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# ON CHARACTER AMENABILITY OF BEURLING AND SECOND DUAL ALGEBRAS

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Abstract. We investigate the character amenability of Beurling algebras. Also, for a Banach algebra $A$, we studied the relations between the character amenability of $A''$ and Arens regularity of $A$ and the role of topological centres in the character amenability of $A''$.

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## 1. Introduction

The notion of amenability in Banach algebra was initiated by Johnson in [15]. Since then, amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [19] and [21].

In [10], Ghahramani and Loy introduced generalized notions of amenability with the hope that it will yield Banach algebra without bounded approximate identity which nonetheless had a form of amenability. All known approximate amenable Banach algebras have bounded approximate identities until recently when Ghahramani and Read in [13] give examples of Banach algebras which are boundedly approximately amenable but which do not have bounded approximate identities. This answers a question open since the year 2004 when Ghahramani and Loy founded the notion of approximate amenability.

Let $A$ be a Banach algebra over $\mathbb{C}$ and $\varphi : A \to \mathbb{C}$ be a character on $A$, that is, an algebra homomorphism from $A$ into $\mathbb{C}$, and let $\Phi_A$ denote the character space of $A$ (that is the set of all characters on $A$). In [22], Monfared introduced the notion of character amenable Banach algebras. His definition of this notion requires continuous derivations from $A$ into dual Banach $A$-bimodules to be inner, but only those modules are concerned where either of the left or right module action is defined by characters on A. As such character amenability is weaker than the classical amenability introduced by Johnson in [15], so all amenable Banach algebras are character amenable.

In [20], Mewomo and Okoli applied the concept of approximate amenability to that of character amenability and introduced the notions of approximate left character amenability, approximately right character amenability and approximately character amenability. They developed general theory on these notions and studied them for Banach algebras defined over locally compact groups and second duals of Banach algebras.

Various aspects of cohomologies of Beurling algebras have been studied by several authors, most notably are Gronbaek [9], Dales and Lau [5] and Grahramani, Loy and Zhang [12]. Beurling algebras are $L^1$-algebras associated with locally compact groups $G$ with an extra weights $\omega$ on the groups. It is shown in [9] that the Beurling algebra $L^1(G,\omega)$ is amenable as a Banach algebra if and only if $G$ is amenable as a locally compact group and $\{\omega(t)\omega(t^{-1}) : t \in G\}$ is bounded.

In [11], Ghahramani, Loy and Willis considered the possibility of the second dual of a Banach algebra being either amenable or weakly amenable.

In particular, they showed that for a Banach algebra $A$, the amenability of the second dual $A''$ of $A$ necessitates the amenability of A, and similarly for weak amenability provided $A$ is a left ideal in $A''$.

In this work, we study the character amenability of Beurling algebras $L^1(G,\omega)$ and that of the second dual $A''$ by focussing on the following questions:

1. under what condition on the weight on the locally compact group is the Beurling algebra character amenable?
2. Does the character amenability of the second dual $A''$ imply that $A$ is Arens regular?

The first question is natural. We show that under certain assumptions on $A''$, the answers to second question is positive. We also explore the roles of topological centres in the character amenability of $A''$.

## 2. Preliminaries

First, we recall some standard notions; for further details, see [4], [5] and [19].

Let $A$ be an algebra. The character space of $A$ is denoted by $\Phi_A$. Let $X$ be an $A$-bimodule. A *derivation* from $A$ to $X$ is a linear map $D : A \to X$ such that

$$D(ab) = D(a) \cdot b + a \cdot D(b) \quad (a, b \in A)\,.$$

For example, for $x \in X$, the map $\delta_x : A \to X$ defined by
$\delta_x(a) = a \cdot x - x \cdot a \quad (a \in A)$ is a derivation; derivations of this form are called the *inner derivations.*

Let $A$ be a Banach algebra, and let $X$ be an $A$-bimodule. Then $X$ is a Banach $A$-bimodule if $X$ is a Banach space and if there is a constant $k > 0$ such that

$$\|a \cdot x\| \leq k \|a\| \|x\|, \quad \|x \cdot a\| \leq k \|a\| \|x\| \quad (a \in A, x \in X).$$

