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$\rho-$ CLOSED SETS

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Abstract. Our goal in this paper is to introduce the relatively new notions of ρ–closed and ρ-generalized closed sets. Several properties and connections to other well-known weak and strong closed sets are discussed. ρ-generalized continuous and ρ-generalized irresolute functions and their basic properties and relations to other continuities are explored.

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1. Introduction

Let $(X, \mathfrak{T})$ be a topological space (or simply, a space). If $A \subseteq X$, then the closure of A and the interior of A will be denoted by $Cl(A)$ and $Int(A)$, respectively. A subset $A \subseteq X$ is called *semi-open* [7] if there exists an open set $O \in \mathfrak{T}$ such that $O \subseteq A \subseteq Cl(O)$. Clearly A is a semi-open set if and only if $A \subseteq Cl(Int(A))$. A complement of a semi-open set is called *semi-closed* . A is called *preopen* [10] if $A \subseteq Int(Cl(A))$. A is called *preclosed [14]* if $Cl(Int(A)) \subseteq A$ and regular-closed *[14]* if $A = Cl(Int(A))$. A is a *generalized closed* (= *g-closed*) set [8] if $A \subseteq U$and $U \in \mathfrak{T}$ implies that $A \subseteq U$. For more on the preceding notions, the reader is referred to [2, 3, 6, 9, 11, 12, 13].

A function $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called *g-continuous* [1] if $f^{-1}(V)$ is g-closed in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$ and *contra-semi-continuous* [4] if $f^{-1}(V)$ is semi-open in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$.

We introduce the relatively new notions of $\rho-$closed sets, which is closely related to the class of closed subsets. We show that the collection of all $\rho-$open subsets of a space $(X, \mathfrak{T})$ forms a topology that is cofiner than $\mathfrak{T}$ and we investigate several characterizations of $\rho-$open and $\rho-$closed notions via the operations of interior and closure. In section 3, we introduce the notion of ρ-generalized closed sets and study connections to other weak and strong forms of generalized closed sets. In addition several interesting properties and constructions of ρ-generalized closed sets are discussed. Section 4 is devoted to introducing and studying ρ-generalized continuous and ρ-generalized irresolute functions and connections to other similar forms of continuity.

2. ρ-closed sets

We begin this section by introducing the notions of $\rho-$open and $\rho-$closed subsets.

Definition 1. *Let A be a subset of a space $(X, \mathfrak{T})$. The ρ-*interior *of A is the union of all open subsets of X whose closures are contained in $Int(A)$, and is denoted by $Int_\rho(A)$. The $\rho-$*closure *of A is $Cl_\rho(A) = \{x \in X : Cl(U) \cap Int(A) \neq \varnothing, U \in \mathfrak{T}, x \in U\}$. A is called ρ-*open *if $A = Int_\rho(A)$. The complement of a $\rho-$open subset is called $\rho-$*closed.

It is easy to see that a subset A of a space X is $\rho-$open if and only if for every point $x \in A$, there exists an open set U containing x such that $Cl(U) \subseteq Int(A)$. Clearly $Int_\rho(A) \subseteq Int(A) \subseteq A$ and hence every $\rho-$open set is open and thus every $\rho-$closed set is closed, but the converses needs not be true.

Example 1. *Let $X = \{a, b, c\}$ and $\mathfrak{T} = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$. Set $A = \{a, c\}$. Then A is open but not $\rho-$open as $Int_\rho(A) = \varnothing$.*

Next, we show that the collection of all $\rho-$open subsets of a space $(X, \mathfrak{T})$ forms a topology $\mathfrak{T}_\rho$ that is finer than $\mathfrak{T}$.

