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SLANT SUBMANIFOLDS OF A CONFORMAL SASAKIAN MANIFOLD

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Abstract. In this paper, we introduce a conformal Sasakian manifold and obtain some results on slant submanifolds of a conformal Sasakian manifold. In particular, we characterize three-dimensional slant submanifolds of a conformal Sasakian manifold via covariant derivative of T^2 , T and N where T is the tangent projection and N is the normal projection over the submanifold of a conformal Sasakian manifold.

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1. Introduction

I. Vaisman in [6], introduced the conformal changes of almost metric structures as follows. If M is a $(2n+1)$-dimensional differentiable manifold endowed with an almost contact metric structure (φ, ξ, η, g), a conformal change of the metric g leads to a metric which is no more compatible with the almost contact structure (φ, ξ, η). This can be corrected by a convenient change of ξ and η which implies rather strong restrictions. Using this definition, we introduce a new type of almost contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ on a $(2n+1)$-dimensional manifold $\overline{M}$ which is said to be a conformal Sasakian structure if the structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is conformal related to a Sasakian structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$.

In 1990, B.Y.Chen generalized the concepts of holomorphic and totally real immersions into the slant immersions in contact geometry [3]. Afterwards, slant submanifolds have been studied by many geometers. In particular, In 1996, A. Lotta, extended the concept of slant immersions of a Riemannian manifold into an almost contact metric manifold as follows[4]. Let $\overline{M}$ be an almost contact metric manifold with structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ and let M be an immersed submanifold of $\overline{M}$. For any $p \in M$ and $X \in T_pM$, if the vectors X and ξ are linearly independent, the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between φX and T_pM is well-defined. If $\theta(X)$ is constant and does not depend on the choice of $p \in M$ and $X \in T_pM$, we say that M is slant in $\overline{M}$ and the constant angle $\theta(X)$ is called the slant angle of M in $\overline{M}$.

Latter, J.L.Cabrerizo, A.Carriazo, L.M.Fernandez and M.Fernandez, studied and characterized slant submanifolds of a K-contact and a Sasakian manifold [2]. Our aim in the present note is to study the slant submanifolds of a conformal Sasakian manifold.

The paper is organized as follows. In section 2, we review some basic definitions, formulaes and results on submanifolds' theory and almost contact metric manifolds. We also recall some results on slant submanifolds of an almost contact metric manifold which are useful for further study . In section 3, we introduce a conformal Sasakian manifold and give an example and some properties of submanifols of it. Section 4 is devoted to study and characterize of three-dimensional slant submanifolds of a conformal Sasakian manifold via covariant derivative of T^2 , T and N where T is the tangent projection and N is the normal projection over the submanifold of a conormal Sasakian manifold.

2. Preliminaries

2.1. Submanifolds

Let (M, g) be a submanifold of a Riemannian manifold $(\widetilde{M}, \widetilde{g})$ where g is the induced metric on M. Then, the Gauss and Weingarten formulas are given respectively by

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y) \quad and \quad \widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{1}$$

for any $X, Y \in TM$ and $V \in T^{\perp}M$, where $\widetilde{\nabla}$, ∇ and $\nabla^{\perp}$ are the Riemannian, induced Riemannian and induced normal connections in $\widetilde{M}$, M and the normal bundle $T^{\perp}M$ of M, respectively, and h is the second fundamental form of M related to the shape operator A by

$$g(A_V X, Y) = g(h(X, Y), V). \tag{2}$$

The equation of Gauss is given by

$$\begin{aligned} R(X, Y, Z, W) &= \widetilde{R}(X, Y, Z, W) + \widetilde{g}(h(X, W), h(Y, Z)) \\ &\quad -\widetilde{g}(h(X, Z), h(Y, W)), \end{aligned} \tag{3}$$

for all $X, Y, Z, W \in TM$, where $\widetilde{R}$ and R are the curvature tensors of $\widetilde{M}$ and M, respectively.

For any $X \in TM$, we write

$$\varphi X = TX + NX, \tag{4}$$

where TX (resp. NX) is the tangential (resp. normal) component of φX. Similarly, for any $V \in T^{\perp}M$, we have

$$\varphi V = tV + nV, \tag{5}$$

where tV(resp. nV) is the tangential (resp. normal) component of φV. Moreover, we have the following relations

$$\begin{aligned} g(X, tV) &= -g(NX, V), \\ g(nV, W) &= -g(V, nW), \\ g(tnV, X) &= -g(V, NTX), \end{aligned} \tag{6}$$

for any $X \in TM$ and $V \in TM^{\perp}$.

The submanifold M is said to be invariant (anti-invariant) if $\varphi X \in TM$($\varphi X \in T^{\perp}M$), for any $X \in TM$.

