$Q \to \infty$. Denoting by

$$\varphi(\textit{conditions}) = \begin{cases} 1, & \text{if } u, v \text{ satisfy all } \textit{conditions} \\ 0, & \text{else} \end{cases},$$

the characteristic function for a given set of *conditions* (equalities or inequalities in variables $u$ and $v$), the density $g(u,v)$ is given by:

$$\begin{aligned} g(u,v) =& \varphi\big(1 < u + 2v,\ 1 < 2u + v\big) + \sum_{k=2}^{\min\left(\frac{u+v}{1-v}, \frac{u+v}{1-u}\right)} \frac{1}{k} \\ & + \frac{1}{2}\,\varphi\big(2u + v = 1,\ \text{if } 0 < u < 1/3 \\ & \qquad\qquad \text{or } \ u + 2v = 1,\ \text{if } 1/3 < u < 1\big) \\ & + \frac{1}{2k}\,\varphi\big(k = \tfrac{u+v}{1-v} \geq 2,\ \ \text{if } \tfrac{k}{k+2} < u < 1, \\ & \qquad\qquad \text{or } k = \tfrac{u+v}{1-u} \geq 2,\ \ \text{if } \tfrac{k-1}{k+1} < u < \tfrac{k}{k+2}\big) \\ & + \frac{k+2}{4k(k+1)}\,\varphi\big(u = v = \tfrac{k}{k+2},\ k \geq 1\big)\,. \end{aligned}$$

(Here $k$ is a positive integer.) The function $g(u,v)$ is locally constant on a subset of the unit square $(0,1)^2$ with measure 1. Its shape is that of a stairway ascending to infinity as an harmonic sum at $(1,1)$.

In Figures 4.1 and 4.2 one can compare the distribution of neighbor odd denominators for a small $Q$ with the density on the superimposed quadrangles $\mathcal{V}_k$. In the case of the even denominators the corresponding distribution has a prominent density for even small values of $Q$ (see Figure 4.3).

## 5. **Other facets of Farey fractions**

Here we present other aspects of the gaps between Farey fractions. The first one refers to the frequency with which different gaps appear, the second is concerned with the distribution of the index of members of $\mathfrak{F}_Q$ and the third one deals, more generally, with the distribution of angles of all the lines determined by the origin with lattice points inside a domain in $\mathbb{R}^2$.

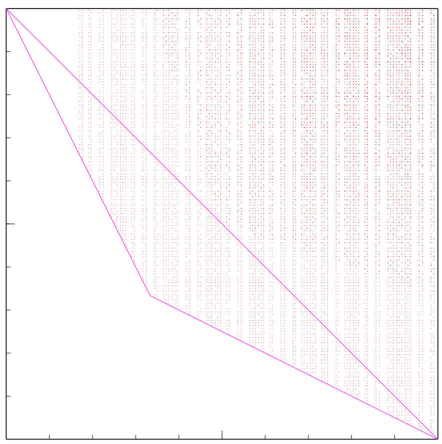

Figure 4.1: The set $\mathcal{C}_{500,1,2}$.

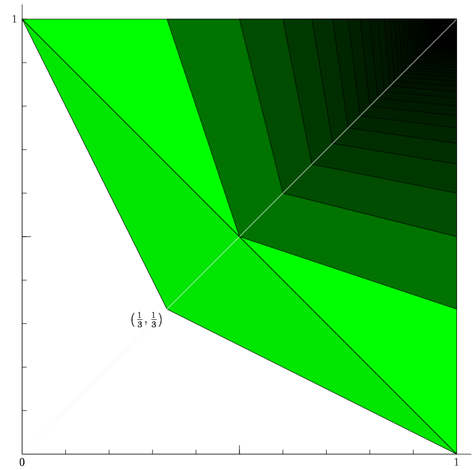


Figure 4.2: The set $\mathcal{O} = \mathcal{T} \cup \bigcup_{k=1}^{\infty} \mathcal{V}_k$.

5.1. **Jumping Champions of $\mathfrak{F}_Q$.** Finding the maximum or the minimum of a certain sequence may be both interesting and difficult. Two remarkable examples require to find $X_n$, the length of the longest increasing subsequence of a random permutation of the integers from 1 to $n$, and respectively to find $\lambda_n$, the largest eigenvalue of a $n \times n$ random GUE matrix. Recently in [BDJ1999] and [TW2000] it was shown that $X_n$ and $\lambda_n$ share the same distribution and in [RW2002] are presented many other appearances of the same distribution in limit theorems from widely different areas.

Being confined both in magnitude to the interval $[0, 1]$ and in size of the denominators to $[1, Q]$, the definition of $\mathfrak{F}_Q$ makes trivial the question on how large is the maximum or the minimum spacing between neighbor fractions. Of course, these are $1/Q$ and $1/Q(Q-1)$, respectively, attained by the distances of the fraction $(Q-1)/Q$ to its neighbors. A finer way to analyze the distribution of the gaps is to find the frequency with which they appear. The most encountered difference is called the *jumping champion*, or shortly `LJC`. For the Farey sequence Cobeli, Ford and Zaharescu [CFZ2002] have estimated the size of a set of possible candidates for `LJC` and studied the arithmetic structure of `LJC`.

