\mathbf{R})$, so from (2.4) we
obtain

$$\lim_{\varepsilon \to 0} \frac{d^k}{dx^k} f_\varepsilon(x) = \delta^{(k)}(x). \tag{2.5}$$

Particularly, for 1 = *k* we have

$$\lim_{\varepsilon\to 0}\left[-\frac{2}{\pi}\frac{x\varepsilon}{\left(x^2+\varepsilon^2\right)^2}\right]=\delta'(x)\,. \quad (2.6)$$

From here we obtain

$$\lim_{\varepsilon\to 0}\left(-\frac{2}{\pi}\frac{x^2\varepsilon}{\left(x^2+\varepsilon^2\right)^2}\right)=-\delta(x) \quad (2.7)$$

Indeed, because $x\delta(x)=0$, by differentiation we obtain $x\delta'(x)=-\delta(x)$ and thus we have

$$\lim_{\varepsilon\to 0}\left(-\frac{2}{\pi}\frac{x^2\varepsilon}{\left(x^2+\varepsilon^2\right)^2}\right)=x\lim_{\varepsilon\to 0}\left(-\frac{2}{\pi}\frac{x\varepsilon}{\left(x^2+\varepsilon^2\right)^2}\right)=x\delta'(x)=-\delta(x)\,.$$

We introduce the function

$$h(x)=\frac{2}{\pi}\frac{1}{\left(x^2+1\right)^2},\ x\in\mathbf{R}$$

which has the property

$$\int_{\mathbf{R}}h(x)dx=1\,.$$

Consequently, on the basis of the proposition 2.1 we obtain

$$\lim_{\varepsilon\to 0}h_\varepsilon(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}h\left(\frac{x}{\varepsilon}\right)=\delta(x),$$

hence

$$\lim_{\varepsilon\to 0}\frac{2}{\pi}\frac{\varepsilon^3}{\left(x^2+\varepsilon^2\right)^2}=\delta(x)\,. \quad (2.8)$$

Taking into account (2.6) we obtain the relation

$$\lim_{\varepsilon\to 0}\left(-\frac{2}{\pi}\frac{x\varepsilon^2}{\left(x^2+\varepsilon^2\right)^2}\right)=0\,, \quad (2.6)$$

which is true both in the distributions and in classical sense.
Denoting by

$$g_\varepsilon(x)=-\frac{2}{\pi}\frac{x\varepsilon}{\left(x^2+\varepsilon^2\right)^2},\ \varepsilon>0$$

we can writen

$$\lim_{\varepsilon\to 0}g_\varepsilon(x)=\delta'(x) \quad (2.9)$$

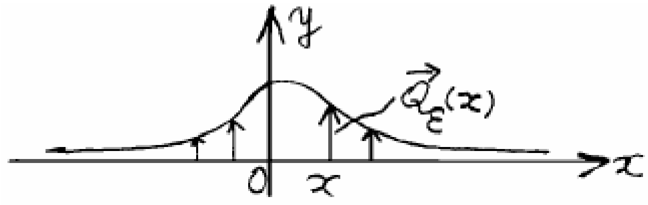

Fig. 1

The graph of the function $f_\varepsilon, \varepsilon > 0$ is represented in Fig. 1 and can be interpreted as the forces field $\vec{Q}_\varepsilon(x) = f_\varepsilon(x)\vec{j}$ which acts perpendicular on the axis Ox . The intensity of the field of forces $\vec{Q}_\varepsilon(x)$ on the unit of length is $f_\varepsilon(x)$. For the resultant vector $\vec{R\varepsilon}$ and resultant moment $\vec{M\varepsilon}$ with respect to the point *O* of the forces field we obtain

$$\vec{R}_\varepsilon = \int_{\mathbf{R}} \vec{Q}_\varepsilon(x)dx = \frac{\varepsilon}{\pi}\vec{j}\int_{\mathbf{R}} \frac{dx}{x^2+\varepsilon^2} = \vec{j},$$
$$\vec{M}_\varepsilon = \int_{\mathbf{R}} \left[x\,\vec{i} \times \vec{Q}_\varepsilon(x) \right] dx = \frac{\varepsilon}{\pi}\left(\vec{i}\times\vec{j}\right)\int_{\mathbf{R}} \frac{x\,dx}{x^2+\varepsilon^2} = 0. \quad (2.10)$$

From here it results that the action of unit force $\vec{j}$ applied perpendicular to O can be approximated with the action the forces field $\vec{Q}_\varepsilon$. This approximation from point of view of the mechanical effect, will be the better the smaller will be $\varepsilon > 0$.