2.2. Almost contact metric manifold

Let $(\widetilde{M}, g)$ be an odd-dimensional Riemannian manifold. Then $\widetilde{M}$ is said to be an almost contact metric manifold [1] if there exist on $\widetilde{M}$ a tensor φ of type $(1,1)$, a vector field ξ (structure vector field), and a 1-form η satisfying

$$\begin{aligned} &\varphi^2 X = -X + \eta(X)\xi, \quad g(X, \xi) = \eta(X), \\ &g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \end{aligned} \tag{7}$$

for any $X, Y \in T\widetilde{M}$. The 2-form Φ is called the fundamental 2-form in $\widetilde{M}$ and the manifold is said to be a contact metric manifold if $\Phi = d\eta$.

The almost contact structure of $\widetilde{M}$ is said to be normal if $[\varphi, \varphi] + 2d\eta \otimes \xi = 0$, where $[\varphi, \varphi]$ is the Nijenhuis torsion of φ. A Sasakian manifold is a normal contact metric manifold. It is easy to show that an almost contact metric manifold is a Sasakian manifold if and only if

$$(\nabla_X \varphi)Y = g(X, Y)\xi - \eta(Y)X,$$

for any $X, Y \in T\widetilde{M}$, where ∇ is the Riemannian connection in $\widetilde{M}$.

2.3. Slant submanifold of a almost contact metric manifold

Let $\overline{M}$ is an almost contact metric manifold of dimension $2n+1$ with structure $(\overline{\varphi}, \overline{g}, \overline{\xi}, \overline{\eta})$. We say that an immersed submanifold M of an almost contact metric manifold $\overline{M}$ is slant in $\overline{M}$ if for any $p \in M$ and any $X \in T_pM$ such that X, ξ are linearly independent, the angle $\theta(X) \in [0, \frac{\pi}{2}]$ between ϕX and T_pM is a constant θ, that is θ does not depend on the choice of X and $p \in M$. θ is called the slant angle of M in $\overline{M}$. Invariant and anti-invariant submanifolds are slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively [4]. A slant submanifold which is not invariant nor anti-invariant is called a proper slant submanifold.

By Q, we mean T^2 and we define the covariant derivative of Q, T and N as

$$\begin{aligned} (\nabla_X Q)Y &= \nabla_X(QY) - Q(\nabla_X Y), \\ (\nabla_X T)Y &= \nabla_X(TY) - T(\nabla_X Y), \\ (\nabla_X N)Y &= \nabla^{\perp}_X(NY) - N(\nabla_X Y), \end{aligned} \tag{8}$$

for any $X, Y \in TM$, where ∇ and $\nabla^{\perp}$ are the induced Riemannian and induced normal connections in M and the normal bundle $T^{\perp}M$ of M, respectively.

We have the following theorem which characterize slant submanifolds of an almost contact metric manifold.

Theorem 1. *[2] Let M be a submanifold of an almost contact metric manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a constant $\lambda \in [0, 1]$ such that*

$$T^2 = -\lambda(I - \eta \otimes \xi). \tag{9}$$

Furthermore, in such case, if θ is the slant angle of M, it satisfies that $\lambda = \cos^2\theta$.

Corollary 2. *[2] Let M be a submanifold of an almost contact metric manifold $\overline{M}$,with slant angle θ. Then,for any $X, Y \in TM$, we have*

$$g(TX, TY) = \cos^2\theta(g(X, Y) - \eta(X)\eta(Y)), \tag{10}$$

$$g(NX, NY) = \sin^2\theta(g(X, Y) - \eta(X)\eta(Y)). \tag{11}$$

We recall the following results for latter use.

Lemma 3. *[4] Let M be a slant submanifold of an almost contact metric manifold $\overline{M}$. Denote by θ the slant angle of M. Then, at each point p of M, $Q|_{\mathcal{D}}$ has only one eigenvalue $\lambda_1 = -\cos^2\theta$.*

Lemma 4. *[5] Let M be a 3-dimensional slant submanifold of an almost contact metric manifold $\overline{M}$. Suppose that M is not anti invariant. If $p \in M$, then in a neighborhood of p there exist vector fields e_1, e_2 tangent to M, such that $\{\xi, e_1, e_2\}$ is a local orthonormal frame satisfying*

$$Te_1 = (\cos\theta)e_2, \quad Te_2 = -(\cos\theta)e_1. \tag{12}$$

3. Conformal Sasakian manifolds

A $(2n+1)$-dimensional Riemannian manifold $\overline{M}$ endowed with the almost contact metric structure $(\overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ is called a conformal Sasakian manifold if for a C^∞ function $f : \overline{M} \to \mathbb{R}$, there are

$$\widetilde{g} = exp(f)\overline{g}, \qquad \widetilde{\xi} = (exp(-f))^{\frac{1}{2}}\overline{\xi}, \qquad \widetilde{\eta} = (exp(f))^{\frac{1}{2}}\overline{\eta}, \qquad \widetilde{\varphi} = \overline{\varphi},$$

such that $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$ is a Sasakian manifold