For a set $\mathcal{M} = \{\gamma_1, \ldots, \gamma_M\}$ of real numbers ordered increasingly, let $D(\mathcal{M}) = \{\gamma_{i+1} - \gamma_i \; : \; 1 \le i \le M-1\}$ be the set of gaps between consecutive elements. The elements of $D(\mathcal{M})$ are arranged in ascending order, keeping in the list all the differences with their mulitplicities. The *champion of* $\mathcal{M}$ is the element of $D(\mathcal{M})$ with the highest multiplicity. When more numbers have the same highest multiplicity in $D(\mathcal{M})$,

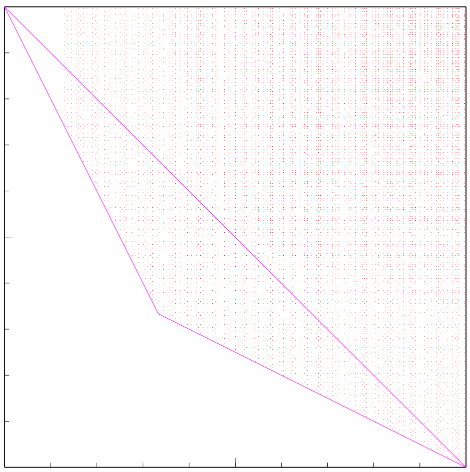

Figure 4.3: The set $\mathcal{C}_{500,0,2}$.

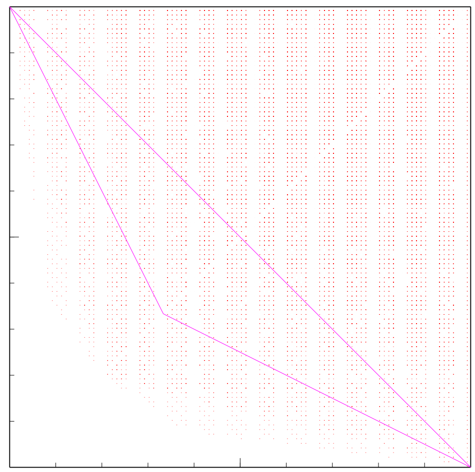

Figure 4.4: The set $\mathcal{C}_{1200,3,12}$.

those numbers share the position of champion.

Depending on $\mathcal{M}$, finding the LJC of a set may prove a complex task. Certainly this is the case when $\mathcal{M} = \mathcal{P}_n$, the set of primes less than or equal to $n$. Odlyzko, Rubinstein and Wolf [ORW1999] give empirical and heuristic evidences based on the $r$-tuple conjecture that the LJC for primes are the primorials $6,\ 30,\ 210,\ 2310, \dots$, except for a few small values of $n$, when LJC are $1$, $2$ or $4$. Similarities seem to exist between the arithmetical properties of the elements of $D(\mathcal{P}_n)$ and $D(\mathfrak{F}_Q)$, but in the case of Farey fractions the results are proved unconditionally.

Since $\mathfrak{F}_Q$ is symmetric with respect to $1/2$, we take $\mathcal{M}_Q = \mathfrak{F}_Q \cap [0, 1/2]$. Then $|D(\mathcal{M}_Q)| = (|\mathfrak{F}_Q| - 1)/2$. For $Q$ small, $1 \le Q \le 9$, all the elements of $D(\mathcal{M}_Q)$ are distinct, so they all share the position of champion. For $Q = 10$, we have

$$\mathcal{M}_{10} = \left\{0, \frac{1}{10}, \frac{1}{9}, \frac{1}{8}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{2}{9}, \frac{1}{4}, \frac{2}{7}, \frac{3}{10}, \frac{1}{3}, \frac{3}{8}, \frac{2}{5}, \frac{3}{7}, \frac{4}{9}, \frac{1}{2}\right\}$$

and

$$D(\mathcal{M}_{10}) = \left\{\frac{1}{10}, \frac{1}{90}, \frac{1}{72}, \frac{1}{56}, \frac{1}{42}, \frac{1}{30}, \frac{1}{45}, \frac{1}{36}, \frac{1}{28}, \frac{1}{70}, \frac{1}{30}, \frac{1}{24}, \frac{1}{40}, \frac{1}{35}, \frac{1}{63}, \frac{1}{18}\right\}.$$

The gap $1/30$ appears twice in $D(\mathcal{M}_{10})$, so it is the LJC of $\mathcal{F}_{10}$. In Table 1 are listed some champions with a record number of appearances through different values of $Q \le 400$, in other words 'Champs among champions'. One can see that a condition for a number to be a champion is to be highly composite.

For any two consecutive elements of $\mathfrak{F}_Q$, say $\frac{a'}{q'}$, $\frac{a''}{q''}$, the gap between them is $\frac{a''}{q''} - \frac{a'}{q'} = \frac{1}{q'q''}$, so the basic set to look at is $\mathcal{T}_Q$, the set of the pairs of consecutive denominators of Farey fractions. This is given by

$$\mathcal{T}_Q = \big\{(q_1, q_2) \,:\, 1 \le q_1, q_2 \le Q,\ q_1 + q_2 > Q,\ \gcd(q_1, q_2) = 1\big\}\,.$$

Table 1: Selected champions with a record number of appearances.