Consequently, by definition the limit in D'(**R**)namely

$$\lim_{\varepsilon \to 0} \vec{Q}_\varepsilon(x) = \frac{\varepsilon}{\pi}\vec{j}\lim_{\varepsilon \to 0}\frac{\varepsilon}{x^2+\varepsilon^2},$$

represents the mathematical expression of the unit force $\vec{j}$ applied in the point *O*. Taking into account (2.4) we obtain

$$\lim_{\varepsilon \to 0} \vec{Q}_\varepsilon(x) = \vec{j}\,\delta(x). \quad (2.11)$$

If instead of the force $\vec{j}$ applied in *O* we have the force P $\vec{j}$, then it can be represented by the distribution

$$P\vec{j}\,\delta(x) = P\vec{j}\lim_{\varepsilon \to 0}\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}. \quad (2.12)$$

We shall next show that the distribution (2.6), from mechanical point of view, describes the action of a unit concentrated couple applied in origin and with the clockwise sense of rotation (Fig.2), hence

$$\lim_{\varepsilon\to 0}\vec{j}\left(-\frac{2}{\pi}\frac{x\varepsilon}{\left(x^2+\varepsilon^2\right)^2}\right)=\vec{j}\,\delta'(x).$$
(2.13)

With this end in view we shall admit that the following parallel forces field acts perpendicularly on the axis *Ox*

$$\vec{q}_\varepsilon(x)=\frac{d}{dx}\vec{Q}_\varepsilon(x)=f'_\varepsilon(x)\,\vec{j}.$$
(2.14)

(Fig. 2) where $q_\varepsilon(x)$ has the expression (2.9).

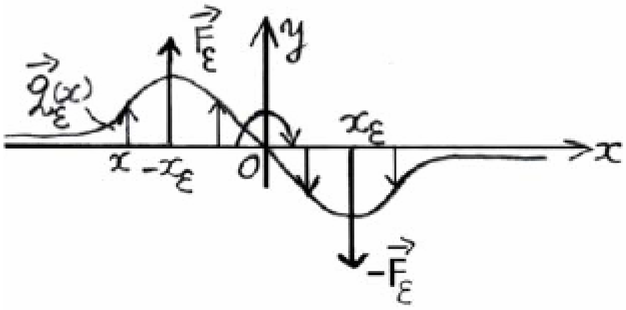

Fig. 2

Let $\vec{F}_\varepsilon$ and $-\vec{F}_\varepsilon$ be the resultant of the parallel forces corresponding to the semi-straight line $(-\infty,0]$ and $(\infty,0]$. These forces act in the points $-x_\varepsilon$ and x_ε, which represent the abscissas of the center of the parallel forces from the two semi-straight lines. The ensemble of forces ($\vec{F}_\varepsilon$,-$\vec{F}_\varepsilon$) determines a couple whose moment has the value $M_\varepsilon=2x_\varepsilon F_\varepsilon$
> >
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is clockwise.

For the values and we obtain the expression

$$F_\varepsilon=-\int_0^\infty q_\varepsilon(x)dx=\frac{2\varepsilon}{\pi}\int_0^\infty\frac{xdx}{\left(x^2+\varepsilon^2\right)^2}=\frac{1}{\varepsilon\pi},\quad x_\varepsilon=\frac{1}{\int_0^\infty q_\varepsilon(x)dx}\cdot\int_0^\infty xq_\varepsilon(x)dx=\frac{\pi\varepsilon}{2}$$

because

$$\int_0^\infty x q_\varepsilon(x)dx = -\frac{2\varepsilon}{\pi}\int_0^\infty \frac{x^2 dx}{(x^2+\varepsilon^2)^2} = -\frac{2\varepsilon}{\pi}\cdot\frac{\pi}{4\varepsilon} = -\frac{1}{2}.$$