Example 1. *Let $\mathbb{R}^{2n+1}$ be the (2n+1)-dimensional Euclidean space endowed with the almost contact metric structure $(\overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ defined by*

$$\overline{\varphi}\Big(\sum_{i=1}^{n}(X_i\frac{\partial}{\partial x^i} + Y_i\frac{\partial}{\partial y^i}) + Z\frac{\partial}{\partial z}\Big) = \sum_{i=1}^{n}\big(Y_i\frac{\partial}{\partial x^i} - X_i\frac{\partial}{\partial y^i}\big) + \sum_{i=1}^{n} Y_i y^i\frac{\partial}{\partial z},$$

$$\begin{aligned}
\overline{g} &= \exp(-f)\{\overline{\eta}\otimes\overline{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\},\\
\overline{\eta} &= (\exp(-f))^{\frac{1}{2}}\{\frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i)\},\\
\overline{\xi} &= (\exp(f))^{\frac{1}{2}}\{2\frac{\partial}{\partial z}\},
\end{aligned}$$

where $f = \sum_{i=1}^{n}(x^i)^2 + (y^i)^2 + z^2$.

It is easy to show that $(\mathbb{R}^{2n+1}, \overline{\varphi}, \overline{\xi}, \overline{\eta}, \overline{g})$ is not a Sasakian manifold, but $\mathbb{R}^{2n+1}$ with the structure $(\widetilde{\varphi}, \widetilde{\xi}, \widetilde{\eta}, \widetilde{g})$ given by

$$\begin{aligned}
\widetilde{\varphi} &= \overline{\varphi},\\
\widetilde{g} &= \overline{\eta}\otimes\overline{\eta} + \frac{1}{4}\sum_{i=1}^{n}\{(dx^i)^2 + (dy^i)^2\},\\
\widetilde{\eta} &= \frac{1}{2}(dz - \sum_{i=1}^{n} y^i dx^i),\\
\widetilde{\xi} &= 2\frac{\partial}{\partial z},
\end{aligned}$$

is a Sasakian space form with the $\widetilde{\varphi}$-sectional curvature equal to -3.

Let $\widetilde{\nabla}$ and $\overline{\nabla}$ are the Riemannian connections on $\overline{M}$ with respect to the metrics $\widetilde{g}$ and $\overline{g}$, respectively. Using Koszul formula, we derive the following relation between the connections $\widetilde{\nabla}$ and $\overline{\nabla}$

$$\widetilde{\nabla}_X Y = \overline{\nabla}_X Y + \frac{1}{2}\{\omega(X)Y + \omega(Y)X - \overline{g}(X,Y)\omega^\sharp\}, \tag{13}$$

for any $X, Y \in T\overline{M}$, where $\omega(X) = X(f)$ and $\overline{g}(\omega^\sharp, X) = \omega(X)$.

By using (13), we get the relation between the curvature tensors of $(\overline{M}, \overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ and $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$ as follow

$$\begin{aligned}\exp(-f)\widetilde{R}(X,Y,Z,W) &= \overline{R}(X,Y,Z,W) + \frac{1}{2}\{B(X,Z)\overline{g}(Y,W) \\ &\quad -B(Y,Z)\overline{g}(X,W) + B(Y,W)\overline{g}(X,Z) \\ &\quad -B(X,W)\overline{g}(Y,Z)\} \\ &\quad +\frac{1}{4}\|\omega^\sharp\|^2\{\overline{g}(X,Z)\overline{g}(Y,W) \\ &\quad -\overline{g}(Y,Z)\overline{g}(X,W)\}, \end{aligned} \tag{14}$$

for any $X, Y, Z, W \in T\overline{M}$, such that $B = \overline{\nabla}\omega - \frac{1}{2}\omega \otimes \omega$ and $\overline{R}$ and $\widetilde{R}$ are the curvature tensors of $(\overline{M}, \overline{\varphi}, \overline{\eta}, \overline{\xi}, \overline{g})$ and $(\overline{M}, \widetilde{\varphi}, \widetilde{\eta}, \widetilde{\xi}, \widetilde{g})$, respectively.

