Acta Universitatis Apulensis
ISSN: 1582-5329

No. 37/2014
pp. 141-159

Q-TYPE SPACES AND COMPACT COMPOSITION OPERATORS

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Abstract. In this paper, we study composition operators on Bloch space and $Q_K(p,q;n)$ spaces. We give a Carleson measure characterization on $Q_K(p,q;n)$ spaces, then we use this Carleson measure characterization of the compact compositions on $Q_K(p,q;n)$ spaces to show that every compact composition operator on $Q_K(p,q;n)$ spaces is compact on Bloch space.

2000 *Mathematics Subject Classification*: Primary $47B33$; Secondary $46E15$.

Keywords: keywords, phrases. $Q_K(p,q;n)$ spaces, Carleson measure, α-Bloch space.

1. Introduction

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk in the complex plane $\mathbb{C}$ and let $\partial\mathbb{D}$ be its boundary. Let $H(\mathbb{D})$ denote the class of all analytic functions on $\mathbb{D}$. For $0 < \alpha < \infty$. The α-Bloch space $\mathcal{B}^\alpha$ is the space of analyti functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha |f'(z)| < \infty.$$

It becomes a Banach space with norm

$$|f(0)| + \|f\|_{\mathcal{B}^\alpha}.$$

For $a \in \mathbb{D}$ the Möbius transformation $\varphi_a(z)$ is defined by

$$\varphi_a(z) = \frac{a-z}{1-\bar{a}z}, \ \text{ for } z \in \mathbb{D}.$$

For a point $a \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disk $D(a,r)$ with pseudo-hyperbolic center a and pseudo-hyperbolic radius r is defined by $D(a,r) = \varphi_a(rD)$. The pseudo-hyperbolic disk $D(a,r)$ is also an Euclidean disk: its Euclidean center and Euclidean radius are $\frac{(1-r^2)a}{1-r^2|a|^2}$ and $\frac{(1-|a|^2)r}{1-r^2|a|^2}$, respectively (see [41]). Let A denote the normalized Lebesgue area measure on $\mathbb{D}$, and for a Lebesgue measurable set $K_1 \subset \mathbb{D}$, denote by $|K_1|$ the measure of K_1 with respect to A. It follows immediately that:

$$|D(a,r)| = \frac{(1-|a|^2)^2}{(1-r^2|a|^2)^2} r^2.$$

The following identity is easily verified:

$$1 - |\varphi_a(z)|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\bar{a}z|^2} = (1-|z|^2)|\varphi_a'(z)|.$$

For $a \in \mathbb{D}$, the substitution $z = \varphi_a(w)$ results in the Jacobian change in measure given by $dA(w) = |\varphi_a'(z)|^2 dA(z)$. For a Lebesgue integrable or a non-negative Lebesgue measurable function h on $\mathbb{D}$, we thus have the following change-of-variable formula:

$$\int\limits_{D(0,r)} h(\varphi_a(w))dA(w) = \int\limits_{D(a,r)} h(z)\left(\frac{1-|\varphi_a(z)|^2}{1-|z|^2}\right)^2 dA(z) \ . \tag{1}$$

Note that $\varphi_a(\varphi_a(z)) = z$, thus $\varphi_a^{-1}(z) = \varphi_a(z)$. For $a, z \in \mathbb{D}$ and $0 < r < 1$, the pseudo-hyperbolic disc $D(a,r)$ is defined by $D(a,r) = \{z \in \mathbb{D} : |\varphi_a(z)| < r\}$. Denote by

$$g(z,a) = \log\left|\frac{1-\bar{a}z}{z-a}\right| = \log\frac{1}{|\varphi_a(z)|}$$

the Green's function of $\mathbb{D}$ with logarithmic singularity at $a \in \mathbb{D}$.
Two quantities A_f and B_f, both depending on an analytic function f on $\mathbb{D}$, are said to be equivalent, written as $A_f \approx B_f$, if there exists a finite positive constant C not depending on f such that for every analytic function f on $\mathbb{D}$ we have:

$$\frac{1}{C} B_f \le A_f \le C B_f.$$

If the quantities A_f and B_f, are equivalent, then in particular we have $A_f < \infty$ if and only if $B_f < \infty$.
Note: we say $K_1 \lesssim K_2$ (for two functions K_1 and K_2) if there is a constant $C > 0$ such that $K_1 \le C K_2$.

Definition 1. *[35] If E is any set, we define the characteristic function χ_E of the set E to be the function given by*

$$\chi_E(z) = \begin{cases} 1 & if \quad z \in E \\ \\ 0 & if \quad z \notin E. \end{cases}$$

The function $\chi_E(z)$ is measurable if and only if E is measurable.