In consequence, for the moment $M_\varepsilon=2x_\varepsilon F_\varepsilon$ of the couple $(\vec{F}_\varepsilon, -\vec{F}_\varepsilon)$ we obtain the value $M_\varepsilon=1$, irrespective of the parameter $\varepsilon>0$. From mechanical point of view the action of the couple $(\vec{F}_\varepsilon, -\vec{F}_\varepsilon)$ in the distributions space. **D**'(**R**) will be represented by the action of the load

$$\vec{h}_\varepsilon(x) = \vec{F}_\varepsilon\,\delta(x+x_\varepsilon) - \vec{F}_\varepsilon\,\delta(x-x_\varepsilon) = \frac{1}{\varepsilon\pi}\vec{j}\left[\delta\left(x+\frac{\pi\varepsilon}{2}\right)-\delta\left(x-\frac{\pi\varepsilon}{2}\right)\right]$$

By definition the limit in D'(**R**), namely $\vec{q}\,(x) = \lim_{\varepsilon\to 0}\vec{h}_\varepsilon(x)$, is named concentrated moment in the origin, has the value equal with unit and its acts clockwise.

Therefore, for the concentrated moment $\vec{q}\,(x)$ we obtain the expression

$$\vec{q}(x) = \lim_{\varepsilon\to 0} h_\varepsilon(x) = \vec{j}\lim_{\varepsilon\to 0}\frac{\delta\left(x+\frac{\pi\varepsilon}{2}\right)-\delta\left(x-\frac{\pi\varepsilon}{2}\right)}{\pi\varepsilon}$$

For the limit calculus we shall use the definition of the derivative of the distribution $f \in D'(R)$, namely

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}.$$

Consequently, we obtain

$$\vec{q}(x) = \vec{j}\frac{1}{2}\lim_{\varepsilon\to 0}\frac{\delta\left(x+\frac{\pi\varepsilon}{2}\right)-\delta(x)}{\frac{\pi\varepsilon}{2}} + \vec{j}\frac{1}{2}\lim_{\varepsilon\to 0}\frac{\delta\left(x-\frac{\pi\varepsilon}{2}\right)-\delta(x)}{-\frac{\pi\varepsilon}{2}} = \vec{j}\frac{1}{2}\delta'(x)+\vec{j}\frac{1}{2}\delta'(x) = \vec{j}\delta'(x),$$

hence

$$\vec{q}(x) = \vec{j}\,\delta'(x) \qquad (2.15)$$

This expression constitutes the representation in the distributions space D'(**R**)of a concentrated moment of unit value applied in the point *O*, which determines a clockwise rotation. In the case when the value of the concentrated moment is M>0, then this will be represented as in Fig. 3 and its expression will be $\vec{q} = M\,\vec{j}\,\delta'(x)$.