From (13) it follows that

$$\overline{\nabla}_X\overline{\xi} = -(\exp(f))^{\frac{1}{2}}\overline{\varphi}X + \frac{1}{2}\{\overline{\eta}(X)\omega^\sharp - \omega(\overline{\xi})X\}, \tag{15}$$

$$\begin{aligned}(\overline{\nabla}_X\overline{\varphi})Y &= (\exp(f))^{\frac{1}{2}}\{\overline{g}(X,Y)\overline{\xi} - \overline{\eta}(Y)X\} \\ &- \frac{1}{2}\{\omega(\overline{\varphi}Y)X - \omega(Y)\overline{\varphi}X + \overline{g}(X,Y)\overline{\varphi}\omega^\sharp \\ &\quad -\overline{g}(X,\overline{\varphi}Y)\omega^\sharp\}. \end{aligned} \tag{16}$$

From (16), we get

$$\begin{aligned}(\nabla_X T)Y &= (\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\ &\quad +A_{NY}X + th(X,Y) - \frac{1}{2}\{\omega(TY)X - \omega(Y)TX \\ &\quad +g(X,Y)T\omega^\sharp - g(X,TY)\omega^\sharp\}, \end{aligned} \tag{17}$$

and

$$\begin{aligned}(\nabla_X N)Y &= nh(X,Y) - h(X,TY) \\ &\quad +\frac{1}{2}\{\omega(Y)NX - g(X,Y)N\omega^\sharp + g(X,TY)\omega^{\sharp\perp}\}, \end{aligned} \tag{18}$$

where, $g = \overline{g}|_M, \xi = \overline{\xi}|_M, \eta = \overline{\eta}|_M$ and $\varphi = \overline{\varphi}|_M$.

4. Slant submanifolds of conformal Sasakian manifolds

By the following proposition, we characterize slant submanifolds of a conformal Sasakian manifold with $\nabla Q = 0$.

Proposition 1. *Let M be a slant submanifold of a conformal Sasakian manifold $\overline{M}$ such that $\omega^\sharp \in TM^\perp$ and $\xi \in TM$. Then, Q is parallel if and only if M is an anti-invariant submanifold.*

Proof. From (9), we have

$$Q\nabla_X Y = -(\cos^2\theta)\nabla_X Y + (\cos^2)\theta\eta(\nabla_X Y)\xi, \tag{19}$$

where $X, Y \in TM$ and θ is the slant angle of M. By taking the covariant derivative of (9), we get

$$\begin{aligned} \nabla_X QY &= -(\cos^2\theta)\nabla_X Y + (\cos^2)\theta\eta\nabla_X Y\xi \\ &\quad +(\cos^2\theta)g(Y, \nabla_X\xi)\xi + (\cos^2\theta)\eta(Y)\nabla_X\xi. \end{aligned} \tag{20}$$

Therefore, $\nabla Q = 0$ if and only if $\nabla_X\xi = 0$ for any $X \in TM$. Now, we get the result from (15).

Theorem 5. *Let M be a submanifold of a conformal Sasakian manifold $\overline{M}$ such that $\xi \in TM$ and the tangent bundle decomposes as $TM = \mathcal{D} \oplus < \xi >$. Then, M is slant if and only if*

1. *The endomorphism $Q|_{\mathcal{D}}$ has only one eigenvalue at each point of M.*

2. *There exists a function $\lambda : M \mapsto [0, 1]$ such that*

$$\begin{aligned} (\nabla_X Q)Y &= \lambda\big\{(\exp(f))^{\frac{1}{2}}\{g(X, TY)\xi - \eta(Y)TX\} \\ &\quad -\frac{1}{2}\{\omega(\xi)g(X, Y)\xi - \eta(X)\omega(Y)\xi \\ &\quad +\omega(\xi)\eta(Y)X - \eta(X)\eta(Y){\omega^\sharp}^T\}\big\} \end{aligned} \tag{21}$$

for any $X, Y \in TM$.

Moreover, in this case, if θ is the slant angle of M, we have $\lambda = \cos^2\theta$.

Proof. Statement 1 gets from Lemma 3. Now, we prove the statement 2. Let M is a slant submanifold, then by using (19) and (20) we have

$$(\nabla_X Q)Y = (\cos^2\theta)(g(Y, \nabla_X \xi)\xi + \eta(Y)\nabla_X \xi). \tag{22}$$

By putting (15) in (22) we find (21).