Definition 2. *[38] Let f be an analytic function on $\mathbb{D}$ and let $0<p<\infty$. If*

$$\|f\|_p^p = \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty,$$

then f belongs to the Hardy space H^p. If $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)| < \infty$, then f belongs to the Hardy space H^∞. Moreover, $f \in H^2$ if and only if

$$\int_{\mathbb{D}} |f'(z)|^2 (1-|z|^2) dA(z) < \infty.$$

Definition 3. *[47] Let f be an analytic function in $\mathbb{D}$ and let $0<\alpha<\infty$. If*

$$\|f\|_{\mathcal{B}^\alpha} = \sup_{z\in\mathbb{D}} (1-|z|^2)^\alpha |f'(z)| < \infty,$$

then f belongs to the α-Bloch space $\mathcal{B}^\alpha$. The space $\mathcal{B}^1$ is called the Bloch space $\mathcal{B}$.

Definition 4. *[41, 42] Let f be an analytic function in $\mathbb{D}$ and let $1<p<\infty$. If*

$$\|f\|_{B_p}^p = \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2} dA(z) < \infty,$$

then f belongs to the Besov space B_p .

Definition 5. *(see [17] and [18]) For $0 \le p < \infty$, the spaces Q_p are defined by*

$$Q_p = \{f \in H(\mathbb{D}) : \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g^p(z,a) dA(z) < \infty\},$$

where the weight function $g(z,a) = \log\left|\frac{1-\bar{a}z}{a-z}\right|$ is defined as the composition of the Möbius transformation φ_a and the fundamental solution of the two-dimensional real Laplacian.

Definition 6. *[44] Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing function and let f be an analytic function in $\mathbb{D}$ then $f \in Q_K$ if*

$$\|f\|_{Q_K}^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^2 K(g(z,a)) dA(z) < \infty.$$

Definition 7. *[45] Let* $K : [0,\infty) \to [0,\infty)$ *be a right continuous and nondecreasing function. For* $0 < p < \infty$ *and* $-2 < q < \infty$*, we say that a function* f *analytic in* $\mathbb{D}$ *belongs to the space* $Q_K(p,q)$ *if*

$$\|f\|^p_{Q_K(p,q)} \quad = \quad \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f'(z)\right|^p \left(1-|z|^2\right)^q K(g(z,a))dA(z) < \infty$$

where $dA(z)$ *is the Euclidean area element on* $\mathbb{D}$.

Remark 1. *It should be noted that* $Q_K(p,q)$ *spaces are more general many classes of analytic functions. If* $p = 2, q = 0$*, we have that* $Q_K(p,q) = Q_p$ *(see [22, 44]). If* $K(t) = t^s$*, then* $Q_K(p,q) = F(p,q,s)$ *(see [46]) that* $F(p,q,s)$ *is contained in* $\frac{q+2}{p}-$*Bloch space.*

For $0 < p < \infty$ and $-2 < q < \infty$, we define the nth derivative $Q_K(p,q;n)$ as follows;

$$\|f\|^p_{Q_K(p,q;n)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \left|f^{(n)}(z)\right|^p \left(1-|z|^2\right)^q K(g(z,a))dA(z) < \infty$$

In this paper, we will study $Q_K(p,q;n)$ spaces with a right continuous and nondecreasing function $K : [0,\infty) \to [0,\infty)$. This choice for the weight function is due to some technical reasons. Also, we assume throughout the paper that

$$\int_0^1 (1-r^2)^{-2} K(\log\frac{1}{r}) r dr < \infty.$$

We can define an auxiliary function as follows:

$$\varphi_K(s) = \sup_{0<t\leq 1} \frac{K(st)}{K(t)}, \qquad 0 < s < \infty,$$

we assume that

$$\int_0^1 \varphi_K(s)\frac{ds}{s} < \infty \qquad \text{(see [23])}, \tag{2}$$

and

$$\int_1^\infty \varphi_K(s)\frac{ds}{s^2} < \infty \qquad \text{(see [23])}. \tag{3}$$

From now we take the above weight function K satisfies the following properties :
(a) K is nondecreasing on $[0,\infty)$,
(b) K is second differentiable on $(0,1)$,
(c) $\int\limits_0^{\frac{1}{e}} K(\log(\frac{1}{r}))rdr < \infty$,
(d) $K(t) = K(1) > 0,\ t \geq 1$ and
(e) $K(2t) \approx K(t),\ \ t \geq 0.$