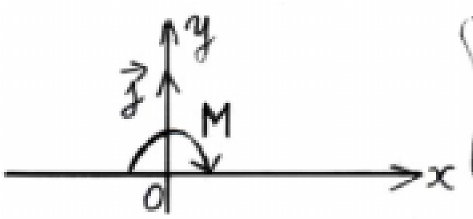

Fig. 3

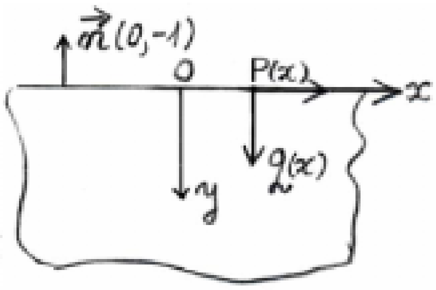

Fig. 4

Elastic half-plane

We consider the homogeneous, isotropic elastic half-plane with respect to the orthogonal reference system *Oxy* and we will denote with p(x),q(x)$\in$ D(**R)** the load expresses by distributions which act on the boundary *Ox* of the elastic half-plane, about the axes *Ox* and *Oy* (Fig.
4). As a result of the action of the loads *p* and *q*, in every point (x,y) $\in$ R$\in R\times R^{+}$ from the interior of the half-plane, a stress state is created characterized by the stress matrix $(\sigma)=\begin{pmatrix}\sigma_{xx}\\ \sigma y\\ \sigma_{yy}\end{pmatrix}$. The normal stresses $\sigma_{xx}(x,y),\sigma_{yy}(x,y)$ and the tangential stresses $\sigma_{xx}(x,y),\sigma_{yy}(x,y)$ are considered distributions from D'(**R**) with respect to the variable x$\in$ **R,** depending on the parameter *y* >0.

Taking into account [1], we call generalized problem in stresses for the elastic half-plane, the determination of the distributions (σ_{xx}, σ_{yy}, σ_{xy} $\in D'$(**R**) **R** depending on the parameter y>0, and which verify the equilibrium equations

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{xy}}{\partial y}=0,\ \frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}=0 \qquad (2.16)$$

continuity equations

$$\Delta\left(\sigma_{xx}+\sigma_{yy}\right)=0,\ (x,y)\in\mathbf{R}\times\mathbf{R}^{+},\ \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}, \qquad (2.17)$$

and the boundary conditions (the conditions on the boundary *Ox*)

$$\lim_{y\to+0}\sigma_{xy}(x,y)=-p(x),\ \lim_{y\to+0}\sigma_{yy}(x,y)=-q(x),\ x\in\mathbf{R} \qquad (2.18)$$

We introduce the following matrix to write the solution of the above problem

$$(U)=\begin{pmatrix} u_{11}u_{12} \\ u_{21}u_{22} \\ u_{31}u_{32} \end{pmatrix}, \quad (Q)=\begin{pmatrix} p \\ q \end{pmatrix}, \tag{2.19}$$

named the matrix of the fundamental solution corresponding to the equations (2.16), (2.17), (2.18) and the matrix of the loads which act on the elastic half-plane, respectively.

The generalized solution in stresses of the elastic half-plane has the expression

$$(\sigma)=(U)*(Q) \tag{2.20}$$

where the symbol „*" represents the convolution product with respect to $x\in R$, and the function type distributions u_{ij} $(x,y),(x,y) \in \mathbf{R}\times\mathbf{R}^{+}$ $i=1,2,3$, $j=1,2$ corresponding to the matrix (u) have the expressions

$$u_{11}=-\frac{2}{\pi}\frac{x^3}{\left(x^2+y^2\right)^2},\ u_{21}=u_{12}=-\frac{2}{\pi}\frac{x^2y}{\left(x^2+y^2\right)^2}$$
$$u_{31}=u_{22}=-\frac{2}{\pi}\frac{y^2x}{\left(x^2+y^2\right)^2},\ u_{32}=-\frac{2}{\pi}\frac{y^3}{\left(x^2+y^2\right)^2}. \tag{2.21}$$

By direct calculus the relations are verified

$$\frac{\partial u_{11}}{\partial x}+\frac{\partial u_{21}}{\partial y}=\frac{\partial u_{12}}{\partial x}+\frac{\partial u_{22}}{\partial y}=\frac{\partial u_{21}}{\partial x}+\frac{\partial u_{31}}{\partial y}=\frac{\partial u_{22}}{\partial x}+\frac{\partial u_{32}}{\partial y}=0,$$
$$\Delta\left(u_{11}+u_{31}\right)=\Delta\left(u_{12}+u_{32}\right)=0, \tag{2.22}$$

where the derivatives are considered in the distributions sense, and being a parameter. $0>y$

We observe that in the basis of the formulas (2.6'), (2.7) and (2.8) we have

$$\lim_{y\to+0}u_{21}(x,y)=\lim_{y\to+0}\left[-\frac{2}{\pi}\frac{x^2y}{\left(x^2+y^2\right)^2}\right]=-\delta(x),$$
$$\lim_{y\to+0}u_{32}(x,y)=\lim_{y\to+0}\left[-\frac{2}{\pi}\frac{y^3}{\left(x^2+y^2\right)^2}\right]=-\delta(x),$$
$$\lim_{y\to+0}u_{22}(x,y)=\lim_{y\to+0}u_{31}(x,y)=\left(\lim_{y\to+0}y\right)\left[\lim_{y\to+0}\left(-\frac{2}{\pi}\frac{xy}{\left(x^2+y^2\right)^2}\right)\right]=0\cdot\delta'(x)=0. \tag{2.23}$$