By renorming $X$, we can suppose that $k = 1$. For example, $A$ itself is Banach $A$-bimodule, and $X'$, the dual space of a Banach $A$-bimodule $X$, is a Banach $A$-bimodule with respect to the module operations specified for by

$$\langle x, a \cdot \lambda \rangle = \langle x \cdot a, \lambda \rangle, \quad \langle x, \lambda \cdot a \rangle = \langle a \cdot x, \lambda \rangle \quad (x \in X)$$

for $a \in A$ and $\lambda \in X'$; we say that $X'$ is the *dual module* of $X$.

Let $A$ be a Banach algebra, and let $X$ be a Banach $A$-bimodule. Then $\mathcal{Z}^1(A, X)$ is the space of all continuous derivations from $A$ into $X$, $\mathcal{N}^1(A, X)$ is the space of all inner derivations from $A$ into $X$, and the first cohomology group of $A$ with coefficients in $X$ is the quotient space

$$\mathcal{H}^1(A, X) = \mathcal{Z}^1(A, X)/\mathcal{N}^1(A, X).$$

The Banach algebra $A$ is *amenable* if $\mathcal{H}^1(A, X') = \{0\}$ for each Banach $A$-bimodule $X$.

A derivation $D : A \to X$ is *approximately inner* if there is a net $(x_v)$ in $X$ such that

$$D(a) = \lim_v (a \cdot x_v - x_v \cdot a) \quad (a \in A),$$

the limit being taken in $(X, \|.\|)$. That is, $D(a) = \lim_v \delta_{x_v}(a)$, where $(\delta_{x_v})$ is a net of inner derivations. The Banach algebra $A$ is *approximately amenable* if, for each Banach $A$-bimodule $X$, every continuous derivation $D : A \to X'$ is approximately inner.

We let $\mathcal{M}^A_{\varphi_r}$ denote the class of Banach $A$- bimodule $X$ for which the right module action of $A$ on $X$ is given by $x \cdot a = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$, and $\mathcal{M}^A_{\varphi_l}$ denote the class of Banach $A$- bimodule $X$ for which the left module action of $A$ on $X$ is given by $a \cdot x = \varphi(a)x \quad (a \in A, x \in X, \varphi \in \Phi_A)$. If the right module action of $A$ on $X$ is given by $x \cdot a = \varphi(a)x$, then it is easy to see that the left module action of $A$ on the dual module $X'$ is given by $a \cdot f = \varphi(a)f \quad (a \in A, f \in X', \varphi \in \Phi_A)$. Thus, we note that $X \in \mathcal{M}^A_{\varphi_r}$ (resp. $X \in \mathcal{M}^A_{\varphi_l}$) if and only if $X' \in \mathcal{M}^A_{\varphi_l}$ (resp. $X' \in \mathcal{M}^A_{\varphi_r}$).

Let $A$ be a Banach algebra and let $\varphi \in \Phi_A$, we recall from [14], see also [22] that (i) $A$ is left $\varphi$-amenable if every continuous derivation $D : A \to X'$ is inner for every

$X \in \mathcal{M}_{\varphi_r}^A$;
(ii) $A$ is right $\varphi$-amenable if every continuous derivation $D : A \to X'$ is inner for every $X \in \mathcal{M}_{\varphi_l}^A$;
(iii) $A$ is left character amenable if it is left $\varphi$-amenable for every $\varphi \in \Phi_A$;
(iv) $A$ is right character amenable if it is right $\varphi$-amenable for every $\varphi \in \Phi_A$;
(v) $A$ is character amenable if it is both left and right character amenable;
(vi) $A$ is left $\varphi$-contractible if every continuous derivation $D : A \to X$ is inner for every $X \in \mathcal{M}_{\varphi_r}^A$;
(vii) $A$ is right $\varphi$-contractible if every continuous derivation $D : A \to X$ is inner for every $X \in \mathcal{M}_{\varphi_l}^A$;
(viii) $A$ is left character contractible if it is left $\varphi$-contractible for every $\varphi \in \Phi_A$;
(ix) $A$ is right character contractible if it is right $\varphi$-contractible for every $\varphi \in \Phi_A$;
(x) $A$ is character contractible if it is both left and right character contractible.