A linear subspace X of $\mathcal{B}$ with a semi norm $\|.\|_X$ is Möbius invariant if for all Möbius transformation ϕ and all $f \in X$, $f \circ \phi \in X$ and $\|f \circ \phi\|_X = \|f\|_X$, there exists a positive constant λ such that

$$\|f\|_{\mathcal{B}} \leq \lambda \|f\|_X.$$

It is easy to see that $\mathcal{B}$ is a Möbius invariant space.
A Möbius invariant Banach space X is a Möbius invariant subspace of the Bloch space with a seminorm $\|.\|_X$, whose norm is

$$f \to \|f\|_X \quad \text{or} \quad f \to |f(0)| + \|f\|_X.$$

Rubel and Timoney showed in [34] that $\mathcal{B}$ is the largest Möbius invariant Banach space that possesses a decent linear functional. It is clear that $Q_K(p,q;n)$ is a Banach space with the norm $\|f\| = |f(0)| + \|f\|_{K,p,q;n}$ where $p \geq 1$. If $q+2 = p$, $Q_K(p,q;n)$ is Möbius invariant, i.e.,

$$\|f \circ \varphi_a\| = \|f\|_{K,p,q;n} \quad \text{for all } a \in \mathbb{D}.$$

For a subarc $I \subset \partial\mathbb{D}$, let

$$S(I) = \{r\xi \in \mathbb{D} : 1 - |I| < r < 1,\ \xi \in I\}.$$

If $|I| \geq 1$ then we set $S(I) = \mathbb{D}$. For $0 < p < \infty$, we say that a positive measure $d\mu$ is a p-Carleson measure on $\mathbb{D}$ if

$$\sup_{I \subset \partial\mathbb{D}} \frac{\mu(S(I))}{|I|^p} < \infty.$$

Here and henceforth $\sup\limits_{I \subset \partial\mathbb{D}}$ indicates the supremum taken over all subarcs I of $\partial\mathbb{D}$. Note that $p = 1$, gives the classical Carleson measure (cf. [21]). For several studies about Carleson measure and p-Carleson measure on some different classes of holomorphic Banach function spaces, we refer to [9, 16, 19] and others.

From [12, 14, 23], we know that a positive Borel measure μ on $\mathbb{D}$ is called a K-Carleson measure if

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}} \mu(S(I)) < \infty,$$

where the supremum is taken over all subarcs I of $\partial\mathbb{D}$, and

$$\mu(S(I)) = \int_{S(I)} K\left(\frac{1-|z|}{|I|}\right) d\mu(z).$$

Also, μ is said to be a compact K-Carleson measure if

$$\|\mu\|_K < \infty \text{ and } \lim_{|I|\to 0} \mu(S(I)) = 0,$$

where the supremum is taken over all subarcs $I \subset \partial\mathbb{D}$. Here, for the subarc I of $\partial\mathbb{D}$, $|I|$ is the length of I and

$$S(I) = \{r\xi : \xi \in I, 1-|I| < r < 1\}$$

is the corresponding Carleson box based on I.

Let ϕ be an analytic self-map of unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ and let $dA(z)$ be the Euclidean area element on $\mathbb{D}$. Associated with ϕ, the composition operator C_ϕ is defined by

$$C_\phi f = f \circ \phi.$$

The problem of boundedness and compactness of C_ϕ has been studied in many Banach spaces of analytic functions and the study of such operators has recently attracted the most attention.
Shapiro in [36], using Nevanlinna counting function, characterized the compact composition operator on H^2 as follows:
C_ϕ is a compact operator on H^2 if and only if

$$\lim_{|w|\to 1} \frac{N_\phi(w)}{-\log|w|} = 0.$$

MacCluer in [29], Madigan in [31], Roan in [33], and Shapiro in [37] have characterized the boundedness and compactness of C_ϕ in "small" spaces. In "large" spaces, MacCluer and Shapiro proved in [30] that C_ϕ is compact on Bergman spaces if and only if ϕ does not have an angular derivative at any point of $\partial\mathbb{D}$. Madigan and Matheson proved in [32] that C_ϕ is compact on the Bloch space if and only if

$$\lim_{|\phi(z)|\to 1} \frac{|\phi'(z)|(1-|z|^2)}{1-|\phi(z)|} = 0.$$

They also proved that if C_ϕ is compact on $\mathcal{B}$ then it can not have an angular derivative at any point of $\partial\mathbb{D}$. Tjani (see [43]) studied compact composition operators on the Besov spaces. Bourdon, Cima and Matheson in [20] and Smith in [40] investigated the same problem on $BMOA$. Li and Wulan in [28] gave some characterizations of compact composition operators on Q_K and $F(p,q,s)$ spaces. Very recently in [9, 10] there are some studies of boundedness and compactness of composition operators on some weighted analytic Besov spaces. On the other hand there are some studies on hyperbolic function spaces see [6, 7, 11, 27]

In this paper we study compact composition operator on the spaces $Q_K(p,q;n)$. Also we will discuss some important properties of these spaces, then we give a Carleson measure characterization of the compact composition operator C_ϕ on $Q_K(p,q;n)$ spaces.