| 1/Champion | Decomposition | No. of appearances | The values of Q |
|---|---|---|---|
| **6** | $2 \cdot 3$ | 3 | 3, 4, 6 |
| **12** | $2^2 \cdot 3$ | 4 | 4–6, 12 |
| **30** | $2 \cdot 3 \cdot 5$ | 7 | 6–12 |
| **70** | $2 \cdot 5 \cdot 7$ | 7 | 11–17 |
| **210** | $2 \cdot 3 \cdot 5 \cdot 7$ | 10 | 17, 21–28, 30 |
| **390** | $2 \cdot 3 \cdot 5 \cdot 13$ | 6 | 30–32, 39–41 |
| **420** | $2^2 \cdot 3 \cdot 5 \cdot 7$ | 12 | 28–32, 35–41 |
| **546** | $2 \cdot 3 \cdot 7 \cdot 13$ | 10 | 41–50 |
| **840** | $2^3 \cdot 3 \cdot 5 \cdot 7$ | 11 | 41, 47–50, 55–60 |
| **1260** | $2^2 \cdot 3^2 \cdot 5 \cdot 7$ | 17 | 47–50, 55, 59–70 |
| **2310** | $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ | 27 | 70–96 |
| **6930** | $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$ | 43 | 126–168 |
| **8190** | $2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 13$ | 38 | 130–153, 167–180 |
| **10010** | $2 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 40 | 167–206 |
| **18018** | $2 \cdot 3^2 \cdot 7 \cdot 11 \cdot 13$ | 70 | 201–270 |
| **30030** | $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ | 87 | 231–233, 269–352 |
| **39270** | $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$ | 56 | 269–272, 349–400 |

We denote by $h(D, Q)$ the number of gaps of length $1/D$ in $\mathcal{M}_Q$. Then $h(D, Q)$ is the multiplicity of $1/D$ in $D(\mathcal{M}_Q)$, and this can be written as:

$$\begin{aligned} h(D, Q) =& \Big|\big\{(q_1, q_2) \in \mathcal{T}_Q \,:\, q_1 q_2 = D,\ q_1 < q_2\big\}\Big| \\ =& \Big|\big\{q | D \,:\, \gcd(q, \tfrac{D}{q}) = 1,\ \tfrac{D}{q} < q \le Q,\ \tfrac{D}{q} + q > Q\big\}\Big|\,. \end{aligned}$$

This gives a partition of $D(\mathcal{M}_Q)$, therefore

$$\big|D(\mathcal{M}_Q)\big| = \sum_{D \ge 1} h(D, Q)\,.$$

Then any champion verifies:

$$M(Q) = \max_D h(D, Q)\,.$$

The size of LJC is given by

$$M(Q) = \exp\left(2\log 2 \frac{\log Q}{\log\log Q} + O\left(\frac{\log Q}{(\log\log Q)^2}\right)\right)\,.$$

Next we look at the set of champions (strictly speaking inverses of champions):

$$\mathrm{Champs}(Q) = \{D : \ h(D,Q) = M(Q)\}.$$

The characteristic of $\mathrm{Champs}(Q)$ is that its elements have close to the maximum possible number of prime factors for integers of their size. Thus, if $D \in \mathrm{Champs}(Q)$ then

$$\omega(D) = 2\frac{\log Q}{\log\log Q} + O\left(\frac{\log Q}{(\log\log Q)^2}\right),$$

where $\omega(n)$ is the number of distinct prime factors of $n$. As a consequence, most of the prime factors of a champion are small. Indeed, one can show that the largest prime factor of any $D \in \mathrm{Champs}(Q)$, is $\ll (\log Q)^3$.

Various other results related to the champions of $\mathfrak{F}_Q$ are proved in [CFZ2002]. Specifically, it is studied $H(Q)$, the number of distinct gaps (participants in the competition for `LJC` ) and $G_r(Q)$, the number of gaps with multiplicity $\geq r$. For $H(Q)$ it is shown that

$$\frac{Q^2}{(\log Q)^\theta}\exp\left(-c_1\sqrt{\log\log Q\log\log\log Q}\right) \ll H(Q) \ll \frac{Q^2}{(\log Q)^\theta}\cdot\frac{1}{\sqrt{\log\log Q}},$$

where $\theta = 1 - \frac{1+\log\log 2}{\log 2}$ and $c > 0$ is constant. Depending on $r$, different bounds for $G_r(Q)$ are also proved. An example is one given in a closed form:

$$G_r(Q) = \frac{Q^2}{(\log Q)^{\theta+o(1)}},$$

which is valid when $r = e^{o\left(\frac{\log\log Q}{\log\log\log Q}\right)}$.

5.2. **Moments of the index of Farey fractions.** Here we present some asymptotic formulae concerning the distribution of the index of Farey fractions of order $Q$ as $Q \to \infty$. First let us have a closer look at the definition of the index. With the notations introduced in Section 3.4, one sees that $T(x,y) = (y, ky - x)$ for any $(x,y) \in \mathcal{T}_k$. Starting with $k_1(x,y) = \left[\frac{1+x}{y}\right]$, for $j \geq 2$ one has recursively

$$k_j(x,y) = (k_{j-1}\circ T)(x,y).$$

Also, for $j \geq 1$

$$L_{j+1}(x,y) = k_j(x,y)L_j(x,y) - L_{j-1}(x,y),$$

with $L_0(x,y) = x$, $L_1(x,y) = y$.