Let $G$ be a locally compact group. We denote by $L^1(G)$ the group algebra of $G$. This is the Banach space

$$\{f : G \to \mathbb{C}, f \ measurable \ : \|f\|_1 := \int_G |f(t)| d\mu(t) < \infty\},$$

where $\mu$ denotes left Haar measure on $G$ and we equate functions that are equal almost everywhere with respect to $\mu$. The product on $L^1(G)$ is defined by

$$(f * g)(t) = \int_G f(s)g(s^{-1}t)d\mu(s) \quad (t \in G, f, g \in L^1(G)).$$

$(L^1(G), *, \|f\|_1)$ is a Banach algebra. In the case where $G$ is discrete, we write $l^1(G)$ for $L^1(G)$. For details, see [4].

A weight on a locally group $G$ is a continuous function $\omega : G \to (0, \infty)$ such that

$$\omega(st) \leq \omega(s)\omega(t) \quad (s, t \in G).$$

A weight $\omega$ on $G$ is said to be symmetric if $\omega(t^{-1}) = \omega(t) \quad (t \in G)$. Two weights $\omega_1$ and $\omega_2$ on $G$ are said to be equivalent if there exist constants $k_1 > 0$ and $k_2 > 0$ such that

$$k_1\omega_1(t) \leq \omega_2(t) \leq k_2\omega_2(t) \quad (t \in G).$$

For a continuous weight $\omega$ on $G$, we define the weighted spaces

$$L^1(G, \omega) := \{f \ Borel \ measurable \ : \|f\|_{L^1(G,\omega)} = \|\omega f\|_{L^1(G)} < \infty\},$$

and

$$L^\infty(G, \frac{1}{\omega}) := \{f \ Borel \ measurable \ : \|f\|_{L^\infty(G,\frac{1}{\omega})} = \|\frac{f}{\omega}\|_{L^\infty(G)} < \infty\}.$$

$L^1(G,\omega)$ and $L^\infty(G,\frac{1}{\omega})$ are isometric to $L^1(G)$ and $L^\infty(G)$ respectively. Also, $L^\infty(G,\frac{1}{\omega})$ is the dual of $L^1(G,\omega)$ with the duality pairing

$$\langle f,g\rangle = \int_G f(t)g(t)d\mu(t) \quad (f \in L^1(G,\omega), g \in L^\infty(G,\frac{1}{\omega})),$$

where $\mu$ is the left Haar measure on $G$.

With the multiplication and norm $\|.\|_{L^1(G,\omega)}$, $L^1(G,\omega)$ becomes a Banach algebra and the algebra $L^1(G,\omega)$ is called the Beurling algebra on $G$. $L^1(G,\omega_1)$ and $L^1(G,\omega_2)$ are isomorphic as Banach algebras whenever $\omega_1$ and $\omega_2$ are equivalent. For further details see [5] and [12].

## 3. General Results

The following result is very useful in the proof of our main result in this section and it's proof follows from [22, Theorem 2.6 (iv)] and the fact that any statement about left character amenability turns into an analogous statement about right character amenability.

**Proposition 1.** Let $A$ be a Banach algebra. Suppose $A$ is character amenable. Then
(i) $A$ has a bounded approximate identity
(ii) the unitization algebra $A^\sharp$ is character amenable.

**Proposition 2.** Let $A$ be a character amenable Banach algebra and $I$ a closed ideal in $A$. Then the following statements are equivalent:
(i) $I$ has a bounded approximate identity
(ii) $I$ is character amenable.

*Proof.* Follows from [14, Lemma 3.2] and the fact that any statement about left character amenability turns into an analogous statement about right character amenability.

The main result in this section is given below and is useful in establishing our results on Beurling algebras.

