Acta Universitatis Apulensis No. 36/2013
ISSN: 1582-5329 pp. 7-13

SOLUBLE GROUPS WITH SOME NEARLY $\mathcal{F}$-SUPPLEMENTED SUBGROUPS

X. Zhang, C. Li

Abstract. Suppose that G is a finite group and $\mathcal{F}$ a class of finite groups. A subgroup H of G is said to be nearly $\mathcal{F}$-supplemented in G if there exists a subgroup T of G such that $HT \trianglelefteq G$ and $(H \cap T)H_G/H_G$ is contained in the $\mathcal{F}$-hypercenter $Z_\infty^{\mathcal{F}}(G/H_G)$ of G/H_G. By using this new concept, we establish some new criteria for a group G to be soluble.

2000 *Mathematics Subject Classification*: 20D10, 20D20.

Keywords: nearly $\mathcal{F}$-supplemented subgroups, Sylow subgroups, soluble groups.

1. Introduction

All groups mentioned in this paper are considered to be finite. We use conventional notions and notation, as in [5, 16].

For a class $\mathcal{F}$ of groups, a chief factor H/K of a group G is called $\mathcal{F}$-central if $[H/K](G/C_G(H/K)) \in \mathcal{F}$ (see [5]). The symbol $Z_\infty^{\mathcal{F}}(G)$ denotes the $\mathcal{F}$-hypercenter of a group G, that is, the product of all such normal subgroups H of G whose G-chief factors are $\mathcal{F}$-central. A subgroup H of G is said to be $\mathcal{F}$-hypercenter in G if $H \leq Z_\infty^{\mathcal{F}}(G)$. A class $\mathcal{F}$ of groups is called a formation if it is closed under a homomorphic image and a subdirect product. It is clear that every group G has a smallest normal subgroup (called $\mathcal{F}$-residual of G and denoted by $G^{\mathcal{F}}$) with quotient in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if it contains every group G with $G/\Phi(G) \in \mathcal{F}$. We use $\mathcal{S}$ to denote the formation of all soluble groups.

Recall that a subgroup H of G is said to be c-supplemented [12] in G if there exists a subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. By using c-supplemented subgroups, people have obtained many interesting results; see, for example, [2, 7, 8, 12, 13], etc.

In 2008, $\mathcal{F}$-supplemented subgroups were introduced by W. Guo [6]. Let G be a finite group and $\mathcal{F}$ a class of finite groups. A subgroup H of G is said to be $\mathcal{F}$-supplemented in G if there exists a subgroup T of G such that $G = HT$ and

$(H\cap T)H_G/H_G$ is contained in the $\mathcal{F}$-hypercenter $Z_\infty^{\mathcal{F}}(G/H_G)$ of G/H_G. Obviously, this concept is a generalization of c-supplemented subgroups, $\mathcal{F}_n$-supplemented subgroups [14] and $\mathcal{U}_c$-normal subgroups [1]. More results about $\mathcal{F}$-supplemented subgroups can been found in [10, 11, 15].

In order to generalize above mentioned subgroups, we give a new concept as follows:

Definition 1. *Let H be a subgroup of G and $\mathcal{F}$ a class of finite groups. We say that H is nearly $\mathcal{F}$-supplemented in G if there exists a subgroup T of G such that $HT \trianglelefteq G$ and $(H\cap T)H_G/H_G$ is contained in the $\mathcal{F}$-hypercenter $Z_\infty^{\mathcal{F}}(G/H_G)$ of G/H_G.*

In present article, we use some nearly $\mathcal{F}$-supplemented subgroups to characterize the solubility of finite groups.

2. Preliminaries

Lemma 1. *Let A, B and K be subgroups of a group G.*

(1) *If $(|G:A|, |G:B|) = 1$, then $G = AB$* [5, Lemma 3.8.1].

(2) *If $(|G:A|, |G:B|) = 1$ and K is normal in G, then $K = (K\cap A)(K\cap B)$* [5, Lemma 3.8.2].