Conversely, let $\lambda_1(p)$ is the only eigenvalue of $Q|_{\mathcal{D}}$ for any point $p \in M$ and $Y \in \mathcal{D}$ is the eigenvector associated to λ_1, then we have

$$QY = \lambda_1 Y. \tag{23}$$

By taking covariant derivative of (23) and using (21), we get

$$\begin{aligned} X(\lambda_1) + \lambda_1 \nabla_X Y &= Q(\nabla_X Y) + \lambda\{(\exp(f))^{\frac{1}{2}} g(X, TY)\xi \\ &\quad -\frac{1}{2}\big(\omega(\xi)g(X,Y)\xi - \eta(X)\omega(Y)\xi\big)\}, \end{aligned} \tag{24}$$

for any $X \in TM$. Now, we conclude that $X(\lambda_1) = 0$ and then λ_1 is contact, since $\xi, \nabla_X Y$ and $Q\nabla_X Y$ is perpendicular to Y. For proving that M is slant we refer to Theorem 4.3 in [2].

Corollary 6. *Let M be a three-dimensional submanifold of a conformal Sasakian manifold $\overline{M}$ such that $\xi \in TM$. Then, M is slant if and only if there exists a function $\lambda : M \mapsto [0,1]$ such that*

$$\begin{aligned} (\nabla_X Q)Y &= \lambda\big\{(\exp(f))^{\frac{1}{2}}\{g(X,TY)\xi - \eta(Y)TX\} \\ &\quad -\frac{1}{2}\{\omega(\xi)g(X,Y)\xi - \eta(X)\omega(Y)\xi \\ &\quad +\omega(\xi)\eta(Y)X - \eta(X)\eta(Y)\omega^{\sharp^T}\}\big\}, \end{aligned} \tag{25}$$

for any $X, Y \in TM$. Moreover, in this case, if θ is the slant angle of M, we have $\lambda = \cos^2\theta$.

Proof. If $dimM = 3$, then $Q|_{\mathcal{D}}$ has only one eigenvalue at each point of M. Therefore, the result follows by Theorem 5.

Theorem 7. *Let M be a three-dimensional proper slant submanifold of a conformal Sasakian manifold $\overline{M}$ such that $\xi \in TM$, then*

$$\begin{aligned} (\nabla_X T)Y &= (\cos^2\theta)(\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\ &\quad +\frac{1}{2}\{\omega(\xi)g(X,TY)\xi - \eta(X)\omega(TY)\xi \\ &\quad +\omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T}\}, \end{aligned} \tag{26}$$

for any $X, Y \in TM$, where θ is the slant angle of M.

Proof. Let $X, Y \in TM$ and $p \in M$. Let $\{\xi, e_1, e_2\}$ is the orthonormal frame in a neighborhood U of p given by Lemma 4. Put $\xi|_U = e_0$ and let α_i^j be the structural $1-$forms defined by

$$\nabla_X e_i = \sum_{j=0}^{2} \alpha_i^j(X) e_j. \tag{27}$$

In view of orthonormal frame $\{\xi, e_1, e_2\}$, we have

$$Y = \eta(Y)e_0 + g(Y, e_1)e_1 + g(Y, e_2)e_2, \tag{28}$$

thus, we get

$$\begin{aligned}(\nabla_X T)Y &= \eta(Y)(\nabla_X T)e_0 + g(Y, e_1)(\nabla_X T)e_1 \\ &\quad + g(Y, e_2)(\nabla_X T)e_2.\end{aligned} \tag{29}$$

Therefore, for obtaining $(\nabla_X T)Y$, we have to get $(\nabla_X T)e_0, (\nabla_X T)e_1$ and $(\nabla_X T)e_2$. By applying (15), we get

$$\begin{aligned}(\nabla_X T)e_0 &= \nabla_X T e_0 - T\nabla_X e_0 \\ &= (\exp(f))^{\frac{1}{2}} T^2 X + \frac{1}{2}\{\omega(\xi)TX - \eta(X)T\omega^{\sharp^T}\}.\end{aligned} \tag{30}$$

Moreover, by using (12) we obtain

$$\begin{aligned}(\nabla_X T)e_1 &= \nabla_X T e_1 - T\nabla_X e_1 \\ &= \nabla_X((\cos\theta)e_2) - T(\alpha_1^0(X)e_0 + \alpha_1^1(X)e_1 \\ &\quad + \alpha_1^2(X)e_2) \\ &= (\cos\theta)\alpha_2^0(X)e_0,\end{aligned} \tag{31}$$

and analogously

$$(\nabla_X T)e_2 = -(\cos\theta)\alpha_1^0(X)e_0. \tag{32}$$

By substituting (30), (31) and (32) into (29),we have

$$\begin{aligned}(\nabla_X T)Y &= (\exp(f))^{\frac{1}{2}}\eta(Y)T^2X + \frac{1}{2}\big\{\eta(Y)\omega(\xi)TX \\ &\quad -\eta(X)\eta(Y)T\omega^{\sharp^T}\big\} + (\cos\theta)\big\{g(Y, e_1)\alpha_2^0(X) \\ &\quad -g(Y, e_2)\alpha_1^0(X)\big\}e_0.\end{aligned} \tag{33}$$