2. Characterizations for $Q_K(p,q;n)$ spaces

In this section, we characterize analytic Bloch space by $Q_K(p,q;n)$ spaces. The main result is a general Besov-type characterization for $Q_K(p,q;n)$ spaces whih generalizes a Stroethoff theorem.

Theorem 1. *Let f be an analytic function in $\mathbb{D}$. Let $0<r<1,\ 0<p<\infty$ and $K:[0,\infty)\to[0,\infty)$ and let either $\alpha>0$ and $n\in\mathbb{N}$ or $\alpha>1$ and $n=0$. Then the following quantities are equivalent:*

(A) $\|f\|^p_{\mathcal{B}^\alpha}<\infty$.

(B) For $0<p<\infty$, we have

$$\sup_{a\in\mathbb{D}}\frac{1}{\big|D(a,r)\big|^{1-(\frac{\alpha+n-1}{2})p}}\int_{D(a,r)}|f^{(n)}(z)|^p dA(z)<\infty.$$

(C) For $0<p<\infty$, we have

$$\sup_{a\in\mathbb{D}}\int_{D(a,r)}|f^{(n)}(z)|^p\Big(1-|z|\Big)^{(\alpha+n-1)p-2}dA(z)<\infty.$$

(D) For $0<p<\infty$ and $-2<q<\infty$, we have

$$\sup_{a\in\mathbb{D}}\int_{D(a,r)}|f^{(n)}(z)|^p\big(1-|z|\big)^{(\alpha+n-1)p-2}K(1-|\varphi_a(z)|)\,dA(z)<\infty.$$

(E) For $0 < p < \infty$, *we have*

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \left(\log \frac{1}{|z|}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2 \, dA(z) < \infty.$$

(F)

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(z)|^p \left(1-|z|\right)^{(\alpha+n-1)p-2} K\big(g(z,a)\big) \, dA(z) < \infty, \quad \textit{if and only if}$$

$$\int_0^1 (1-r^2)^{-2} K(\log \frac{1}{r}) r dr < \infty. \tag{4}$$

Proof. Let $0 < r < 1$, $0 < p < \infty$ and $K : [0,\infty) \to [0,\infty)$. Because for every analytic function g on $\mathbb{D}$, $|g|^p$ is a subharmonic function, we have

$$\big|g(0)\big|^p \leq \frac{1}{\pi r^2} \int_{D(0,r)} \big|g(w)\big|^p \, dA(w).$$

Set $g = f^{(n)} \circ \varphi_a$, we obtain that

$$\begin{aligned} \big|f^{(n)}(a)\big|^p &\leq \frac{1}{\pi r^2} \int_{D(0,r)} \big|f^{(n)} \circ \varphi_a(w)\big|^p dA(w) \\ &= \frac{1}{\pi r^2} \int_{D(a,r)} \big|f^{(n)}(z)\big|^p \frac{(1-|\varphi_a(z)|^2)^2}{(1-|z|^2)^2} dA(z). \end{aligned}$$

Since,

$$\frac{1-|\varphi_a(z)|^2}{1-|z|^2} = |\varphi_a'(z)| \,, \quad \text{where} \quad \frac{1-|\varphi_a(z)|^2}{1-|z|^2} \leq \frac{4}{1-|a|^2} \quad a, z \in \mathbb{D}.$$

Then, we obtain that

$$\big|f^{(n)}(a)\big|^p \leq \frac{16}{\pi r^2 (1-|a|^2)^2} \int_{D(a,r)} \big|f^{(n)}(z)\big|^p dA(z).$$

Therefore, by $(1-|a|^2)^2 \approx (1-|z|^2)^2 \approx |D(a,r)|$, for $z \in D(a,r)$, we deduce that

$$|f^{(n)}(a)|^p (1-|a|)^p \leq \frac{16(1-|a|)^p}{\pi r^2 (1-|a|^2)^2} \int_{D(a,r)} |f^{(n)}(z)|^p dA(z).$$