On the other hand, noticing that for any $q', q'', q'''$, consecutive denominators of fractions from $\mathfrak{F}_Q$ we have $T\big(\frac{q'}{Q}, \frac{q''}{Q}\big) = \big(\frac{q''}{Q}, \frac{q'''}{Q}\big)$, we get $k_1\big(\frac{q_{j-1}}{Q}, \frac{q_j}{Q}\big) = \big[\frac{Q+q_{j-1}}{q_j}\big]$, and then, for any $r \in \mathbb{N}$ it follows that,

$$\begin{aligned} k_{r+1}\big(\tfrac{q_{j-1}}{Q}, \tfrac{q_j}{Q}\big) &= k_1 \circ T^r\big(\tfrac{q_{j-1}}{Q}, \tfrac{q_j}{Q}\big) = k_1\big(\tfrac{q_{j+r-1}}{Q}, \tfrac{q_{j+r}}{Q}\big) \\ &= \big[\tfrac{Q+q_{j+r-1}}{q_{j+r}}\big]\,. \end{aligned}$$

Finally, one should notice that the index of a Farey fraction $\gamma$ is the integer that reduces the fraction that gives $\gamma$ as the mediant of its neighbor fractions in $\mathfrak{F}_Q$. Indeed, the index satisfy the equalities:

$$\nu(\gamma_j) = \left[\frac{Q+q_{j-1}}{q_j}\right] = k_1\left(\frac{q_{j-1}}{Q}, \frac{q_j}{Q}\right) = \frac{q_{j+1}+q_{j-1}}{q_j} = \frac{a_{j+1}+a_{j-1}}{a_j}\,.$$

The index has interesting properties. Hall and Shiu [HS2001] discovered some very remarkable exact formulae involving the index of Farey fractions, and they also proved a number of asymptotic results. Boca, Gologan and Zaharescu [BGZ2002] found asymptotic formulae for the moments of the index. They proved that

$$\left|\sum_{\gamma\in\mathfrak{F}_Q} \nu(\gamma)^\alpha - 2B_\alpha N(Q)\right| \ll_\alpha \begin{cases} Q\log Q, & \text{for } \alpha<1,\\ Q\log^2 Q, & \text{for } \alpha=1,\\ Q^\alpha \log Q, & \text{for } 1<\alpha<2. \end{cases} \tag{5.1}$$

Here $B_\alpha$ is a constant given by

$$B_\alpha = \iint\limits_{\mathcal{T}} \left[\frac{1+s}{t}\right]^\alpha ds\,dt = \sum_{k=1}^\infty k^\alpha \operatorname{Area}(\mathcal{T}_k) \ll \sum_{k=1}^\infty k^{\alpha-3} < \infty\,.$$

The moment of order 2 was calculated in [HS2001] with an error term of size $O(Q\log^2 Q)$. Similar formulae for moments of order $\alpha \in (0, 3/2)$ were also established for the Farey fractions from a subinterval of $[0,1]$.

In [HS2001] and [BGZ2002] it is also investigated the twisted sum

$$S_{h,t}(Q) = \sum_{\gamma_j\in\mathfrak{F}_Q\cap[0,t]} \nu_Q(\gamma_j)\nu_Q(\gamma_{j+h})\,,$$

where $0 \le t \le 1$. Boca, Gologan and Zaharescu [BGZ2002] proved that for any integer $h \ge 1$ and $0 \le t \le 1$, we have

$$S_{h,t}(Q) = tA(h)N(Q) + O_{h,\epsilon}\big(Q^{3/2+\epsilon}\big)\,,$$

where $A(h) = 2\iint_{\mathcal{T}} k_1(s,t)k_{h+1}(s,t)\,ds\,dt$ are rational positive constants of size $\ll 1+\log h$. In the case $t=1$, they get a better error term of size $O(Q\log^2 Q)$.

5.3. **The distribution of lattice points visible from the origin.** Let $\Omega \subset \mathbb{R}^2$ be a domain that contains the origin with respect to which it is star-shaped. The boundary of $\Omega$ is parametrized by $x = r_\Omega(\theta)\cos\theta$, $y = r_\Omega(\theta)\sin\theta$ and $r_\Omega(\theta)$ is supposed to be continuous and piecewise $C^1$ on $[0, 2\pi]$. For any $Q \geq 1$, let $\Omega_Q = \{(Qx, Qy)\colon\ (x,y) \in \Omega\}$ be the $Q$-dilation of $\Omega$. Then visible from the origin are those lattice points from $\Omega_Q$ with relatively prime coordinates. Let

$$\mathfrak{F}(\Omega, Q) = \left\{(a,q) \in \Omega_Q\colon a, q \in \mathbb{Z},\ (a,q) = 1\right\}.$$

Then the cardinality of $\mathfrak{F}(\Omega, Q)$ is

$$N = N(\Omega, Q) = \#\mathfrak{F}(\Omega, Q) \sim \frac{\mathrm{Area}(\Omega)Q^2}{\zeta(2)},$$

as $Q \to \infty$. Boca, Cobeli and Zaharescu [BCZ2000] found the nearest neighbor distribution of angles of all the lines determined by the origin with points from $\mathfrak{F}(\Omega, Q)$. Without restricting the generality, one may assume that $\Omega$ is included in the second octant. Let $\pi/4 = \theta_0 < \theta_1 < \cdots < \theta_N = \pi/2$ be the angles corresponding to points from $\mathfrak{F}(\Omega, Q)$. The angles are normalized to $N = \tilde\theta_0 = 4N\theta_0/\pi < \tilde\theta_1 = 4N\theta_1/\pi < \cdots < \tilde\theta_N = 4N\theta_N/\pi = 2N$, to get a sequence of $N$ points with average consecutive spacing equal to 1. Then the first spacing measure is defined by

$$\mu_{\Omega,Q} := \frac{1}{N(\Omega,Q)} \sum_{j=1}^{N(\Omega,Q)} \delta_{\tilde\theta_j - \tilde\theta_{j-1}}.$$