Now, we obtain $\alpha_1^0(X)$ and $\alpha_2^0(X)$ as follow

$$\begin{aligned}\alpha_1^0(X) &= g(\nabla_X e_1, e_0)\\ &= Xg(e_1, e_0) - g(e_1, \nabla_X e_0)\\ &= -(\exp(f))^{\frac{1}{2}}(\cos\theta)g(e_2, X) + \frac{1}{2}\{\omega(\xi)g(e_1, X)\\ &\quad -\eta(X)\omega(e_1)\},\end{aligned} \tag{34}$$

and analogously we get

$$\begin{aligned}\alpha_2^0(X) &= (\exp(f))^{\frac{1}{2}}(\cos\theta)g(e_1, X)\\ &\quad +\frac{1}{2}\{\omega(\xi)g(e_2, X) - \eta(X)\omega(e_2)\}.\end{aligned} \tag{35}$$

By using (34) and (35) in (33) we have

$$\begin{aligned}(\nabla_X T)Y &= (\cos^2\theta)\{(\exp(f))^{\frac{1}{2}}(-X\eta(Y) + \eta(X)\eta(Y)\xi)\\ &\quad +g(e_1, X)g(e_1, Y)\xi + g(e_2, X)g(e_2, Y)\xi\}\\ &\quad +\frac{1}{2}\{\eta(Y)\omega(\xi)TX - \eta(X)\eta(Y)T\omega^{\sharp T}\\ &\quad +(\cos\theta)\big(g(e_2, X)g(e_1, Y)\omega(\xi)\xi\\ &\quad -g(e_2, Y)g(e_1, X)\omega(\xi)\xi\\ &\quad +g(e_2, Y)\omega(e_1)\eta(X)\xi\\ &\quad -g(e_1, Y)\omega(e_2)\eta(X)\xi\big)\}.\end{aligned} \tag{36}$$

In view of (28), we get

$$g(e_1, X)g(e_1, Y) + g(e_2, X)g(e_2, Y) = g(X, Y) - \eta(X)\eta(Y), \tag{37}$$

and from (12) and (37), we obtain

$$\begin{aligned}&g(e_1, Y)g((\cos\theta)e_2, X) - g(e_2, Y)g((\cos\theta)e_1, X)\\ &= -g(Y, e_1)g(e_1, TX) - g(TX, e_2)g(Y, e_2)\\ &= -g(TX, Y),\end{aligned} \tag{38}$$

and

$$\begin{aligned}&g(e_1, \omega^\sharp)g((\cos\theta)e_2, X) - g(e_2, \omega^\sharp)g((\cos\theta)e_1, Y)\\ &= -g(TY, e_1)g(e_1, \omega^\sharp) - g(TY, e_2)g(e_2, \omega^\sharp)\\ &= -\omega(TY).\end{aligned} \tag{39}$$

Now, by substituting (37),(38) and (39) in (36), we get (26).

The next result characterizes three-dimensional slant submanifolds in term of the shape operator.

Theorem 8. *Let M be a three-dimensional slant submanifold of a conformal Sasakian manifold $\overline{M}$ such that $\xi \in TM$. Then, there exists a function $C : M \mapsto [0,1]$ such that*

$$\begin{aligned} A_{NX}Y &= A_{NY}X + C(\exp(f))^{\frac{1}{2}}(\eta(X)Y - \eta(Y)X) \\ &\quad +\omega(\xi)g(TX,Y)\xi + g(X,TY)\omega^{\sharp} + \frac{1}{2}\{\eta(X)\omega(TY)\xi \\ &\quad -\eta(Y)\omega(TX)\xi + \eta(X)\omega(\xi)TY - \eta(Y)\omega(\xi)TX \\ &\quad -\omega(X)TY + \omega(Y)TX + \omega(TX)Y - \omega(TY)X\}, \end{aligned} \tag{40}$$

for any $X, Y \in TM$. Moreover, in this case, if θ is the slant angle of M, then we have $C = \sin^2\theta$.

Proof. Let $X, Y \in TM$ and M is a slant submanifold. From (17) and Theorem 7, we have

$$\begin{aligned} th(X,Y) &= -A_{NY}X + (\lambda - 1)(\exp(f))^{\frac{1}{2}}\{g(X,Y)\xi - \eta(Y)X\} \\ &\quad +\frac{1}{2}\{\omega(\xi)g(X,TY)\xi - \eta(X)\omega(TY)\xi \\ &\quad +\omega(\xi)\eta(Y)TX - \eta(X)\eta(Y)T\omega^{\sharp^T} + \omega(TY)X \\ &\quad -\omega(Y)TX + g(X,Y)T\omega^{\sharp} - g(X,TY)\omega^{\sharp}\}, \end{aligned} \tag{41}$$

Now, by using the fact that $h(X,Y) = h(Y,X)$, we obtain (40).