Since $(1-|a|)^2 \approx (1-|a|^2)^2$, then

$$\begin{aligned} |f^{(n)}(a)|^p(1-|a|)^p &\leq \frac{16}{\pi r^2(1-|a|)^{2-p}} \int\limits_{D(a,r)} |f^{(n)}(z)|^p dA(z) \\ &\leq \frac{16\lambda}{\pi r^2 \big|D(a,r)\big|^{1-\frac{p}{2}}} \int\limits_{\mathbb{D}} |f^{(n)}(z)|^p dA(z) \\ &= \frac{M(r)}{\big|D(a,r)\big|^{1-\frac{p}{2}}} \int\limits_{\mathbb{D}} |f^{(n)}(z)|^p dA(z), \end{aligned}$$

where λ is a positive constant and $M(r)=\frac{16\lambda}{\pi r^2}$ is a constant depending on r. Thus the quantity (A) is less than or equal to a constant times the quantity (B).
From $|D(a,r)| \approx (1-|z|^2)^2$ for all $z \in D(a,r)$, it is obvious that $(B) \approx (C)$.
By $1-|\varphi_a(z)|^2 > 1-r^2$ and $1-|\varphi_a(z)| > 1-r$ for $z \in D(a,r)$, we thus obtain

$$\begin{aligned} &\int\limits_{D(a,r)} |f^{(n)}(z)|^p(1-|z|)^{(\alpha+n-1)p-2} dA(z) \\ &= \int\limits_{D(a,r)} |f^{(n)}(z)|^p(1-|z|)^{(\alpha+n-1)p-2}\, \frac{K(1-|\varphi_a(z)|^2)}{K(1-|\varphi_a(z)|^2)} dA(z) \\ &\leq \frac{1}{K(1-r^2)} \int\limits_{D(a,r)} |f^{(n)}(z)|^p(1-|z|)^{(\alpha+n-1)p-2}\, K(1-|\varphi_a(z)|^2) dA(z). \end{aligned}$$

Hence, the quality (C) is less than or equal to a constant times (D). By $1-|\varphi_a(z)|^2 \leq 2g(z,a)$ for all $z,a \in \mathbb{D}$, we obtain that the quantity (D) is less than or equal to a constant times (F).
The equivalent between the quantity (F) and quantity (A) follows from Wulan and Zhou (see [45]).
Now, from the inequality $1-|z|^2 \leq 2\log\frac{1}{|z|}$ for every $z \in \mathbb{D}$, putting $K(1-|\varphi_a(z)|) = (1-|\varphi_a(z)|)^2$ in (D), we see the quantity (D) is less than or equal to (E). Finally, let

$$\begin{aligned} I(a) &= \int\limits_{D(a,r)} \big|f^{(n)}(z)\big|^p \left(\log\frac{1}{|z|}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2\, dA(z) \\ &= \Bigg(\int\limits_{\mathbb{D}_{\frac{1}{4}}} + \int\limits_{\mathbb{D}\setminus\mathbb{D}_{\frac{1}{4}}}\Bigg) \big|f^{(n)}(z)\big|^p \left(\log\frac{1}{|z|}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2\, dA(z) \\ &= I_1(a) + I_2(a), \end{aligned}$$

where for $z \in \mathbb{D}_{\frac{1}{4}} = \{z : |z| < \frac{1}{4}\}$, $|\varphi_a'(z)|^2 = \frac{(1-|a|^2)}{|1-\bar{a}z|^4} \leq \frac{1}{(1-|z|)^4} \leq (\frac{4}{3})^4$, then we obtain

$$\begin{aligned}
I_1(a) &= \int\limits_{\mathbb{D}_{\frac{1}{4}}} \left|f^{(n)}(z)\right|^p \left(\log \frac{1}{|z|}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2 \, dA(z) \\
&\leq \|f\|^p_{\mathcal{B}^\alpha} \int\limits_{\mathbb{D}_{\frac{1}{4}}} \left(\frac{\log \frac{1}{|z|}}{(1-|z|)}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2 \, dA(z) \\
&\leq \|f\|^p_{\mathcal{B}^\alpha} \left(\frac{4}{3}\right)^{(\alpha+n-1)p+4} \int\limits_{\mathbb{D}_{\frac{1}{4}}} \left(\log \frac{1}{|z|}\right)^{(\alpha+n-1)p} dA(z) \\
&= \left(\frac{4}{3}\right)^{(\alpha+n-1)p+4} C(p) \|f\|^p_{\mathcal{B}^\alpha},
\end{aligned}$$

where

$$C(p) = \int\limits_{\mathbb{D}_{\frac{1}{4}}} \left(\log \frac{1}{|z|}\right)^{(\alpha+n-1)p} dA(z) < \infty.$$