(3) *$K\cap AB = (K\cap A)(K\cap B)$ if and only if $KA\cap KB = K(A\cap B)$* [3, Lemma A.1.2].

A formation $\mathcal{F}$ is said to be S-closed (S_n-closed) if it contains all subgroups (all normal subgroups, respectively) of all its groups. The following lemma is well known.

Lemma 2. *Let G be a group and $A \leq G$. Let $\mathcal{F}$ be a non-empty saturated formation. Then*

(1) *If A is normal in G, then $AZ_\infty^{\mathcal{F}}(G)/A \leq Z_\infty^{\mathcal{F}}(G/A)$.*

(2) *If $\mathcal{F}$ is S-closed, then $Z_\infty^{\mathcal{F}}(G)\cap A \leq Z_\infty^{\mathcal{F}}(A)$.*

(3) *If $\mathcal{F}$ is S_n-closed and A is normal in G, then $Z_\infty^{\mathcal{F}}(G)\cap A \leq Z_\infty^{\mathcal{F}}(A)$.*

(4) *If $G\in\mathcal{F}$, then $Z_\infty^{\mathcal{F}}(G) = G$.*

In view of Lemma 2, we can get the following lemma easily.

Lemma 3. *Let G be a group and $H \leq M \leq G$.*

(1) *If H is nearly $\mathcal{F}$-supplemented in G and $\mathcal{F}$ is S-closed, then H is nearly $\mathcal{F}$-supplemented in M.*

(2) *Suppose that $H \trianglelefteq G$. Then M/H is nearly $\mathcal{F}$-supplemented in G/H if and only if M is nearly $\mathcal{F}$-supplemented in G.*

(3) *If $H \trianglelefteq G$, then for every nearly $\mathcal{F}$-supplemented subgroup E of G with $(|H|, |E|)=1$, HE/H is nearly $\mathcal{F}$-supplemented in G/H.*

Lemma 4 ([4, Theorem A]). *Suppose that G has a Hall π-subgroup, where π is a set of odd primes. Then all Hall π-subgroups of G are conjugate.*

3. Main results

Theorem 5. *A group G is soluble if and only if every Sylow subgroup of G is nearly $\mathcal{S}$-supplemented in G.*

Proof. The necessity is obvious. We need only prove the sufficiency. Suppose that the assertion is false and let G be a counterexample of minimal order. Then:

(1) $P_G = 1$ for any prime p dividing $|G|$ and any Sylow p-subgroup P of G.

If there exists a Sylow p-subgroup P of G such that $P_G \neq 1$, then by Lemma 3(1), it is easy to see that G/P_G satisfies the hypothesis of the theorem. Hence the minimal choice of G implies that G/P_G is soluble, and so G is soluble, a contradiction.

(2) $Z_\infty^{\mathcal{S}}(G) = 1$.

If $Z_\infty^{\mathcal{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which is contained in $Z_\infty^{\mathcal{S}}(G)$. Obviously, N is abelian. With the same argument as step (1), we have that G is soluble, a contradiction.

(3) If $1 \neq N \trianglelefteq G$, then G/N is soluble.

Let M/N be a Sylow p-subgroup of G/N, where $p||G/N|$. Then, obviously $M/N = PN/N$, where P is a Sylow p-subgroup of G. By the hypothesis, there exists a subgroup K of G such that $PK \trianglelefteq G$ and $P \cap K = 1$. It follows that $(PN)(NK) = N(PK) \trianglelefteq G$. Since

$$(|PK \cap N : N \cap K|, |PK \cap N : N \cap P|) = 1,$$

by Lemma 1(1),

$$PK \cap N = (K \cap N)(P \cap N).$$

Thus $N = (P \cap K)N = PN \cap KN$ by Lemma 1(3). This implies that

$$(PN \cap KN)(PN)_G/(PN)_G = N(PN)_G/(PN)_G = 1 \subseteq Z_\infty^{\mathcal{S}}(G/(PN)_G).$$

Therefore, $M = PN$ is nearly $\mathcal{S}$-supplemented in G. By Lemma 3(2), we have $M/N = PN/N$ is nearly $\mathcal{S}$-supplemented in G/N. This shows that G/N satisfies the hypothesis of the theorem. The minimal choice of G implies that G/N is soluble.