Theorem 1. *If $(X, \mathfrak{T})$ is a space, then $(X, \mathfrak{T}_\rho)$ is a space such that $\mathfrak{T} \supseteq \mathfrak{T}_\rho$.*

Proof. We only need to show $(X, \mathfrak{T}_\rho)$ is a space. Clearly $\varnothing$ and X are ρ-open. If $A, B \in \mathfrak{T}_\rho$, then $A = Int_\rho(A)$ and $B = Int_\rho(B)$. Now $Int_\rho(A \cap B) = \cup\{U \in \mathfrak{T} : Cl(U) \subseteq Int(A \cap B)\} = \cup\{U \in \mathfrak{T} : Cl(U) \subseteq Int(A) \cap Int(B)\}$. Thus $Int_\rho(A \cap B) \supseteq Int_\rho(A) \cap Int\rho(B) = A \cap B$. Therefore, $A \cap B = Int_\rho(A \cap B)$ and so $A \cap B \in \mathfrak{T}_\rho$.

If $\{A_\alpha : \alpha \in \Delta\}$ is a collection of ρ-open subsets of X, then for every $\alpha \in \Delta$, $Int_\rho(A\alpha) = A_\alpha$. Hence

$$
\begin{aligned}
Int_\rho(\cup_{\alpha\in\Delta} A\alpha) &= \cup\{U \in \mathfrak{T} : Cl(U) \subseteq Int(\cup_{\alpha\in\Delta} A\alpha)\} \\
&\supseteq \cup\{U \in \mathfrak{T} : Cl(U) \subseteq \cup_{\alpha\in\Delta} Int(A\alpha)\} \\
&\supseteq \cup\{U \in \mathfrak{T} : Cl(U) \subseteq A\alpha\} \text{ for every } \alpha \in \Delta \\
&= Int_\rho(A\alpha) \text{ for every } \alpha \in \Delta \\
&= A\alpha \text{ for every } \alpha \in \Delta.
\end{aligned}
$$

Hence $\cup_{\alpha\in\Delta} A\alpha \subseteq Int_\rho(\cup_{\alpha\in\Delta} A\alpha)$ and thus $\cup_{\alpha\in\Delta} A\alpha$ is $\rho-$open.

Next we show that $A \subseteq Cl_\rho(A)$ needs not be true.

Example 2. *Let* $X = \{a, b, c, d\}$ *and* $\mathfrak{T} = \{\varnothing, X, \{a, b\}, \{c, d\}, \{a\}, \{a, c, d\}\}$*. Set* $A = \{a, b, c\}$*. Then* $c \in A$*, but* $c \notin Cl_\rho(A)$ *since* $c \in \{c, d\} \in \mathfrak{T}$*, but* $Cl(\{c, d\}) \cap Int(A) = \varnothing$*.*

One might think that a subset A of a space X is $\rho-$closed if and only if $A = Cl_\rho(A)$, but this is not true as shown in the next example.

Example 3. *Consider the space in Example 1 and set* $A = \{b, c\}$*. Since* $\{a\}$ *is an open set containing* a*,* $Cl(\{a\}) = \{a, c\}$ *and* $Int(A) = \{b\}$*, we have* $Cl(\{a\}) \cap Int(A) = \varnothing$*. Namely we have shown* $a \notin Cl_\rho(A)$*. Since for any open set* U *containing* b*,* $U = \{b\}$*,* $\{a, b\}$ *or* X *and so* $Cl(U) = \{b, c\}$ *or* X *and* $Cl(U) \cap Int(A) \neq \varnothing$*, then* $b \in Cl_\rho(A)$*. Similarly,* $c \in Cl_\rho(A)$ *and so* $A = Cl_\rho(A)$*. On the other hand,* A *is not* $\rho-$*closed as for every point* $a \in X\backslash A = \{a\}$*, let* U *be any open set containing* a*. Then* $U = \{a\}$*,* $\{a, b\}$ *or* X *and as* $Cl(\{a\}) = \{a, c\}$ *and* $Cl(\{a, b\}) = Cl(X) = X$*, we have* $Cl(U) \subsetneq Int(X\backslash A) = Int(\{a\}) = \{a\}$*. This implies* $X\backslash A$ *is not* $\rho-$*open and hence* A *is not* $\rho-$*closed.*

Lemma 2. *For any subset* A *of* X*,*

(i) $Int(A) \subseteq Cl_\rho(A)$*.*

(ii) $Int(A) = \varnothing$ *if and only if* $Cl_\rho(A) = \varnothing$*.*

Proof. (i) $x \notin Cl_\rho(A)$ implies that there exists an open set U containing x such that $Cl(U) \cap Int(A) = \varnothing$. Hence $x \notin Int(A)$.