Now, for $z \in \mathbb{D} \backslash \mathbb{D}_{\frac{1}{4}}$, we know that $\log \frac{1}{|z|} \leq 4(1-|z|^2) \leq 8(1-|z|)$, then

$$\begin{aligned}
I_2(a) &\leq 8 \int\limits_{\mathbb{D} \backslash \mathbb{D}_{\frac{1}{4}}} \left|f^{(n)}(z)\right|^p \left(\log \frac{1}{|z|}\right)^{(\alpha+n-1)p} |\varphi_a'(z)|^2 \, dA(z) \\
&\leq 8^p \|f\|^p_{\mathcal{B}^\alpha} \int\limits_{\mathbb{D} \backslash \mathbb{D}_{\frac{1}{4}}} |\varphi_a'(z)|^2 dA(z) \leq \lambda \|f\|^p_{\mathcal{B}^\alpha}
\end{aligned}$$

where λ is a positive constant. Hence, the quantity (E) is less than or equal to a constant times (A). The proof is complete.

Remark 2. *It is still an open problem to generalize Theorem 1 in Clifford analysis. For more details on some classes of quaternion function spaces, we refer to ([1],[2], [3], [4], [5], [8] [13],[15], [24], [25], [26]) and others.*

The following lemma is proved by Tjani in [43]:

Lemma 2. *[43] Let X, Y be two Banach spaces of analytic functions on $\mathbb{D}$. Suppose (i) the point evaluation functionals on X are continuous.*

(ii) the closed unit ball of X is a compact subset of X in the topology of uniform convergence on compact sets.
(iii) $T : X \to Y$ is continuous when X and Y are given the topology of uniform convergence on compact sets.
Then T is a compact operator if and only if given a bounded sequence (f_n) in X such that $f_n \to 0$ uniformly on compact sets, then the sequence (Tf_n) converges to zero in the norm of Y.

Recall that a linear operator $T : X \to Y$ is said to be compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of the space of all analytic functions $H(\mathbb{D})$, we call that T is compact from X to Y if and only if for each bounded sequence (x_n) in X, the sequence $(Tx_n) \in Y$ contains a subsequence converging to some limit in Y.

3. Composition operators

Using Riesz Factorization theorem and Vitali's convergence theorem, Shapiro and Taylor showed in [39] that, C_ϕ is compact on H^p, for some $0 < p < \infty$ if and only if C_ϕ is compact on H^2. Moreover, Shapiro solved the compactness problem for composition operators on H^p using the Nevanlinna counting function

$$N_\phi(w) = \sum_{\phi(z)=w} -\log|w| \quad \text{(see [26])}.$$

The counting function for the Besov space B_p is

$$N_p(w,\phi) = \sum_{\phi(z)=w} \Big(|\phi'(z)|(1-|z|^2)\Big)^{p-2} \quad \text{for } w \in \mathbb{D},\ p > 1 \quad \text{(see [39])}.$$

In [28], Li and Wulan gave a modification of the Nevanlinna type counting function on $F(p,q,s)$ spaces as follows:

$$N_{p,q,s,\phi}(w) = \sum_{\phi(z)=w} |\phi'(z)|^{p-2}(1-|z|^2)^q g^s(z,a) \quad \text{(see [28])}$$

for $w \in \phi(\mathbb{D})$, $2 \le p < \infty$, $-2 < q < \infty$ and $0 < s < \infty$.
Now, we give the following definition:

Definition 8. *The counting function for the $Q_K(p,q)$ spaces is*

$$N_{K,p,q,\phi}(w) = \sum_{\phi(z)=w} |\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a)),$$

for $w \in \phi(\mathbb{D})$, $2 \le p < \infty$, $-2 < q < \infty$ *and* $K : [0,\infty) \to [0,\infty)$.