Taking into account (2.22) it results that the equations (2.16) and (2.17) are satisfied. We shall show that the boundary conditions (2.18) are satisfied. Indeed, from (2.20) results

$$\lim_{y\to+0}\sigma_{xy}(x,y)=\left(\lim_{y\to+0}u_{21}(x,y)\right)*p(x)+\left(\lim_{y\to+0}u_{22}(x,y)\right)*q(x),$$
$$\lim_{y\to+0}\sigma_{yy}(x,y)=\left(\lim_{y\to+0}u_{31}(x,y)\right)*p(x)+\left(\lim_{y\to+0}u_{32}(x,y)\right)*q(x). \qquad (2.24)$$

On the basis of the formulas (2.23) we obtain

$$\lim_{y\to+0}\sigma_{xy}(x,y)=-\delta(x)*p(x)=-p(x),\ \lim_{y\to+0}\sigma_{yy}(x,y)=-\delta(x)*q(x)=-q(x),$$

which shows that the boundary conditions are satisfied. Hence, the generalized solution in stresses of the half-plane is given by the matrix equality (2.20).

The elastic half-space

The generalized problem in displacements of the isotropic homogeneous elastic half-space consists in determination of the displacements u(x,y,z), v(x,y,z) $\in$ D'(**R**^{2}**)**, z>0 being a parameter that verifies the equilibrium equations

$$(\lambda+\mu)\frac{\partial\varepsilon_v}{\partial x}+\mu\Delta u=0,\ (\lambda+\mu)\frac{\partial\varepsilon_v}{\partial y}+\Delta v=0,\ (\lambda+\mu)\frac{\partial\varepsilon_v}{\partial z}+\Delta w=0, \qquad (2.25)$$

and the boundary conditions

$$\lim_{z\to+0}\sigma_{xz}(x,y,z)=-p(x,y),$$
$$\lim_{z\to+0}\sigma_{yz}(x,y,z)=-q(x,y),$$
$$\lim_{z\to+0}\sigma_{zz}(x,y,z)=-r(x,y), \qquad (2.26)$$

where p,q,r$\in$*D'(***R**^{2}**)**

In the equations (2.25) λ and μ represent the Lamé's elastic constants,

$$\varepsilon_v=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

the volume deformation, and

$$\Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$$

the Laplace's operator.

We mention that the quantities *r q p* that appear in the equation (2.26) represent the loads on the three coordinate axes written by distributions from D'(**R**^{2}) which act on the boundary *Oxy* of the elastic half-space (Fig. 5).

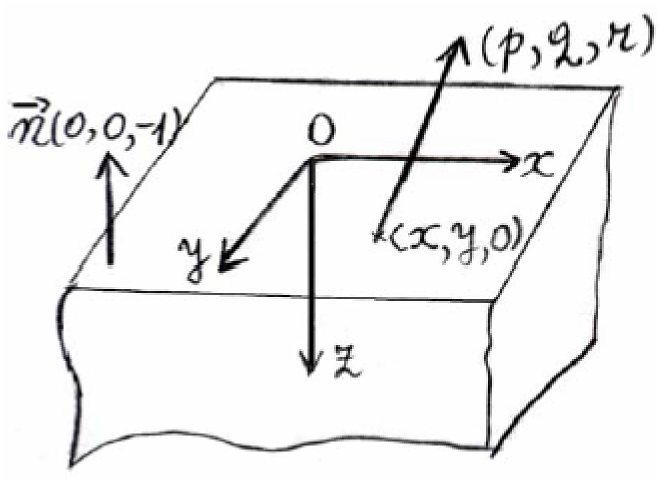

Fig. 5

The quantities σ_{zz} , σ_{yz}, σ_{xz} represent, obviously, stresses on a point (x,y,z),z>0 from the interior of the elastic half-space expressed by the distributions from D'($\mathbf{R}^2$), depending by the parameter z>0.