(4) Final contradiction.

Since $\mathcal{S}$ is closed under subdirect product, by step (3), G has only one minimal normal subgroup, N say. For any prime p dividing the order of N, we claim that every Sylow p-subgroup N_p of N is complemented in N. In fact, let P be a Sylow p-subgroup of G such that $N_p \leq P$. Then, obviously, $N_p = N \cap P$. By the hypothesis, there exists a subgroup K of G such that $PK \trianglelefteq G$ and $P \cap K = 1$. The unique minimal normality of N implies that $N \leq PK$. Since $(|PK : K|, |PK : P|) = 1$, $N = (N \cap P)(N \cap K) = N_p(N \cap K)$ by Lemma 1(2). Then $N_p \cap (N \cap K) = (P \cap N) \cap (N \cap K) = 1$. This shows that every Sylow p-subgroup of N is complemented in N. Hence N is soluble by Hall's theorem [9], which induces that G is soluble. This contradiction completes the proof.

Corollary 6 ([12, Theorem 2.4])**.** *A group G is soluble if and only if every Sylow subgroup of G is c-supplemented in G.*

Corollary 7 ([6, Theorem 4.2])**.** *A group G is soluble if and only if every Sylow subgroup of G is $\mathcal{S}$-supplemented in G.*

Theorem 8. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is nearly $\mathcal{S}$-supplemented in G, then G is soluble.*

Proof. Suppose that the assertion is false and let G be a counterexample of minimal order. Then by the well known Feit-Thompson's theorem, we have that $p = 2$. We now proceed the proof by the following steps.

(1) $O_2(G) = 1$.

Assume that $L = O_2(G) \neq 1$. Obviously, P/L is a Sylow 2-subgroup of G/L. Let M/L be a maximal subgroup of P/L. Then M is a maximal subgroup of P. By the hypothesis and Lemma 3(2), M/L is nearly $\mathcal{S}$-supplemented in G/L. The minimal choice of G implies that G/L is soluble. Consequently, G is soluble. This contradiction shows that step (1) holds.

(2) $O_{2'}(G) = 1$.

Assume that $E = O_{2'}(G) \neq 1$. Then, obviously, PE/E is a Sylow 2-subgroup of G/E. Suppose that M/E is a maximal subgroup of PE/E. Then there exists a maximal subgroup T of P such that $M = TE$. By the hypothesis and Lemma 3(3), $M/E = TE/E$ is nearly $\mathcal{S}$-supplemented in G/E. The minimal choice of G implies that G/E is soluble. By the well known Feit-Thompson's theorem, we know that E is soluble. It follows that G is soluble, a contradiction.

(3) P is not cyclic.

If P is cyclic, then G is 2-nilpotent by [16, Theorem 10.1.9]. This implies that G is soluble, a contradiction.

(4) If $1 \neq N \trianglelefteq G$, then N is not soluble and $G = PN$.

If N is soluble, then $O_2(N) \neq 1$ or $O_{2'}(N) \neq 1$. Since $O_2(N)$ char $N \trianglelefteq G$, $O_2(N) \leq O_2(G)$. Analogously $O_{2'}(N) \leq O_{2'}(G)$. Hence $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, which contradicts step (1) or step (2). Therefore N is not soluble. Assume that $PN < G$. By Lemma 3(1), every maximal subgroup of P is nearly $\mathcal{S}$-supplemented in PN. Thus PN satisfies the hypothesis. By the minimal choice of G, PN is soluble and so N is soluble. This contradiction shows that $G = PN$.