**Theorem 3.** *Let $A$ be a Banach algebra. Suppose $A$ is character amenable and $I$ is a closed ideal of codimension one in $A$. Then*
*(i) $I$ has a bounded approximate identity*
*(ii) $I$ is character amenable.*

*Proof.* (i) Since $A$ is character amenable, then it has a bounded approximate identity $(e_\alpha)$ by Proposition 3.1(i). Also, since $I$ is a closed ideal of codimension one in $A$, we have $I = Ker\varphi$ for some $\varphi \in \Phi_A$. Let $A^\sharp = A \oplus \mathbb{C}e$ be the unitization of $A$, where $e$ is the adjoined unit. We can extend $\varphi \in \Phi_A$ to $\widetilde{\varphi} \in \Phi_{A^\sharp}$ by setting

$$\widetilde{\varphi}(a + ze) = \varphi(a) + z \quad (a \in A, z \in \mathbb{C}).$$

Thus $J = Ker\widetilde{\varphi}$ has codimension one as a subspace of $A^\sharp$ and $A^\sharp = J \oplus \mathbb{C}e \cong J^\sharp$. Thus $J$ is character amenable and so has a bounded approximate identity $(j_\beta)$ by Proposition 3.1(i), since $A$ is character amenable.
For each $\beta$, set

$$j_\beta = a_\beta + c_\beta e,$$

for some $a_\beta \in A$ and $c_\beta \in \mathbb{C}$ such that $c_\beta = -\varphi(a_\beta)$. Also, $\varphi(e_\alpha) \to 1$ since $(e_\alpha)$ is a bounded approximate identity for $A$. Let $f_\alpha = \varphi(e_\alpha)^{-1}e_\alpha$, then $(f_\alpha)$ is also a bounded approximate identity for $A$ and

$$\varphi(f_\alpha) = \varphi(\varphi(e_\alpha)^{-1}e_\alpha) = \varphi(e_\alpha)^{-1}\varphi(e_\alpha) = 1$$

for all $\alpha$. Let

$$i_{\beta,\alpha} = a_\beta + c_\beta f_\alpha,$$

then $(i_{\beta,\alpha})$ is a bounded net in $I$. We know that $j_\beta b \to b$ and $bj_\beta \to b$ for all $b \in I$. Thus,

$$i_{\beta,\alpha}b = j_\beta b + (c_\beta f_\alpha b - c_\beta b) = j_\beta b + c_\beta(f_\alpha b - b) \to b$$

and similarly

$$bi_{\beta,\alpha} \to b.$$

Hence $(i_{\beta,\alpha})$ is a bounded approximate identity for $I$.
(ii) This follows from (i) and Proposition 3.2.

## 4. Results on Beurling algebras

In this section, we shall consider the character amenability properties of Beurling algebras. We briefly recall the following definitions and notations.

Let $G$ be a locally compact group, we write $M(G)$ for the space of all (finite) complex, regular Borel measures on $G$, $C_b(G)$ for bounded continuous functions on $G$. For $x \in G$, the point mass $\delta_x \in M(G)$ at $x$ is defined through

$$\langle f, \delta_x \rangle := f(x) \quad (f \in C_0(G)).$$

A function $f \in C_b(G)$ is called left uniformly continuous if the map

$$G \to C_b(G), \quad x \to \delta_x * f$$

is continuous (with respect to the norm topology on $C_b(G)$), $f$ is right uniformly continuous if the map

$$G \to C_b(G), \quad x \to f * \delta_x$$

is continuous and is uniformly continuous if it is both left and right uniformly continuous. We denote

$$UC(G) = \{f \in C_b(G) : f \text{ is uniformly continuous}\}.$$

**Proposition 4.** Let $\omega$ be a weight on the locally compact group $G$ and let $\Omega(t) = \omega(t)\omega(t^{-1})$ for all $t \in G$. Then the following statements are equivalent:
(i) The Beurling algebra $L^1(G,\Omega)$ is character amenable
(ii) The Beurling algebra $L^1(G,\omega)$ is amenable
(iii) The group $G$ is amenable and $Sup\{\Omega(t) : t \in G\} < \infty$ (i.e. $\Omega$ is bounded on $G$).