The main result of [BCZ2000] says that as $Q \to \infty$ the sequence $(\mu_{\Omega,Q})_{Q\geq 1}$ converges weakly to a probability measure $\mu_Q$. Let $G_\Omega(t)$ be the repartition function of $\mu_Q$. This is obtained as the limit as $Q \to \infty$ of the proportion of differences $\tilde\theta_j - \tilde\theta_{j-1}$ that are $\geq t$. Explicitly, this can be written as $G_\Omega(t) = \int_t^\infty d\mu_\Omega(x)$ and

$$\mu_\Omega([t,\infty)) = \frac{1}{\mathrm{Area}\ (\Omega)} \int\limits_{\pi/4}^{\pi/2} r_\Omega^2(\theta)\, \eta_\Omega(t,\theta)\, d\theta,$$

in which the kernel $\eta_\Omega(t,\theta)$ is given by

$$\eta_\Omega(t,\theta) = \begin{cases} \frac{1}{2}, & \text{for } r^2(\theta)t \in (0,\lambda), \\ \frac{\lambda}{r^2(\theta)t} - \frac{1}{2} - \frac{\lambda}{r^2(\theta)t}\, \log \frac{\lambda}{r^2(\theta)t)}, & \text{for } r^2(\theta)t \in [\lambda, 4\lambda), \\ \frac{\lambda}{r^2(\theta)t} - \frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{4\lambda}{r^2(\theta)t}} & \\ \qquad + \frac{2\lambda}{r^2(\theta)t}\, \log \frac{2}{1+\sqrt{1-\frac{4\lambda}{r^2(\theta)t}}}, & \text{for } r^2(\theta)t \in [4\lambda, \infty). \end{cases}$$

where $\lambda := \lambda(\Omega) = \frac{4\mathrm{Area}(\Omega)}{\pi\zeta(2)}$ and $r(\theta) := r_\Omega(\theta)$. It is shown that the support of $\mu_\Omega$ is included in the interval $[24\mathrm{Area}(\Omega)/(\pi^3 M^2), \infty]$, where $M = M(\Omega) = \sup_\theta r(\theta)$.

Two sets of particular interest are the $Q$-dilations of the Farey triangle, (identified with $\mathfrak{F}_Q$) and the unit disk $\mathcal{D}$. The later example provides a density function with the larger vanishing interval beyond zero, that is the strongest repulsion occurs among the lines determined by the origin and lattice points from $Q\mathcal{D}$. Moreover, the density corresponding to the measure $\mu_{\mathcal{D}}$ equals $g_1(t)$.

## 6. $\mathfrak{F}_Q$ in Mathematical Physics

Lately, Number Theory and in particular the Farey sequence and the Farey tree play an important role in mathematical physics. For historical notes and a generalization of the Farey tree to an extended Farey tree see Lagarias and Tresser [LT1995]. The concept of Farey web was introduced and examined by K. Brucks, J. Ringland and C. Tresser (see [BRT2002]). The results are utilized to solve problems on the organization of frequency-locking. Amoroso [Amo1995] has considered some equivalent conditions to RH. Recently considerable interest raised among physicists the hope to find a direct connection between the Lee-Yang theory of phase transition and the Riemann Hypothesis. Trying to find insights, different statistical mechanics spin chain models based on $\mathfrak{F}_Q$ were introduced and studied by Knauf [Kna1993], [Kna1994], Contucci and Knauf [CK1997], Kleban and Özlük [KO] (see also Landford [LR1996] and Cvitanović [Cvi1992]). Another example is presented by Kholodenko [Kho2001], who shows that the statistical mechanics of Einstein $2 + 1$ gravity on the punctured torus may be modeled by the number-theoretic Farey spin chain partition function $\widehat{Z}(\beta) = \zeta(\beta - 1)/\zeta(\beta)$.

The partition given by the Farey fractions on the interval $[0, 1]$ clearly has some fractal properties. This is studied using a *Farey measure* (see the presentation from Cvitanović [Cvi1992] and Cesaratto and Piacquadio [CP2001] on the behind doors physical motivations). The Farey measure is the unique probability measure on $[0, 1]$ that assigns the equal mass $2^{-n}$ to each of the $2^n$ intervals belonging to the partition of $[0, 1]$ obtained by starting with the fractions $0/1$ and $1/1$ and inserting recursively, in $n$ steps, the mediant of any two consecutive fractions. Cesaratto and Piacquadio [CP1998] compare the Hausdorff spectrum, the computational spectrum, and the Legendre spectrum for the hyperbolic measure, which is induced by the Farey tree partition. Their arguments show that this measure is fundamentally different from any self-similar measure, but it behaves very much like the self-similar measures.

A class of quasiparticles, called fractons, which obey fractal statistics is studied by da Cruz and R. de Oliveira [Cru2000], [CO2000], [Cru2001]. They find that the Hausdorff dimensions of their trajectories, is given by the Farey sequences. Da Cruz also discusses some mathematical properties of the anyons, the duality of their trajectories, the exclusion statistics, and a two-parameter renormalization group flow.

A large potential of applications to problems in electrical engineering appeared after Penner [Pen2002] introduced several new families of wavelets. The expansions of Penner's wavelets rely on equally spaced multiscale sampling methods that depend on $\mathfrak{F}_Q$. Next we present with more details other physical problems related to the Farey sequence.