In the following, we assume that M is a three-dimensional proper slant submanifold of a five-dimensional conformal Sasakian manifold $\overline{M}$ with slant angle θ. Then, for a unit tangent vector field e_1 of M, perpendicular to ξ, we put

$$\begin{aligned} e_2 &= (\sec\theta)Te_1, \quad e_3 = \xi, \\ e_4 &= (\csc\theta)Ne_1, \quad e_5 = (\csc\theta)Ne_2. \end{aligned} \tag{42}$$

It is easy to show that $e_1 = -(sec\theta)Te_2$ and by using Corollary 2, $\{e_1, e_2, e_3, e_4, e_5\}$ form an orthonormal frame such that e_1, e_2, e_3 are tangent to M and e_4, e_5 are normal to M. By using (4) and (7), we have

$$\begin{aligned} te_4 &= -(\sin\theta)e_1, \quad te_5 = -(\sin\theta)e_2, \\ ne_4 &= -(\cos\theta)e_5, \quad ne_5 = -(\cos\theta)e_4. \end{aligned} \tag{43}$$

If we put $h^r_{ij} = g(h(e_i, e_j), e_r)$, $i, j = 1, 2, 3$, $r = 4, 5$, then we have the following lemma

Lemma 9. *In the above conditions, we have*

$$\begin{aligned} h^4_{12} &= h^5_{11}, \quad h^4_{22} = h^5_{12}, \\ h^4_{13} &= h^5_{23} = -(\exp(f))^{\frac{1}{2}} \sin\theta \\ h^4_{32} &= h^4_{33} = h^5_{13} = h^5_{33} = 0. \end{aligned} \tag{44}$$

Proof. By applying formula (40) with $X = e_1$ and $Y = e_2$, we get

$$A_{e_4} e_2 = A_{e_5} e_1 + (\cot\theta)\{\omega(\xi)\xi - \omega^\sharp + \omega(e_1)e_1 + \omega(e_2)e_2\}. \tag{45}$$

By using (2), relation (45) yields to the following results

$$h^4_{12} = h^5_{11}, \quad h^4_{22} = h^5_{12}, \quad h^4_{23} = h^5_{13}. \tag{46}$$

Moreover, by putting $X = e_1$ and $Y = e_3$ in (40), we have

$$A_{e_4} e_3 = -(\exp(f))^{\frac{1}{2}} (\sin\theta) e_1, \tag{47}$$

and from (47), we obtain

$$h^4_{13} = -(\exp(f))^{\frac{1}{2}} \sin\theta, \quad h^4_{23} = h^4_{33} = 0. \tag{48}$$

Finally, we put $X = e_2$ and $Y = e_3$ in (40), then, we have

$$A_{e_5} e_3 = -(\exp(f))^{\frac{1}{2}} (\sin\theta) e_2, \tag{49}$$

and

$$h^5_{13} = -(\exp(f))^{\frac{1}{2}} \sin\theta, \quad h^5_{23} = h^5_{33} = 0. \tag{50}$$

The following result characterizes three-dimensional minimal slant submanifolds in term of ∇N.

Theorem 10. *Let M be a three-dimensional minimal proper slant submanifold of a five-dimensional conformal Sasakian manifold $\overline{M}$ such that $\xi \in TM$,then*

$$\begin{aligned}(\nabla_X N)Y &= (\exp(f))^{\frac{1}{2}}\{2\eta(X)NTY + \eta(Y)NTX\} \\ &+\frac{1}{2}\{\omega(Y)NX - g(X,Y)N\omega^{\sharp} + g(X,TY)\omega^{\sharp\perp}\} \\ &+(\cot^2\theta)\{\omega(X) - \omega(\xi)\eta(X) - \omega(Y)g(X,Y) \\ &-\frac{1}{\cos^2\theta}\omega(TY)g(X,TY)\}NY \qquad (51)\end{aligned}$$

for any $X, Y \in TM$. Conversely, suppose that there is an eigenvalue λ of $Q|_{\mathcal{D}}$ at each point of M such that $\lambda \in (-1, 0)$. In this case, if (51) holds then M is a minimal proper slant submanifold of $\overline{M}$.