*Proof.* By using [9, Theorem 0], it suffices to show that (i) implies (ii). Thus, suppose $L^1(G,\Omega)$ is character amenable and let

$$I_0 = \{f \in L^1(G,\Omega) : \int_G f(x)dx = 0\}$$

be the augmented ideal in $L^1(G,\Omega)$. Since $I_0$ is of codimension one and $L^1(G,\Omega)$ is character amenable, then $I_0$ has a bounded approximate identity $(e_\alpha)$, by Theorem 3.3. Let $f \in L^1(G,\Omega)$ be such that $\int_G f(x)dx = 1$. Then $\delta_x * f - f \in I_0$ for all $x \in G$, where

$$\delta_x * f * e_\alpha - f * e_\alpha - \delta_x * f + f \to 0.$$

That is,

$$\delta_x * (f - f * e_\alpha) - (f - f * e_\alpha) \to 0.$$

Since $\langle f - f * e_\alpha, 1\rangle = 1$, then condition (c) of [9, Theorem 0] holds and so $L^1(G,\omega)$ is amenable.

**Corollary 5.** *Let $G$ be a locally compact group. Suppose $\omega$ is a symmetric weight on $G$. Then $L^1(G,\omega)$ is character amenable if and only if it is amenable.*

*Proof.* Since every amenable Banach algebra is character amenable, we only have to show that $L^1(G,\omega)$ is amenable if it is character amenable. Thus, suppose $L^1(G,\omega)$ is character amenable. Consider the weight $\omega' = \sqrt{\omega}$ on $G$. Since $\omega$ is symmetric, then $\omega = \Omega'$ clearly satisfy $\Omega'(t) = \omega'(t)\omega'(t^{-1})$ for all $t \in G$. Thus by Proposition 4.1, $L^1(G,\omega')$ is amenable. It then follows from [9, Theorem 0] that $L^1(G,\Omega') = L^1(G,\omega)$ is amenable.

**Proposition 6.** Let $G$ be a locally compact group and $\omega$ a weight on $G$. Suppose $\omega$ is bounded away from 0 and $L^1(G,\omega)$ is character amenable. Then $G$ is amenable.

*Proof.* The hypothesis ensure that $L^1(G,\omega) \subset L^1(G)$, and hence $UC(G)$ is an $L^1(G,\omega)$-bimodule. By following [10, Theorem 3.2], there is an invariant mean on UC(G), and so $G$ is amenable.

**Proposition 7.** Let $G$ be a locally compact group and $\omega$ a symmetric weight on $G$. Suppose $\lim_{x\to\infty} \omega(x) = \infty$, then $L^1(G,\omega)$ is not character amenable.

*Proof.* By considering the weight $\omega' = \sqrt{\omega}$ on $G$. Since $\omega$ is symmetric, then $\omega = \Omega'$ clearly satisfy $\Omega'(t) = \omega'(t)\omega'(t^{-1})$ for all $t \in G$. Suppose $L^1(G,\omega)$ is character amenable and $\lim_{x\to\infty} \omega(x) = \infty$, then the argument of Proposition 4.1 implies that $L^1(G,\omega')$ is amenable and hence, $\Omega' = \omega$ is bounded, which is a contradiction.

**Corollary 8.** *Let $G$ be a locally compact group. Suppose a weight $\omega$ on $G$ satisfies*

$$\limsup_{t\in G} \frac{\omega(t^{-1})}{\omega(t)} \leq K \tag{4.1}$$

*for some $K > 0$, and $\lim_{t\to\infty} \omega(t) = \infty$, then $L^1(G,\omega)$ is not character amenable.*

*Proof.* Condition (4.1) implies there are positive constants $k_1$ and $k_2$ such that

$$k_1\omega(t^{-1}) \leq \omega(t) \leq k_2\omega(t^{-1})$$

for all $t \in G$, and so

$$\sqrt{k_1}\omega'(t) \leq \omega(t) \leq \sqrt{k_2}\omega'(t) \tag{4.2}$$

for all $t \in G$, and $\omega'(t) = \sqrt{\omega(t)}$ is symmetric on $G$. (4.2) imply $\omega$ and $\omega'$ are equivalent, and so $L^1(G,\omega') \cong L^1(G,\omega)$.
Thus by Proposition 4.4, $L^1(G,\omega') \cong L^1(G,\omega)$ is not character amenable.