For the determination of the solution of the formulated problem the functions (the function type distributions) play an important part [3]

$$f(x,y,z)=\frac{1}{\sqrt{x^2+y^2+z^2}},\ \overline{f}(x,y,z)=\ln\left(\sqrt{x^2+y^2+z^2}+z\right),\ z>0 \tag{2.27}$$

We observe that these are harmonic functions and satisfy the relations

$$\overline{f}_z{}'=f,\ \Delta f=\Delta\overline{f}=0,\ z>0$$

In the construction of the solution of the elastic half-space problem arises the family of functions

$$h_\varepsilon:\mathbf{R}^2\to\mathbf{R},\ h_\varepsilon(x,y)=\frac{p-2}{2\pi}\frac{\varepsilon^{p-2}}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{p}{2}}},\ \varepsilon>0,\ p>2. \tag{2.28}$$

In consequence of [1], p.167 this family constitute a „Dirac sequence", hence

$$\lim_{\varepsilon\to+0}\frac{p-2}{2\pi}\frac{\varepsilon^{p-2}}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{p}{2}}}=\delta(x,y),\ \varepsilon>0,\ p>2 \tag{2.29}$$

Particularly, for 3 p =3 and we obtain p =5

$$\lim_{\varepsilon\to+0}\frac{\varepsilon}{2\pi}\frac{1}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{3}{2}}}=\delta(x,y),$$

$$\lim_{\varepsilon\to+0}\frac{3\varepsilon^3}{2\pi}\frac{1}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{5}{2}}}=\delta(x,y).$$

(2.30)

Taking into account (2.30) by differentiation it results the relations

$$\delta'_x(x,y)=\lim_{\varepsilon\to+0}\left[-\frac{1}{2\pi}\frac{3x\varepsilon}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{5}{2}}}\right],$$

$$\delta'_y(x,y)=\lim_{\varepsilon\to+0}\left[-\frac{1}{2\pi}\frac{3y\varepsilon}{\left(x^2+y^2+\varepsilon^2\right)^{\frac{5}{2}}}\right].$$

(2.31)

Also, because xδ(x,y)=0 and yδx, y=0 by differentiation of these relations we obtain

$$x\delta'_y(x,y)=0,\quad y\delta'_x(x,y)=0$$

(2.32)

We observe that on the basis of the relation (2.30) we can write

$$\lim_{z\to+0}\frac{1}{2\pi}f'_z=-\delta(x,y),\ \lim_{z\to+0}zf''_{x^2}=\lim_{z\to+0}zf''_{y^2}=\lim_{z\to+0}zf''_{z^2}=0.$$

(2.33)

Also, both in the classic sense and in the distribution sense hold the relations

$$\lim_{z\to+0}zf''_{xy}=\lim_{z\to+0}zf''_{yz}=\lim_{z\to+0}zf''_{xz}=0.$$

(2.34)

Thus, for example, we have

$$zf''_{x^2}=-2\pi x\left[\frac{-3xz}{2\pi\left(x^2+y^2+z^2\right)^{5/2}}\right]+2\pi\left(\frac{1}{2\pi}f'_z\right)$$

Taking into account (2.31) we obtain

$$\lim_{z\to+0}zf''_{x^2}=-2\pi x\lim_{z\to+0}\frac{-3xz}{2\pi\left(x^2+y^2+z^2\right)^{5/2}}+2\pi\lim_{z\to+0}\left(\frac{1}{2\pi}f'_z\right)=-2\pi x\delta'_x-2\pi\delta=-2\pi\left(x\delta'_x+\delta\right)=0$$

Analogously, on the basis of the relations (2.30) and (2.31) the other two relations are proved.