(ii) If $x \in Cl_\rho(A)$, then for every open subset U containing x, $Cl(U) \cap Int(A) \neq \varnothing$. Hence there exists $y \in Cl(U) \cap Int(A)$ and as $Int(A)$ is open, $U \cap Int(A) \neq \varnothing$. Therefore $Int(A) \neq \varnothing$.

Conversely if $Cl_\rho(A) = \varnothing$, then by (i) as $Int(A) \subseteq Cl_\rho(A)$, $Int(A) = \varnothing$.

Lemma 3. *The union of an open set with a* ρ*-open set is open.*

Proof. Let A be an open set and B be a ρ-open set. For all $x \in A \cup B$, $x \in A$ or $x \in B$ and so $x \in Int(A) \subseteq Int(A \cup B)$ or $x \in Int_\rho(B) \subseteq Int_\rho(A \cup B) \subseteq Int(A \cup B)$.

Corollary 4. *The intersection of a closed set with a* ρ*-closed set is closed.*

Lemma 5. *If* A *is a semi-open subset of a space* X*, then* $Cl(A) = Cl_\rho(A)$*.*

Proof. If U is an open set containing x such that $Cl(U) \cap Int(A) \neq \varnothing$, then there exists $y \in Cl(U) \cap Int(A)$. Thus $U \cap Int(A) \neq \varnothing$ and so $U \cap A \neq \varnothing$. Therefore $Cl_\rho(A) \subseteq Cl(A)$.

Conversely if for every open set U containing A we have $U \cap A \neq \varnothing$, $U \cap Int(Cl(A)) \neq \varnothing$, since A is semi-open. Thus there exists $y \in U \cap Int(Cl(A))$ and so $U \cap Int(A) \neq \varnothing$ which implies that $Cl(U) \cap Int(A) \neq \varnothing$. Hence $Cl(A) \subseteq Cl_\rho(A)$.

Corollary 6. *(i) For any subset A of X, $Cl_\rho(A) \subseteq Cl(A)$.*

(ii) If A is a semi-open subset of a space X, then $A \subseteq Cl_\rho(A)$.

Lemma 7. *If A is a $\rho-$closed subset of a space X, then $Cl_\rho(A) \subseteq A$.*

Proof. If A is a $\rho-$closed subset, then A is closed and thus by Corollary 6 (i), $Cl_\rho(A) \subseteq A$.

Next, we show that a preclosed set that is also semi-open equals its ρ-closure.

Theorem 8. *If A is regular closed subset of a space X, then $Cl_\rho(A) \subseteq A$.*

Proof. $Cl_\rho(A) \subseteq Cl(A) \subseteq Cl(Cl(Int(A))) = Cl(Int(A)) \subseteq A$. This together with Corollary 6 implies that $A = Cl_\rho(A)$.

3. ρ-generalized closed sets

In this section, we introduce the notion of ρ-generalized closed set. Moreover, several interesting properties and constructions of these subsets are discussed.

Definition 2. *A subset A of a space X is called* ρ-generalized closed *(ρ-g-closed) if whenever U is an open subset containing A, we have $Cl_\rho(A) \subseteq U$. A is ρ-g-open if $X \backslash A$ is ρ-g-closed.*

Theorem 9. *A subset A of $(X, \mathfrak{T})$ is ρ-g-open if and only if $F \subseteq Int\rho(A)$, whenever $F \subseteq A$ and F is closed in $(X, \mathfrak{T})$.*

Proof. Let A be a ρ-g-open set and F be a closed subset such that $F \subseteq A$. Then $X \backslash A \subseteq X \backslash F$. As $X \backslash A$ is ρ-g-closed and as $X \backslash F$ is open, $Cl_\rho(X \backslash A) \subseteq X \backslash F$. So $F \subseteq X \backslash Cl_\rho(X \backslash A) = Int_\rho(A)$.