## 5. Character amenability of second dual

Let $A$ be a Banach algebra. Then the second dual $A''$ of $A$ is a Banach $A$-bimodule for the maps $(a,\Phi) \to a\cdot\Phi$ and $(a,\Phi) \to \Phi\cdot a$ from $A \times A''$ to $A''$ that extend the product map $A\times A \to A$, $(a,b) \to ab$ on $A$. Arens in [1] defined two products, $\Box$ and $\Diamond$, on the second dual $A''$ of a Banach algebra $A$; $A''$ is a Banach algebra with respect to each of these products, and each algebra contains $A$ as a closed subalgebra. The products are called the *first* and *second Arens products* on $A''$, respectively. For the general theory of Arens products, see [6, 8]. We recall briefly the definitions. For $\Phi \in A''$, we set

$$\langle a, \lambda \cdot \Phi \rangle = \langle \Phi, a \cdot \lambda \rangle, \quad \langle a, \Phi \cdot \lambda \rangle = \langle \Phi, \lambda \cdot a \rangle \quad (a \in A, \lambda \in A'),$$

so that $\lambda \cdot \Phi, \Phi \cdot \lambda \in A'$. Let $\Phi, \Psi \in A''$. Then

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi \cdot \lambda \rangle, \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda \cdot \Phi \rangle \quad (\lambda \in A').$$

Suppose that $\Phi, \Psi \in A''$ and that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$ for nets $(a_\alpha)$ and $(b_\beta)$ in $A$. Then

$$\Phi \Box \Psi = \lim_\alpha \lim_\beta a_\alpha b_\beta \quad and \quad \Phi \Diamond \Psi = \lim_\beta \lim_\alpha a_\alpha b_\beta,$$

where all limits are taken in the weak-$*$ topology $\sigma(A'', A')$ on $A''$.

We recall the definition of the topological centres of the second dual of $A$. For details, see [5] and [6]. Let $A$ be a Banach algebra. The left and right topological centres, $Z_t^{(l)}(A'')$ and $Z_t^{(r)}(A'')$ of $A''$ are

$$Z_t^{(l)}(A'') = \{\Phi \in A'' : \Phi \Box \Psi = \Phi \Diamond \Psi \quad for \quad all \quad \Psi \in A''\},$$

$$Z_t^{(r)}(A'') = \{\Phi \in A'' : \Psi \Box \Phi = \Psi \Diamond \Phi \quad for \quad all \quad \Psi \in A''\},$$

respectively. Clearly $Z_t^{(l)}(A'')$ and $Z_t^{(r)}(A'')$ are closed subalgebras of $A''$ endowed with the Arens products. The Banach algebra $A$ is Arens regular if $Z_t^{(l)}(A'') = Z_t^{(r)}(A'') = A''$.

For a Banach algebra $A$, we denote by $A^{op}$ the opposite Banach algebra to $A$, this Banach algebra has the product $(a, b) \to ba, \quad A \times A \to A$. Every Banach $A$-bimodule $X$ has a canonical $A^{op}$-bimodule structure given by $a \circ x = x \cdot a$ and $x \circ a = a \cdot x \quad (a \in A^{op}, x \in X)$.

The following is from Corollary 6.7 (ii) of [14].

**Proposition 5.1** *Let $A$ be a Banach algebra and $\varphi \in \Phi_A \cup \{0\}$. $A$ is left [right] $\varphi$-amenable if and only if $A^{op}$ is right [left] $\varphi$-contractible.*

Our next result follows from Proposition 5.1 and the definitions of character amenability and character contractibility of $A$.