The above counting functions come up in the change of variables formula in the respective spaces as follows:

For $f \in Q_K(p,q;n)$, $2 \le p < \infty$, $-2 < q < \infty, n \in \mathbb{N}$ and $K : [0,\infty) \to [0,\infty)$, we have

$$\begin{aligned}\|C_\phi f\|^p_{Q_K(p,q;n)} &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |(f\circ\phi)^{(n)}(z)|^p (1-|z|^2)^q K(g(z,a))dA(z)\\ &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(\phi(z))|^p |\phi'(z)|^2 |\phi'(z)|^{p-2}(1-|z|^2)^q K(g(z,a))dA(z)\end{aligned}$$

By making a non-univalent change of variables, we obtain that

$$\|C_\phi f\|^p_{Q_K(p,q;n)} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(w)|^p N_{K,p,q,\phi}(w)dA(w). \tag{5}$$

Now we consider the restriction of C_ϕ to $Q_K(p,q;n)$. Then C_ϕ is bounded operator if and only if there is a positive constant λ such that

$$\|C_\phi f\|^p_{Q_K(p,q;n)} \le \lambda \|f\|^p_{Q_K(p,q;n)} \tag{6}$$

for all $f \in Q_K(p,q;n)$ or, equivalently,

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n)}(w)|^p N_{K,p,q,\phi}(w)dA(w) \le \lambda \|f\|^p_{Q_K(p,q;n)}$$

Here, we shall show that the measures which obey a "generalized" Carleson condition play a role in understanding which analytic function ϕ mapping $\mathbb{D}$ into $\mathbb{D}$ produce bounded composition operators on certain Möbius invariant spaces $X = (Q_K(p,q;n)$ or $\mathcal{B}^\alpha)$. This leads, as in [16], to the following definition of generalized Carleson type measure. Since we are interested in characterizing the compact composition operators, we will also talk about vanishing Carleson measure.

Definition 9. *Let* μ *be a positive measure on* $\mathbb{D}$ *and let* $X = \mathcal{B}^\alpha$ *or* $Q_K(p,q;n)$ *for* $0 < p < \infty$, $-2 < q < \infty, n \in \mathbb{N}$ *and* $K : [0,\infty) \to [0,\infty)$. *Then* μ *is an* (X,K)*-Carleson measure if there is a constant* $A > 0$ *such that*

$$\int_{\mathbb{D}} |f^{(n)}(w)|^p d\mu(w) \le A\|f\|^p_X,$$

for all $f \in X$, *holds.*

We see that C_ϕ is a bounded operator on $Q_K(p,q;n)$ if and only if the measure $N_{K,p,q,\phi}(w)dA(w)$ is a $(Q_K(p,q;n),K)$-Carleson measure.
Now, we give characterization of compact composition operator on $Q_K(p,q;n)$ spaces in terms of Carleson-type measure.

Theorem 3. *Let* $0<p<\infty$ *and* $K:[0,\infty)\to[0,\infty)$. *The following are equivalent:*
(i) μ *is a* $(Q_K(p,(\alpha+n-1)p-2;n),K)$*-Carleson measure,*
(ii) *there is a constant* A *such that* $\mu(S(I))\le A|I|^p$ *for a subarc* $I\subset\partial\mathbb{D}$,
(iii) *there is a constant* C *such that*

$$\int\limits_{\mathbb{D}} |\varphi_a^{(n)}(z)|^p d\mu(z) \ \le\ C \quad for\ \ all\ \ a\in\mathbb{D}.$$

Proof. Suppose (i) holds. Then using Theorem 1 and Definition 9, we obtain

$$\int\limits_{\mathbb{D}} |f^{(n)}(z)|^p d\mu(z) \le C\int\limits_{\mathbb{D}} |f^{(n)}(z)|^p(1-|z|^2)^{(\alpha+n-1)p-2}K(g(z,a))dA(z),$$

for all $f\in Q_K(p,(\alpha+n-1)p-2;n)$. In particular this holds for $f(z)=\varphi_a(z)=\frac{a-z}{1-\bar{a}z}$.
Hence

$$\begin{aligned}\sup_{a\in\mathbb{D}}\int\limits_{\mathbb{D}} |\varphi_a^{(n)}(z)|^p d\mu(z) \ &\le\ C\sup_{a\in\mathbb{D}}\int\limits_{\mathbb{D}} |\varphi_a^{(n)}(z)|^p(1-|z|^2)^{(\alpha+n-1)p-2}K(g(z,a))dA(z)\\ &\le\ C\,\|\varphi_a\|^p_{Q_K(p,(\alpha+n-1)p-2;n)} \ \le\ C\,\lambda,\end{aligned}$$

for all $a\in\mathbb{D}$. This gives (iii).
Suppose that (iii) holds, we shall show that (ii) is true, hence

$$\begin{aligned}C\ &\ge \int\limits_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p d\mu(z) \ge \int\limits_{S(I)}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^p d\mu(z),\\ &\gtrsim \frac{\mu(S(I))}{|I|^p} \ge \frac{\lambda}{|I|^p}\mu(S(I))\end{aligned}$$