Conversely if $X \backslash A \subseteq U$ where U is open, then the closed set $X \backslash U \subseteq A$. Thus $X \backslash U \subseteq Int_\rho(A) = X \backslash Cl_\rho(X \backslash A)$ and so $Cl_\rho(X \backslash A) \subseteq U$.

Next we show that every $\rho-$closed set is $\rho - g-$closed sets. Moreover, the class g-closed sets is properly placed between the classes of semi-open closed sets and $\rho - g-$closed sets. Clearly every closed semi-open set by Lemma 5 is $\rho-$closed. A closed set is trivially $g-$closed and every $g-$closed set is ρ-g-closed by Corollary 6 (i).

The following result follows from Corollary 6 (i) and the fact that every ρ-closed set is closed:

Lemma 10. *Every $\rho-$closed set is $\rho-g-$closed.*

The converse of the preceding result needs not be true.

Example 4. *Consider the space $X=\{a,b,c,d\}$ and $\mathfrak{T}=\{\varnothing,X,\{b\},\{c\},\{b,c\},\{a,b\},\{a,b,c\}\}$. Set $A=\{a\}$. Since $Cl_\rho(A)=\varnothing$, A is $\rho-$g-closed, but A is not $\rho-$closed and not g-closed and hence not closed. Also $B=\{b,d\}$ is a g-closed set that is not closed.*

The following is an immediate result from Lemma 5:

Theorem 11. *If A is a semi-open subset of a space X, the following are equivalent:*

(1) A is $\rho-g-$closed;

(2) A is $g-$closed

Its clear that if $A\subseteq B$, then $Int_\rho(A)\subseteq Int_\rho(B)$ and $Cl_\rho(A)\subseteq Cl_\rho(B)$.

Lemma 12. *If A and B are subsets of a space X, then $Cl_\rho(A\cup B)=Cl_\rho(A)\cup Cl_\rho(B)$ and $Cl_\rho(A\cap B)\subseteq Cl_\rho(A)\cap Cl_\rho(B)$.*

Proof. Since A and B are subsets of $A\cup B$, $Cl_\rho(A)\cup Cl_\rho(B)\subseteq Cl_\rho(A\cup B)$. On the other hand, if $x\in Cl_\rho(A\cup B)$ and U is an open set containing x, then $Cl(U)\cap Int(A\cup B)\neq\varnothing$. Hence either $Cl(U)\cap Int(A)\neq\varnothing$ or $Cl(U)\cap Int(B)\neq\varnothing$. Thus $x\in Cl_\rho(A)\cup Cl_\rho(B)$.

Finally since $A\cap B$ is a subset of A and B, $Cl_\rho(A\cap B)\subseteq Cl_\rho(A)\cap Cl_\rho(B)$.

Corollary 13. *Finite union of ρ-g-closed sets is ρ-g-closed.*

While the finite intersection of ρ-g-closed sets needs not be ρ-g-closed.

Example 5. *Let $X=\{a,b,c,d,e\}$ and $\mathfrak{T}=\{\varnothing,X,\{a,b\},\{c\},\{a,b,c\}\}$. Set $A=\{a,c,d\}$ and $B=\{b,c,e\}$. Then clearly A and B are ρ-g-closed sets since X is their only super open set. But $A\cap B=\{c\}$ is not ρ-g-closed since $\{c\}$ is open and by Lemma 5, $Cl_\rho(\{c\})=Cl(A)=\{c,d,e\}\subsetneq\{c\}$.*

Theorem 14. *The intersection of a ρ-g-closed set with a ρ-closed set is ρ-g-closed.*

Proof. Let A be a ρ-g-closed set and B be a ρ-closed set. Let U be an open set containing $A \cap B$. Then $A \subseteq U \cup X\backslash B$. Since $X\backslash B$ is ρ-open, by Lemma 3, $U \cup X\backslash B$ is open and since A is ρ-g-closed, $Cl_\rho(A \cap B) \subseteq Cl_\rho(A) \cap Cl_\rho(B)$ and by Lemma 7, $Cl_\rho(A \cap B) \subseteq Cl_\rho(A) \cap B \subseteq (U \cup X\backslash B) \cap B = U \cap B \subseteq U$.