6.1. **Chaotic dynamical systems–Thermodynamic averages.** An important tool used to study the paths to chaos are the *thermodynamic averages*, that is sums over configurations or mode-locking intervals. In particular, the interaction of couples of configurations deserves interest, since it preserves the phenomenon in its generality.

Thermodynamic averages for $\mathfrak{F}_Q$ were considered by Hall [Hal1970], S. Kanemitsu, R. Sita Rama Chandra Rao and A. Siva Rama Sarma [KRS1982], S. Kanemitsu, T. Kuzumaki and M. Yoshimoto [KKY2000], Artuso, Cvitanović and Kenny [ACK1989]. These averages are sums over different expressions involving denominators of Farey fractions. One characteristic example are moments of the spacings:

$$T_Q(\alpha,\beta) := \sum_{j=2}^{N(Q)-1} (\gamma_j-\gamma_{j-1})^{\alpha}(\gamma_{j+1}-\gamma_j)^{\beta} = \sum_{j=2}^{N(Q)-1} \frac{1}{q_{j-1}^{\alpha} q_j^{\alpha+\beta} q_{j+1}^{\beta}},$$

for which Hall and Tanenbaum [HT1984] and Hall [Hal1994] get asymptotic formulae. Characteristic for $T_Q(\alpha,\beta)$ is its threshold across the line $\alpha+\beta=2$.

Even earlier several authors obtained various estimates or even identities for sums of functions with variables equal to consecutive denominators of Farey fractions. For example, Robertson [Rob1968] studies the asymptotic behavior of the sum

$$A_Q(\alpha,\beta) := \sum_{j=1}^{N(Q)} q_j^{\alpha} q_{j+1}^{\beta}\,.$$

He showed that as $Q\to\infty$,

$$\frac{A(\alpha,\beta)}{Q^{\alpha+\beta+2}} \sim \frac{6}{\pi^2}\left[\frac{1}{(\alpha+1)(\beta+1)} - \frac{\Gamma(\alpha+1)\Gamma(\beta+1)}{\alpha+\beta+3}\right],$$

for $\alpha>-1$ and $\beta>-2\,(\beta\neq-1)$. Lehner and Newman [LN1968] considered the more general sum

$$S_Q(f) := \sum_{(q',q'')\in\mathcal{T}'_Q} f(q',q''),$$

in which $\mathcal{T}'_Q$ is the set of all pairs of consecutive denominators of fractions from $\mathfrak{F}_Q$ and $f(x,y)$ is a function of two integral variables. Observing the identity

$$S_Q(f) = f(1,1) + \sum_{2\le n\le Q} \sum_{\substack{1\le m\le n\\(m,n)=1}} \big(f(m,n)+f(n,m)-f(m,n-m)\big)\,,$$

they get as a consequence estimates such as

$$\sum_{(q',q'')\in\mathcal{T}'_Q} \frac{Q}{q'q''(q'+q'')} \sim \frac{12\log 2}{\pi^2}\,,$$

or

$$\sum_{(q',q'')\in\mathcal{T}'_Q} \sqrt{q'q''} = \frac{(32-3\pi)Q^3}{12\pi^2} + O\big(Q^2\log Q\big)\,.$$

Among other results, taking $f(x,y)=x^\alpha y^\beta$ with $0\le\alpha,\ \beta$, they show that

$$\frac{A(\alpha,\beta)}{Q^{\alpha+\beta+2}} = c_{\alpha,\beta} + O\left(\frac{\log Q}{Q}\right)\,,$$

where $c_{\alpha,\beta}$ are constants explicitly calculable. The numerator of the error term was lowered by Kanemitsu [Kan1978] to $(\log Q)^{2/3}(\log\log Q)^{1+\varepsilon}$.

A more general perspective on this type of problems follows by counting the lattice points with coprime components from a given region $\Omega\subseteq(0,R)\times(0,R)\subseteq\mathbb{R}^2$. Assume $\partial\Omega$ (the boundary of $\Omega$) is rectifiable and $f$ is a $C^1$ function on $\Omega$. We denote

$$S' = S'(f,\Omega) := \sum_{\substack{(a,b)\in\Omega\cap\mathbb{Z}^2\\(a,b)=1}} f(a,b).$$

Then, using Möbius sumation, Boca, Cobeli and Zaharescu [BCZ2001, Lemma 2] showed

$$\left|S' - \frac{6}{\pi^2}\iint_\Omega f(x,y)\,dx\,dy\right| \ll \left(\Big\|\frac{\partial f}{\partial x}\Big\|_\infty + \Big\|\frac{\partial f}{\partial y}\Big\|_\infty\right)\mathrm{Area}(\Omega)\log R$$

$$+\ \|f\|_\infty\left(R+\mathrm{length}(\partial\Omega)\log R\right),$$

where $\|f\|_\infty = \sup_{(x,y)\in\Omega}|f(x,y)|$. In particular, taking $\Omega=\mathcal{T}$, the Farey triangle, one gets sums of values of a function with two variables given by consecutive denominators of Farey fractions. Moreover one could arrange to restrict the summation only to fractions situated in a certain interval $I\subset[0,1]$ (see Boca et al. [BCZ2001, Lemma 8]). We remark that this setting covers a wide range of situations, since starting with any two consecutive denominators $q',q''$ say, one could get recursively in terms only of $q',q''$ and $Q$ any other denominator of a fraction from $\mathfrak{F}_Q$.