We introduce the matrices of distributions

$$(u)=\begin{pmatrix} u \\ v \\ w \end{pmatrix},\ (U)=-\frac{1}{4\pi\mu}\begin{pmatrix} u_1\ u_2\ u_3 \\ v_1\ v_2\ v_3 \\ w_1\ w_2\ w_3 \end{pmatrix},\ (Q)=\begin{pmatrix} p \\ q \\ r \end{pmatrix} \qquad (2.35)$$

where $u,v,w,u_i,v_i,w_i,i=1,2,3$ are distributions from $D'(\mathbf{R}^2)$ depending by the parameter $z>0$, and p,q,r are distributions from $D'(\mathbf{R}^2)$. The matrices (u)(U),(Q) are called the displacement matrix, the fundamental solution matrix of the equations (2.25), (2.26) and the matrix of the loads which act on the elastic half-space.

According to [2], the generalized solution of the elastic half-space can be written with the help of the convolution product „*” with respect the variable $(x,y)\in\mathbf{R}^2$ under the form

$(u)=(U)*(Q),$ (2.36

where u_i,v_i and $w_i,i=1,2,3$ have the expressions

$$u_1=-2f+z\overline{f}_{x^2}+2\nu\int_z^\infty\overline{f}_{x^2}dz,\ u_2=z\overline{f}_{xy}+2\nu\int_z^\infty\overline{f}_{xy}dz,\ u_3=zf'_x+(1-2\nu)\overline{f}_x$$

$$v_1=z\overline{f}_{xy}+2\nu\int_z^\infty\overline{f}_{xy}dz,\ v_2=-2f+z\overline{f}_{y^2}+2\nu\int_z^\infty\overline{f}_{y^2}dz,\ v_3=zf'_y+(1-2\nu)\overline{f}_y,$$

$$w_1=-(1-2\nu)\overline{f}_x+zf'_x,\ w_2=-(1-2\nu)\overline{f}_y+zf'_y,\ w_3=zf'_z-2(1-\nu)f \qquad (2.37)$$

$$w_1=-(1-2\nu)\overline{f}_x+zf'_x,\ w_2=-(1-2\nu)\overline{f}_y+zf'_y,\ w_3=zf'_z-2(1-\nu)f$$

where represents the Poisson’s coefficient. Between the constants λ,μ and ν there is the relation

$$\lambda+\mu=\frac{\mu}{1-2\nu}.$$

Concerning the stresses σ_{yz} σ_{xz}, and σ_{zz} these have the expressions

$$\sigma_{xz}=-\frac{1}{2\pi}\left[\delta_1^1*p+\delta_2^1*q+\delta_3^1*r\right],$$

$$\sigma_{yz}=-\frac{1}{2\pi}\left[\delta_1^2*p+\delta_2^2*q+\delta_3^2*r\right],$$

$$\sigma_{zz}=-\frac{1}{2\pi}\left[\delta_1^3*p+\delta_2^3*q+\delta_3^3*r\right]. \qquad (2.38)$$

where the functions type distributions has the expressions 3 , 2 , 1 , , = *j i j*

$$\begin{aligned}
&\delta_1^1 = -\left(f_z' - zf_{z^2}''\right),\ \delta_2^1 = zf_{xy}'',\ \delta_3^1 = zf_{xz}'',\\
&\delta_1^2 = zf_{xy}'',\ \delta_2^2 = -\left(f_z' - zf_{y^2}''\right),\ \delta_3^2 = zf_{yz}'', \qquad (2.39)\\
&\delta_1^3 = zf_{xz}'',\ \delta_2^3 = zf_{yz}'',\ \delta_3^3 = -\left(f_z' - zf_{z^2}''\right).
\end{aligned}$$

By direct calculus it is established that the matrix (u) given by (2.36) verifies the equilibrium equations (2.25).

To verify the boundary conditions (2.26), taking into account (2.38), (2.39) and (2.33), (2.34) we have

$$\lim_{z\to+0}\sigma_{xz} = -\frac{1}{2\pi}\left[\lim_{z\to+0}\delta_1^1 * p + \lim_{z\to+0}\delta_2^1 * q + \lim_{z\to+0}\delta_3^1 * r\right] = \left[-\delta(x,y) * p(x,y)\right] = -p(x,y).$$

Analogously, the other two boundary conditions are verified.

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