4. ρ-g-continuous and ρ-g-irresolute functions

Definition 3. *A function $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ is called*

(1) ρ-g-continuous *if $f^{-1}(V)$ is ρ-g-closed in $(X, \mathfrak{T})$ for every closed set V of $(Y, \mathfrak{T}')$,*

(2) ρ-g-irresolute *if $f^{-1}(V)$ is ρ-g-closed in $(X, \mathfrak{T})$ for every ρ-g-closed set V of $(Y, \mathfrak{T}')$.*

Lemma 15. *Let $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ be g-continuous. Then f is $\rho-$g-continuous but not conversely.*

Proof. Follows from the fact that every g-closed set is $\rho-$g-closed.

Example 6. *Consider the space $X = \{a, b, c, d\}$ and $\mathfrak{T} = \{\varnothing, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}\}$. Let $\mathfrak{T}' = \{\emptyset, \{d\}, X\}$. Let $f : (X, \mathfrak{T}) \to (X, \mathfrak{T}')$ be the identity function. Since $f^{-1}(\{a, b, c\} = \{a, b, c\} = Cl_\rho(\{a, b, c\})$, f is ρ-g-continuous, but f is not g-continuous and hence not continuous.*

Even the composition of ρ-g-continuous functions needs not be ρ-g-continuous.

Example 7. *Let f be the function in Example 6. Let $\mathfrak{T}'' = \{\emptyset, \{a, b, d\}, X\}$. Let $g : (X, \mathfrak{T}') \to (X, \mathfrak{T}'')$ be the identity function. It is easily observed that g is also ρ-generalized continuous as the only super set of $\{c\}$ is X. But the composition function $f : (X, \mathfrak{T}) \to (X, \mathfrak{T}'')$ is not ρ-generalized continuous since $\{c\}$ is closed in $(X, \mathfrak{T}'')$, but not ρ-g-closed in $(X, \mathfrak{T})$.*

Corollary 16. *If* $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ *is a continuous and contra-semi-continuous function, then* f *is* ρ*-g-continuous.*

Proof. If V is a closed subset of Y, then as f is continuous $f^{-1}(V)$ is closed and as f is contra-semi-continuous, $f^{-1}(V)$ is semi-open. Thus $f^{-1}(V)$ is ρ-g-closed.

We end this section by giving a necessary condition for ρ-g-irresolute function to be ρ-g-continuous.

Theorem 17. *If* $f : (X, \mathfrak{T}) \to (Y, \mathfrak{T}')$ *is bijective, open and* ρ*-g-irresolute, then* f *is* ρ*-g-continuous.*

Proof. Let V be a closed subset of Y and let $f^{-1}(V) \subseteq O$, where $O \in \mathfrak{T}$. Clearly, $V \subseteq f(O)$. Since $f(O) \in \mathfrak{T}'$ and since V is ρ-g-closed, $Cl_\rho(V) \subseteq f(O)$ and thus $f^{-1}(Cl_\rho(V)) \subseteq O$. Since f is ρ-generalized irresolute and since $Cl_\rho(V)$ is ρ-g-closed in Y, $f^{-1}(Cl_\rho(V)$ is ρ-g-closed. $f^{-1}(Cl_\rho(V) \subseteq Cl_\rho(f^{-1}(Cl_\rho(V))) = f^{-1}(Cl_\rho(V)) \subseteq O$. Therefore, $f^{-1}(V)$ is ρ-g-closed and hence, f is ρ-g-continuous.

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