(5) G has a unique minimal normal subgroup, N say.

By step (4), we see that $G = PN$ for every non-identity normal subgroup N of G. It follows that G/N is soluble. Since $\mathcal{S}$ is a saturated formation, G has a unique minimal normal subgroup N.

(6) $Z_\infty^{\mathcal{S}}(G) = 1$.

If $Z_\infty^{\mathcal{S}}(G) \neq 1$, then we may take a minimal normal subgroup N of G which contained in $Z_\infty^{\mathcal{S}}(G)$. Obviously, N is an elementary Abelian r-subgroup for some prime r, which contradicts steps (1) and (2).

(7) Final contradiction.

Let P_1 be a maximal subgroup of P. By the hypothesis, there exists a subgroup K_1 of G such that $P_1K_1 \trianglelefteq G$ and

$$(P_1 \cap K_1)(P_1)_G/(P_1)_G \subseteq Z_\infty^{\mathcal{S}}(G/(P_1)_G).$$

In view of steps (1) and (6), we get $P_1 \cap K_1 = 1$. This means that $4 \nmid |K_1|$. Hence by [16, Theorem 10.1.9], K_1 has a normal Hall $2'$-subgroup M_1. Evidently, M_1 is also a Hall $2'$-subgroup of P_1K_1 and $M_1 \neq 1$. By steps (4) and (5), $N \leq P_1K_1$ and P_1K_1 is not soluble. Since $N \trianglelefteq G$, $N \trianglelefteq P_1K_1$. It is easy to see that $M_1 \cap N$ is also a Hall $2'$-subgroup of N. Since $G = PN$, we have

$$|G : M_1 \cap N| = |PN : M_1 \cap N| = \frac{|P||N|}{|N \cap P||M_1 \cap N|} = |N : M_1 \cap N||P : P \cap N|$$

is a 2-number. This implies that $M_1 \cap N$ is a Hall $2'$-subgroup of G. Thus $M_1 \cap N = M_1$ is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. For any element $x \in G$, both M_1^x and M_1 are Hall $2'$-subgroups of N. Since any two Hall $2'$-subgroups of a group are conjugate by Lemma 4, M_1^x and M_1 are conjugate in N. Let $H = N_G(M_1)$. By Frattini argument, $G = NH$. Since $(|N : N \cap P|, |N : M_1|) = 1$, $N = (N \cap P)M_1$ by Lemma 1(1). Hence $G = (N \cap P)H$. It follows that

$$P = P \cap (N \cap P)H = (N \cap P)(P \cap H).$$

Since $(|G : P|, |G : M_1|) = 1$, we have $G = PM_1 = PH$ by Lemma 1(1). If $P \cap H = P$, then $P \leq H$ and so $G = H$ has a non-identity normal Hall $2'$-subgroup M_1, which contradicts $O_{2'}(G) = 1$. Thus $P \cap H < P$ and so there exists a maximal subgroup P_2 of P such that $P \cap H \leq P_2$. Then $P = (N \cap P)(P \cap H) = (N \cap P)P_2$. By the hypothesis, there exists a subgroup K_2 of G such that $P_2K_2 \trianglelefteq G$ and $P_2 \cap K_2 = 1$. Using the same argument as above, we can see that K_2 has a non-identity normal

Hall $2'$-subgroup M_2 such that M_2 is a Hall $2'$-subgroup of N and also a Hall $2'$-subgroup of G. Obviously, $G = PM_2$ and $N \leq P_2K_2$. Hence

$$G = PM_2 = PK_2 = (N \cap P)P_2K_2 = P_2K_2.$$

Since both M_1 and M_2 are Hall $2'$-subgroups of G, by Lemma 4 there exists an element $g \in P$ such that $M_2^g = M_1$. Since $(|H : P \cap H|, |H : M_1|) = 1$, $H = (P \cap H)M_1$ by Lemma 1(1). Therefore,

$$G = (P_2K_2)^g = P_2N_G(M_2^g) = P_2N_G(M_1) = P_2H = P_2(P \cap H)M_1 = P_2M_1.$$

It follows that $|G| = |P_2||M_1| < |P||M_1| = |G|$. The final contradiction completes the proof.