The sums $T_Q(\alpha,\beta)$ are cousins of the larger intervals moments between elements of $\mathfrak{F}_Q$, which are defined as

$$S_Q(m,h) := \sum_{j=1}^{N(Q)} (\gamma_{j+h}-\gamma_j)^m\,.$$

(Here we use the the convention $\gamma_{j+N(Q)} = \gamma_j$ for all integers $j$.) Easier to evaluate are those corresponding to intervals of a second, that is when $h = 1$. The sum $S_Q(1,1)$ is immediate, and Borel [Bor1948] used this fact to show that $\sum 1/(qq') = \frac{1}{2}$ for any integer $n$, the summation being taken over all integer pairs $(q,q')$ mutually prime and such that $q \le n$, $q' \le n$, $q+q' > n$. Higher moments were studied by Hall [Hal1970], and later by Kanemitsu [Kan1978] and Kanemitsu, Rao and Sarma [KRS1982] who obtained a smaller error term when $m = 2$ and 3. For $m = 2$ their result is

$$S_Q(2,1) = \frac{12\ln Q}{\pi^2 Q^2} + \frac{12}{\pi^2 Q^2}\Big(\gamma - \frac{\zeta'(2)}{\zeta(2)} + \frac{1}{2}\Big) + O_\varepsilon\bigg(\frac{\ln^{\frac{5}{3}} Q\,(\ln\ln Q)^{1+\varepsilon}}{Q^3}\bigg)\,.$$

For $m \ge 3$, Hall [Hal1970] proves

$$S_Q(m,1) = \frac{2\zeta(m-1)}{Q^m\zeta(m)} + O\Big(\frac{\log^\theta Q}{Q^{m+1}}\Big)\,,$$

where $\theta = 1$ for $m = 3$, $\theta = 0$ if $m \ge 4$. Kanemitsu [Kan1978] and Kanemitsu et al. [KRS1982] improves the error term of Hall to $3\theta\zeta(2)^{-1}Q^{-4}\log Q + O(Q^{-m-1})$. Kanemitsu, Kuzumaki and Yoshimoto [KKY2000] get an even better result with the error term of size $O(Q^{1-2m}\log Q)$ for $m = 2,3,4$ and $O(Q^{-m-3})$ for $m \ge 5$.

Of particular interest are the square moments of larger spacings between Farey fractions. For intervals of a third Hall [Hal1994] proved the estimate

$$S_Q(2,2) = \frac{36\ln Q}{\pi^2 Q^2} + \frac{A}{Q^2} + O\Big(\frac{\ln Q}{Q^{\frac{5}{2}}}\Big),$$

where

$$A = \frac{12}{\pi^2}\Big(3\gamma - \frac{3\zeta'(2)}{\zeta(2)} + \frac{3}{2} + \ln 2 + 2\sum_{k\ge 1}\frac{\zeta(2k)-1}{2k-1}\Big).$$

Answering to Hall's conjecture that a similar formula should be valid for larger intervals, Boca, Cobeli and Zaharescu [BCZ2001] have determined an asymptotic estimation for the larger intervals quadratic moment. Their result is that for any $h \ge 3$,

$$S_Q(2,h) = \frac{12(2h-1)\log Q}{\pi^2 Q^2} + \frac{B(h)}{Q^2} + O_h\bigg(\frac{\log^{1/(h+2)} Q}{Q^{2+1/(h+2)}}\bigg),$$

where $B(h)$ is a constant given explicitly.

In order to get the estimate for the square moment, in [BCZ2001] is evaluated the more general sum involving Farey arcs:

$$S_{Q,I}(h) = \sum_{\gamma_j \in \mathfrak{F}_Q(I)} (\gamma_{j+1} - \gamma_j)(\gamma_{j+h+1} - \gamma_{j+h}),$$

for any interval $I \subset [0,1]$. As a consequence, similar results are obtained for intervals of moments of Farey fractions from $\mathfrak{F}_Q(I)$.

6.2. **Billiards and Farey fractions.** Any effective physical or mathematical model of a system containing a huge number of particles (for example, the order of magnitude of the number of atoms in a liter of gas is about $10^{24}$) will not attempt to predict what each particle is doing individually, but will rather find statistics and averages over states of the particles. Different mathematical models were proposed to describe the brownian motion of the particles of gas in a container, most of them involving measure theory and statistics.

Sinaĭ billiards offer some important problems where number theory, and in particular properties of Farey sequences, can prove to be very valuable tools. Extensive numerical experiments on Sinaĭ billiards (unit cells of a periodic Lorentz gas) were made especially in the last two decades. Artuso, Casati and Guarneri [ACG1996] provide a general discussion, review of previous work and search properties of correlation functions of periodic Sinaĭ billiards.