**Proposition 5.2** *Let $A$ be a Banach algebra. $A$ is character amenable if and only if $A^{op}$ is character contractible.*

**Proposition 5.3** *Let $A$ be a Banach algebra.*
*(i) Suppose $A$ is commutative. Then $(A'', \Box)$ is character amenability if and only if*

*$(A'', \Diamond)$ is character contractible.*
*(ii) Suppose $A$ admits a continuous anti-isomorphism. Then $(A'', \Box)$ is character amenable if and only if $(A'', \Diamond)$ is character contractible.*

*Proof.* (i) Since $A$ is commutative, then $\lambda \cdot \Phi = \Phi \cdot \lambda \quad (\lambda \in A', \Phi \in A'')$, and $\Phi \Box \Psi = \Psi \Diamond \Phi \quad (\Phi, \Psi \in A'')$, and so $(A'', \Diamond) = (A'', \Box)^{op}$.
Thus by Proposition 5.2, $(A'', \Box)$ is character amenable if and only if $(A'', \Diamond)$ is character contractible.
(ii) Let $\tau : A \to A$ be a continuous anti-isomorphism of $A$. Let $\Phi, \Psi \in (A'', \Box)$ and let $(a_\alpha)$ and $(b_\beta)$ be nets in $A$ such that $\Phi = \lim_\alpha a_\alpha$ and $\Psi = \lim_\beta b_\beta$. Let $\tau'' : (A'', \Box) \to (A'', \Diamond)$ be the second dual of $\tau$. Then

$$\tau''(\Phi \Box \Psi) = \lim_\alpha \lim_\beta \tau''(a_\alpha b_\beta)$$

$$= \lim_\alpha \lim_\beta \tau''(b_\beta)\tau''(a_\alpha) = \tau''(\Psi) \Diamond \tau''(\Phi).$$

Thus $\tau''$ is an isomorphism from $(A'', \Box)$ onto $(A'', \Diamond)^{op}$ and so, by (i), $(A'', \Box)$ is character amenable if and only if $(A'', \Diamond)$ is character contractible.

Let $A$ be a Banach algebra and $\varphi \in \Phi_A$. There are various formulations of the notion of $\varphi$-amenability. The following two are from [14]:
The Banach algebra $A$ is left [right] $\varphi$-amenable if and only if either, and hence both of the following holds:
(i) $A$ has a bounded left [right] $\varphi$-approximate diagonal, that is a bounded net $(m_\alpha)$ in $A \hat{\otimes} A$ such that

$$\|m_\alpha \cdot a - \varphi(a) m_\alpha\| \to 0 \quad [\|a \cdot m_\alpha - \varphi(a) m_\alpha\| \to 0] \quad (a \in A)$$

and

$$\varphi(\pi(m_\alpha)) \to 1.$$

(ii) $A$ has a left [right] $\varphi$-virtual diagonal, that is an element $M \in (A \hat{\otimes} A)''$ such that

$$M \cdot a = \varphi(a) M \quad [a \cdot M = \varphi(a) M] \quad (a \in A) \;\; \textit{and} \;\; \pi''(M)(\varphi) = 1,$$

where $\pi : A \hat{\otimes} A \to A$ is the product map defined by $\pi(a \otimes b) = ab \quad (a, b \in A)$.

We denote by $\hat{A}$ the image of $A$ in $A''$ under the canonical mapping and also assume $A''$ has the first Arens product in the next results.

The following Lemma is from [11, Lemma 1.7] and it is very useful in our next result.

**Lemma 5.4** *Let $A$ be a Banach algebra. Then there is a continuous linear mapping $\psi : A'' \hat{\otimes} A'' \to (A \hat{\otimes} A)''$ such that for $a, b, x \in A$ and $m \in A'' \hat{\otimes} A''$ the following holds:*

*(i)* $\psi(\hat{a} \otimes \hat{b}) = \widehat{(a \otimes b)}$
*(ii)* $\psi(m) \cdot x = \psi(m \cdot x)$
*(iii)* $x \cdot \psi(m) = \psi(x \cdot m)$
*(iv)* $\pi''_A(\psi(m)) = \pi_{A''}(m)$.