Corollary 9. *Let M be a maximal subgroup of a group G with $|G : M| = r$, where r is a prime. Let p be the smallest prime dividing $|M|$. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is nearly $\mathcal{S}$-supplemented in G, then G is soluble.*

Proof. If $|G|$ is odd number, then G is soluble by the well known Feit-Thompson's theorem. Now we assume that $2||G|$. If $r = 2$, then M is normal in G. By Lemma 3(1), every maximal subgroup of P is nearly $\mathcal{S}$-supplemented in M. Theorem 3.4 implies that M is soluble. It follows that G is soluble. If $r \neq 2$, then $p = 2$ and P is a Sylow 2-subgroup of G. By using our Theorem 3.4, we obtain that G is soluble.

References

[1] A. Y. Alsheik Ahmad, J. J. Jaraden and A. N. Skiba, *On $\mathcal{U}_c$-normal subgroups of finite groups*, Algebra Colloq. 14 (2007), 25–36.

[2] A. Ballester-Bolinches, Y. Wang and X. Guo, *c-supplemented subgroups of finite groups*, Glasgow Math. J. 42 (2000), 383–389.

[3] K. Doerk and T. Hawkes, *Finite Solvable Groups*, Walter de Gruyter, New York, 1992.

[4] F. Gross, *Conjugacy of odd order Hall subgroups*, Bull. London Math. Soc. 19 (1987), 311-319.

[5] W. Guo, *The theory of classes of groups*, Science Press-Kluwer Academic Publishers, Beijing-New York-Dordrecht-Boston-London, 2000.

[6] W. Guo, *On $\mathcal{F}$-supplemented subgroups of fnite groups*, Manuscripta Math. 127 (2008), 139–150.

[7] X. Guo and K. P. Shum, *Finite p-nilpotent groups with some subgroups c-supplemented*, J. Aust. Math. Soc. 78 (2005), 429–439.

[8] X. Guo and K. P. Shum, *On p-nilpotency of finite groups with some subgroups c-supplemented*, Algebra Colloq. 10 (2003), 259–266.

[9] P. Hall, *A characteristic property of soluble groups*, J. London Math. Soc, 12 (1937), 198–200.

[10] C. Li and X. Yi, *On the semi cover-avoiding property and $\mathcal{F}$-supplementation*, Inter. J. Group Theory, 1 (2012), 21–31.

[11] C. Li and X. Yi, *Finite groups with some $\mathcal{F}$-supplemented subgroups II*, Arab. J. Math. 2 (2013), 51–56.

[12] Y. Wang, *Finite groups with some subgroups of Sylow subgroups c-supplemented*, J. Algebra, 224 (2000), 467–478.

[13] H. Wei, Y. Wang and Y. Li, *c-supplemented maximal and minimal subgroups of Sylow subgroups of finite groups*, Proc. Amer. Math. Soc. 132 (2004), 2197–2204.

[14] N. Yang and W. Guo, *On $\mathcal{F}_n$-supplemented subgroups of finite groups*, Asian-European J. Math. 1 (2008), 619–629.

[15] X. Yi, L. Miao, H. Zhang and W. Guo, *Finite groups with some $\mathcal{F}$-supplemented subgroups*, J. Algebra Appl. 9 (2010), 669–685.

[16] D. J. S. Robinson, *A Course in Theory of Group*, Spinger-Verlag, New York, 1982.

Xuemei Zhang
Department of Basic Sciences,
Yancheng Institute of Technology,
Yancheng, 224051, China
email: *xmzhang807@sohu.com*

Changwen Li
School of Mathematics and Statistics,
Jiangsu Normal University,
Xuzhou, 221116, China
email: *lcw2000@126.com*