we have

$$\mu(S(I))<A|I|^p$$

This gives (ii).
Suppose now that (ii) holds, we shall show that (i) is true, thus completing the implications. For $z=re^{i\theta}$, let

$$E_1(z)=\left\{w:|w-z|<\frac{1-|z|}{2}\right\},$$

$$E_2(z)=\Big\{w:|w-z|<1-|z|\Big\}.$$

Then

$$E_1(z)\subseteq E_2(z)\subseteq S(2(1-|z|),\theta).$$

Further, if $w\in E_1(z)$, then

$$\frac{1}{2}\le\frac{1-|w|}{1-|z|}\le\frac{3}{2}.$$

Let $f\in Q_K(p,(\alpha+n-1)p-2;n)$ because f is analytic we have

$$f^{(n)}(z)=\frac{4}{\pi(1-|z|)^2}\int\limits_{E_1(z)} f^{(n)}(w)dA(w).$$

Therefore by Jensen's inequality (see [35]),

$$|f^{(n)}(z)|^p\le\frac{4}{\pi(1-|z|)^2}\int\limits_{E_1(z)} |f^{(n)}(w)|^p dA(w).$$

Thus,

$$\begin{aligned}\int\limits_{\mathbb{D}}|f^{(n)}(z)|^p d\mu(z) &\le \int\limits_{\mathbb{D}}\frac{4}{\pi(1-|z|)^2}\Big(\int\limits_{E_1(z)}|f^{(n)}(w)|^p dA(w)\Big)d\mu(z)\\ &\le \frac{4}{\pi}\int\limits_{\mathbb{D}}\Big(\int\limits_{E_1(z)}|f^{(n)}(w)|^p\Big(\frac{3}{2(1-|w|)}\Big)^2 dA(w)\Big)d\mu(z)\\ &\le \frac{9}{\pi}\int\limits_{\mathbb{D}}\int\limits_{\mathbb{D}}|f^{(n)}(w)|^p\chi_{E_1(z)}(w)(1-|w|)^{-2}dA(w)d\mu(z)\\ &\le \frac{9}{\pi}\int\limits_{\mathbb{D}}|f^{(n)}(w)|^p(1-|w|)^{-2}\int\limits_{\mathbb{D}}\chi_{E_1(z)}(w)d\mu(z)dA(w).\end{aligned}$$

However, $\chi_{E_1(z)}(w)\le\chi_{S(2(1-|z|),\theta)}(w)$, $z=|z|e^{i\theta}$, since $w\in E_1(z)$ implies that

$$|w-e^{i\theta}|<2(1-|w|).$$

Now applying (ii) and using condition (e), we have

$$\int\limits_{\mathbb{D}}\chi_{E_1(z)}d\mu(z)\le\mu(S(2(1-|w|),\theta))\le A2^q(1-|w|)^p.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{D}} |f^{(n)}(z)|^p d\mu(z) &\leq \frac{9}{\pi} A 2^q \int_{\mathbb{D}} |f^{(n)}(w)|^p (1-|w|)^{p-2} K(g(z,a)) dA(w) \\ &\leq C \int_{\mathbb{D}} |f^{(n)}(w)|^p (1-|w|)^{(\alpha+n-1)p-2} K(g(z,a)) dA(w),\end{aligned}$$

where C is a constant. By Theorem 1; the quantities (C) and (E) are equivalent so, we have

$$\begin{aligned}\int_{\mathbb{D}} |f^{(n)}(z)|^p d\mu(z) &\leq C \int_{\mathbb{D}} |f^{(n)}(w)|^p (1-|w|)^{(\alpha+n-1)p-2} K(g(z,a)) dA(w) \\ &\leq C \ \|f\|^p_{Q_K(p,(\alpha+n-1)p-2;n)},\end{aligned}$$

then,

$$\int_{\mathbb{D}} |f^{(n)}(z)|^p d\mu(z) \leq C \ \|f\|^p_{Q_K(p,(\alpha+n-1)p-2;n)}$$

which is (i). This finishes the proof.

Hence Theorem 3 yields:

Theorem 4. *Let ϕ be an analytic function on $\mathbb{D}$, $0 < p < \infty, n \in \mathbb{N}$ and $K : [0,\infty) \to [0,\infty)$. Then C_ϕ is a bounded operator on $Q_K(p,(\alpha+n-1)p-2;n)$ if and only if*

$$\sup_{a\in\mathbb{D}} \|C_\phi \ \varphi_a\|_{Q_K(p,(\alpha+n-1)p-2;n)} < \infty.$$

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