Techniques employed in the study of $h$-spacings between Farey fractions, in which estimates for Kloosterman sums play an essential role, are used to get results in some billiard problems. Boca, Gologan and Zaharescu [BGZ2001a], [BGZ2001b] and Gologan [Gol2000] considered the billiard problem in a "rectangle with pockets" whose lengths are proportional to the sides and tend to zero. The particle is assumed to move with constant speed equal to 1. Let $D_\epsilon =$ be the rectangular board of width $L_1$ and height $L_2$ with horizontal segments of length $\epsilon L_1$ and vertical segments of length $\epsilon L_2$ removed from each corner, that is, $D_\epsilon = (0,L_1) \times (0,L_2) \cup \{0,L_1\} \times [\epsilon L_2, (1-\epsilon)L_2] \cup [\epsilon L_1, (1-\epsilon)L_1] \times \{0, L_2\}$. The trajectory of the billiard is supposed to begin at one of the corners, changes direction when it reaches side cushions such that the angles of incidence and reflection are equal, and ends when it meets one of the removed segments. For any $\theta \in (0,\pi/2)$, let $l_\epsilon(\theta)$ denote the length of the trajectory of a particle which starts to move from $(0,0)$ under angle $\theta$. This coincides with the length of the trajectory in a right isosceles triangle associated to a system of two equal mass points that move in the interval $[0,1]$, starting each from an endpoint with different velocities and rebounding elastically when they collide and at the endpoints (cf. Sinaĭ [Sin1976, pp. 84–85]). For a more general triangular board shape see Artuso [Art1996], Artuso et al. [ACG1997].

Boca, Gologan and Zaharescu provide asymptotic formulae for the moments of the length of the trajectory and the number of reflections in the side cushions when the lengths of pockets tend to zero. They have shown that for any $[\alpha,\beta]\subset[0,\arctan\frac{L_2}{L_1}]$, any $\delta>0$ and any $r>0$

$$\int_\alpha^\beta l_\epsilon(\theta)^2\,d\theta=\frac{L_1^2(1+2\log 2)(\tan\beta-\tan\alpha)}{\pi^2\epsilon^2}+O_\delta\left(\frac{L_1L_2}{\epsilon^{3/2+\delta}}\right)\quad\text{and}$$

$$\int_\alpha^\beta l_\epsilon(\theta)^r\,d\theta=\frac{C_{r,L_1}(\alpha,\beta)}{\epsilon^r}+O_{\delta,r,L_1,L_2}\left(\frac{1}{\varepsilon^{r-1/6+\delta}}\right),$$

where $C_{r,L_1}(\alpha,\beta)=\frac{12D_rL_1^r}{\pi^2}\int\limits_\alpha^\beta\frac{dx}{\cos^r x}$ and

$$D_r=\frac{1-\frac{1}{2^r}+\ln 2}{r(r+1)}-\frac{1-\frac{1}{2^r}}{r^2}+\frac{1-\frac{1}{2^{r+1}}}{(r+1)^2}+\sum_{k=1}^\infty\frac{(-1)^{k-1}}{k2^k}\left(\frac{\binom{r}{k}}{r}-\frac{\binom{r+1}{k}}{r+1}\right).$$

In particular, one obtains that the average length of the trajectory of the billiard in the unit square is about $\frac{6\ln 2\ln(2+\sqrt{2})}{\pi^2\varepsilon}\approx\frac{0.742792}{\varepsilon}$. Denoting by $R_\epsilon(\theta)$ the number of reflections at the side cushions, the average of $R_\epsilon(\theta)$ along the interval $[\alpha,\beta]$ is:

$$\int_\alpha^\beta R_\epsilon(\theta)\,d\theta=\frac{6(\beta-\alpha+\frac{L_1}{L_2}\log\frac{\cos\alpha}{\cos\beta})\log 2}{\pi^2\epsilon}+O_{\delta,L_1,L_2}\left(\frac{1}{\epsilon^{5/6+\delta}}\right).$$

For the unit interval board, this gives in particular that the average number of reflections is about $\frac{(3\pi+6\ln 2)\ln 2}{\pi^2\epsilon}\approx\frac{0.953987}{\epsilon}$.

Getting more insight into the phenomenon, in [BGZ2001b] it is shown that the existence of the principal term in the formula for the the moment of $l_\epsilon(\theta)$ is due to the convergence of the “level probability measures” $\mu_{\alpha,\beta,\epsilon}$ on $[0,\infty)$, defined by

$$\mu_{\alpha,\beta,\epsilon}\big((-\infty,\lambda]\big):=\frac{1}{\beta-\alpha}\int_\alpha^\beta\chi_{[0,\lambda]}\big(\epsilon l_\epsilon(\theta)\big)\,d\theta=\frac{\big|\{\theta\in[\alpha,\beta]:\ \epsilon l_\epsilon(\theta)\le\lambda\}\big|}{\beta-\alpha}.$$

When $\varepsilon\searrow 0$ the sequence $(\mu_{\alpha,\beta,\epsilon})_\epsilon$ converges weakly to a probability measure $\mu_{\alpha,\beta}$, which is explicitly computable.

## 7. **Some open problems**

In spite of the amount of work done in the last decades many finer properties of the Farey series are yet to be understood. A problem that can not be reached by any techniques known today is to find the relations between the size of the oscillatory error terms of the moments

$$M_Q(r) = \sum_{\gamma \in I} \gamma^r ,$$

when $r \geq 0$ increases. (The summation is over the Farey fractions in an interval $I \subset [0,1]$.) In this way, any nontrivial result on the relations equivalent to RH may prove to be an important breakthrough.

An interesting problem which might be more tractable is to understand the properties of the rationals $A(h)$ in the main term of the twisted sums of indexes formula from Section 5.2.

A large category of problems would require to generalize any of the known results for $\mathfrak{F}_Q$ to $\mathfrak{F}_{Q,c,d}$ or more generally, to sets of Farey fractions with numerators and denominators belonging to given arithmetic progressions. Such results have applications to billiard problems where the trajectory starts from a fixed point with rational coordinates. Numerical computations suggest that in this case the distribution of the length of the trajectory and the number of reflections at the side cushions are different than in the case when the trajectory starts from the origin.

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