Proof. Let M be a minimal proper slant submanifold. Then, in view of Theorem 8 we have (40). Furthermore, we can get by minimality of M and straightforward calculation that (40) satisfies if and only if

$$\begin{aligned}A_V TY &= -A_{nV}Y + (\exp(f))^{\frac{1}{2}}\{2g(Y, tnV)\xi + \eta(Y)tnV\} \\ &+(\cot^2\theta)g(Y, tV)\{\omega^{\sharp} - \omega(\xi)\xi - \omega(Y)Y \\ &-\frac{1}{\cos^2\theta}\omega(TY)TY\}. \qquad (52)\end{aligned}$$

By multiplying (52) to X and using (2) we obtain

$$\begin{aligned}h(X,TY) &= nh(X,Y) - (\exp f)^{\frac{1}{2}}(2(NTY)\eta(X) \\ &+(NTX)\eta(Y)) - (\cot^2\theta)\{\omega(X) - \omega(\xi)\eta(X) \\ &-\omega(Y)g(X,Y) \\ &-\frac{1}{\cos^2\theta}\omega(TY)g(X,TY)\}. \qquad (53)\end{aligned}$$

Now, From (53) and (18) we get (51).

Conversely, let $p \in M$ and $e_1 \in \mathcal{D}$ such that $T^2e_1 = -\cos^2\theta_1 e_1$, where $\theta_1 = \theta(e_1) \in (0, \frac{\pi}{2})$ denotes the angle between φe_1 and T_pM. Now, we define an orthonormal frame $\{e_1, e_2, e_3, e_4, e_5\}$ as follow

$$\begin{aligned}e_2 &= (\sec\theta_1)Te_1, \quad e_3 = \xi, \\ e_4 &= (\csc\theta_1)Ne_1, \quad e_5 = (\csc\theta_1)Ne_2, \qquad (54)\end{aligned}$$

and then we have

$$\begin{aligned}te_4 = -(\sin\theta_1)e_1, \quad ne_4 = -(\cos\theta_1)e_5, \\ te_5 = -(\sin\theta_1)e_2, \quad ne_5 = -(\cos\theta_1)e_4. \qquad (55)\end{aligned}$$

It is obvious that from (51) we can obtain (52). Therefore, we find

$$\begin{aligned} A_{Ne_1}e_2 &= \tan\theta_1 A_{e_4}Te_1 \\ &= \sin\theta_1 A_{e_5}e_1 - (\cos\theta_1)(\omega^\sharp - \omega(\xi)\xi - \omega(e_1)e_1 \\ &\quad -\omega(e_2)e_2), \end{aligned} \tag{56}$$

and

$$\begin{aligned} A_{Ne_2}e_1 &= -\tan\theta_1 A_{e_5}Te_2 \\ &= (\sin\theta_1)A_{e_4}e_2 + (\cos\theta_1)(\omega^\sharp - \omega(\xi)\xi - \omega(e_1)e_1 \\ &\quad -\omega(e_2)e_2), \end{aligned} \tag{57}$$

Thus, we have

$$A_{Ne_1}e_2 = A_{Ne_2}e_1 - (\cos\theta_1)(\omega^\sharp - \omega(\xi)\xi - \omega(e_1)e_1 - \omega(e_2)e_2). \tag{58}$$

Moreover, we get

$$\begin{aligned} A_{Ne_1}e_3 &= (\sin\theta_1)A_{e_4}e_3 = -(\exp f)^{\frac{1}{2}}\sin^2\theta_1 e_1, \\ A_{Ne_2}e_3 &= (\sin\theta_1)A_{e_5}e_3 = -(\exp f)^{\frac{1}{2}}\sin^2\theta_1 e_2. \end{aligned} \tag{59}$$

Therefore, by a direct computation we obtain (40) and by Theorem 8 we deduce that M is proper slant.To prove that M is minimal we must show the following

$$h^4_{11} + h^4_{22} + h^4_{33} = 0$$
$$h^5_{11} + h^5_{22} + h^5_{33} = 0.$$

By taking $Y \in \{e_1, e_2, e_3\}$ and $V \in \{e_4, e_5\}$ in (52), we have

$$\begin{aligned} A_{e_4}e_2 &= A_{e_5}e_1 - (\cot\theta)(\omega^\sharp - \omega(\xi)\xi - \omega(e_1)e_1 - \omega(e_2)e_2), \\ A_{e_4}e_1 &= -A_{e_5}e_2 - 2(\exp f)^{\frac{1}{2}}(\sin\theta)\xi, \\ A_{e_5}e_3 &= -(\exp f)^{\frac{1}{2}}(\sin\theta)e_2, \\ A_{e_4}e_3 &= -(\exp f)^{\frac{1}{2}}(\sin\theta)e_1. \end{aligned} \tag{60}$$

Thus, the result is an obvious consequence of (2) and (60).

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