**Proposition 5.5** *Let $A$ be a Banach algebra and $B$ be a closed subalgebra of $A''$ such that $\hat{A} \subset B$. Suppose $B$ is left [right] $\varphi$-amenable for $\varphi \in \Phi_A$, then $A$ is left [right] $\varphi$-amenable. In particular, if $B$ is left [right] character amenable, then $A$ is left [right] character amenable.*

*Proof.* From the definition of projective tensor norm, we see that when both $B\widehat{\otimes}B$ and $A''\widehat{\otimes}A''$ are equipped with the projective tensor norm, then the map $\tau : B\widehat{\otimes}B \to A''\widehat{\otimes}A''$ defined by

$$\tau(b \otimes c) = b \otimes c \quad (b, c \in B)$$

is norm decreasing.
Let $(m_\alpha)$ be a bounded left $\varphi$-approximate diagonal for $B$ and set $\Gamma = \psi \circ \tau : B\widehat{\otimes}B \to (A\widehat{\otimes}A)''$, where $\psi$ is the continuous linear mapping defined in Lemma 5.3. Then for all $a \in A$, we have

$$\|\Gamma(m_\alpha) \cdot a - \varphi(a)\Gamma(m_\alpha)\| \to 0$$

and

$$\pi''_A(\Gamma(m_\alpha))(\varphi) \to 1.$$

If $M$ is a weak*-cluster point of $(\Gamma(m_\alpha))$ in $(A\hat{\otimes}A)''$, then for each $a \in A$, we have

$$M \cdot a = \varphi(a)M \;\; \textit{and} \;\; \pi''_A(M)(\varphi) = 1.$$

Thus $M$ is a left $\varphi$-virtual diagonal for $A$ and so $A$ is left $\varphi$-amenable.
The right side version can be proved similarly.

**Corollary 5.6** *Let $A$ be a Banach algebra. Suppose $Z_t^{(l)}(A'')$ or $Z_t^{(r)}(A'')$ is left [right] character amenable. Then $A$ is left [right] character amenable.*

We recall that $A'$ is said to factor on the left if $A'A = A'$, [17]. When $A$ has a bounded approximate identity and $A''$ has an identity, then $A'$ factors on the left. The following Lemma is [11, Lemma 1.1].

**Lemma 5.7** *Let $A$ be a Banach algebra such that $A''$ has a bounded approximate identity. Then $A''$ has an identity.*

With these, we have the next result.

**Theorem 5.8** *Let $A$ be a Banach algebra. Suppose that $(A'', \Box)$ is character amenable and $\hat{A}\Box A'' \subset Z_t^{(l)}(A'')$. Then $A$ is Arens regular.*

*Proof.* Since $(A'', \Box)$ is character amenable, then it is both left and right character amenable and so it has a bounded left approximate identity and a bounded right approximate identity [22, Theorem 2.3]. Thus, it has a bounded approximate identity by [2, Proposition 11.3], see also [7, Proposition 2.6]. Also, character amenability of $A''$ necessitates that of $A$ [14, Theorem 3.8] and so $A$ has a bounded approximate identity. Hence $A'$ factors on the left, that is $A' \cdot A = A'$. Let $f \in A'$, then $f = g \cdot a$, for some $g \in A'$ and $a \in A$. Let $\Phi, \Psi \in A''$, and $f \in A'$. Then, since $\hat{a}\Box\Phi \in Z_t^{(l)}(A'')$ and $\hat{a}\Box\Phi = \hat{a}\Diamond\Phi$, we have

$$\langle \Phi\Box\Psi, f\rangle = \langle \Phi\Box\Psi, g\cdot a\rangle = \langle \hat{a}\Box(\Phi\Box\Psi), g\rangle = \langle (\hat{a}\Box\Phi)\Box\Psi, g\rangle$$

$$= \langle (\hat{a}\Box\Phi)\Diamond\Psi, g\rangle = \langle (\hat{a}\Diamond\Phi)\Diamond\Psi, g\rangle$$

$$= \langle \hat{a}\Diamond(\Phi\Diamond\Psi), g\rangle = \langle \Phi\Diamond\Psi, g\cdot a\rangle = \langle \Phi\Diamond\Psi, f\rangle$$

and so, $\Phi\Box\Psi = \Phi\Diamond\Psi$. Thus $A$ is Arens regular.

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