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APPLICATIONS OF A DIFFERENTIAL SUBORDINATION INVOLVING THE CONVOLUTION OF ANALYTIC FUNCTIONS

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Abstract. In the present paper, we derive a subordination theorem involving the convolution of analytic functions. As special cases of our main results, we obtain the sufficient conditions for analytic functions to be parabolic $\phi-$like, parabolic starlike, $\phi-$like and starlike.

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1. Introduction

Let $\mathcal{A}$ denote the class of all functions f analytic in $\mathbb{E} = \{z : |z| < 1\}$, normalized by the conditions $f(0) = f'(0) - 1 = 0$. Therefore, Taylor's series expansion of $f \in \mathcal{A}$, is given by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k.$$

Let the functions f and g be analytic in $\mathbb{E}$. We say that f is subordinate to g written as $f \prec g$ in $\mathbb{E}$, if there exists a Schwarz function ϕ in $\mathbb{E}$ (i.e. ϕ is regular in $|z| < 1, \phi(0) = 0$ and $|\phi(z)| \leq |z| < 1$) such that

$$f(z) = g(\phi(z)), \quad |z| < 1.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p an analytic function in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1), is said to be the best dominant of (1). The best dominant is unique up to a rotation of $\mathbb{E}$.
Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ be two analytic functions, then convolution of $f(z)$ and $g(z)$, written as $(f * g)(z)$ is defined by

$$(f * g)(z) = \sum_{k=0}^{\infty} a_k b_k z^k.$$

A function $f \in \mathcal{A}$ is said to be starlike of order $\alpha (0 \leq \alpha < 1)$ in $\mathbb{E}$ if

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \alpha, z \in \mathbb{E}.$$

Let $\mathcal{S}^*(\alpha)$ denote the class of starlike functions of order α. Write $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of starlike functions.
A function $f \in \mathcal{A}$ is said to be parabolic starlike in $\mathbb{E}$, if

$$\Re \left(\frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right|, z \in \mathbb{E}. \tag{2}$$

The class of parabolic starlike functions is denoted by $\mathcal{S}_{\mathcal{P}}$.
A function $f \in \mathcal{A}$ is said to be strongly starlike of order $\alpha, 0 < \alpha \leq 1$, if

$$\left| \arg \frac{zf'(z)}{f(z)} \right| < \frac{\alpha\pi}{2}, z \in \mathbb{E}. \tag{3}$$

or, equivalently

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z} \right)^{\alpha}, z \in \mathbb{E}.$$

Let $\tilde{\mathcal{S}}(\alpha)$ denote the class of strongly starlike functions of order α.

Let ϕ be analytic in a domain containing $f(\mathbb{E}), \phi(0) = 0$ and $\Re(\phi'(0)) > 0$. Then, the function $f \in \mathcal{A}$ is said to be $\phi-$like in $\mathbb{E}$ if

$$\Re \left(\frac{zf'(z)}{\phi(f(z))} \right) > 0, \quad z \in \mathbb{E}.$$

This concept was introduced by L. Brickman [1]. He proved that an analytic function $f \in \mathcal{A}$ is univalent if and only if f is $\phi-$like for some ϕ. Later, Ruscheweyh [6] investigated the following general class of $\phi-$like functions:

Let ϕ be analytic in a domain containing $f(\mathbb{E})$, where $\phi(0) = 0, \phi'(0) = 1$ and $\phi(w) \neq 0$ for some $w \in f(\mathbb{E})\backslash\{0\}$. Let $q(z)$ be a fixed analytic function in $\mathbb{E}, q(0) = 1$. Then the function $f \in \mathcal{A}$ is called $\phi-$like with respect to q, if

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E}.$$

A function $f \in \mathcal{A}$ is said to be parabolic $\phi-$like in $\mathbb{E}$ if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > \left|\frac{zf'(z)}{\phi(f(z))} - 1\right|, \quad z \in \mathbb{E}. \tag{4}$$

Define the parabolic domain Ω as under:

$$\Omega = \{u + iv : u > \sqrt{(u-1)^2 + v^2}\}.$$

Note that the conditions (2) and (4) are equivalent to the condition that $\dfrac{zf'(z)}{f(z)}$ and $\dfrac{zf'(z)}{\phi(f(z))}$ take values in the parabolic domain Ω respectively.
Ronning [5] and Ma and Minda [2] showed that the function defined by

$$q(z) = 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 \tag{5}$$

maps the unit disk $\mathbb{E}$ onto the parabolic domain Ω. Therefore, the condition (2) and (4) are equivalent to following conditions respectively:

$$\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in \mathbb{E}$$

and

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E},$$

where q(z) is given by (5).
In 2005, Ravichandran et al. [4] proved the following result for $\phi-$ like functions.

Theorem 1. *Let $\alpha \neq 0$ be a complex number and $q(z)$ be convex univalent in $\mathbb{E}$. Define $h(z) = \alpha q^2(z) + (1-\alpha)q(z) + \alpha z q'(z)$. Further assume that $\Re\{\frac{1-\alpha}{\alpha} + 2q(z) + (1 + \frac{zq''(z)}{q'(z)})\} > 0, \quad z \in \mathbb{E}$. If $f \in \mathbb{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))}\left[1 + \frac{\alpha z f''(z)}{f'(z)} + \frac{\alpha z[f'(z) - \{\phi(f(z))\}']}{\phi(f(z))}\right] \prec h(z)$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z)$$

and $q(z)$ is the best dominant.

Later, Shanmugham et al. [7] obtained the following result.

Theorem 2. *Let $q(z) \neq 0$ be analytic and univalent in $\mathbb{E}$ with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$. Let $q(z)$ satisfy*

$$\Re\left[1 + \frac{\alpha q(z)}{\gamma} - \frac{zq'(z)}{q(z)} + \frac{zq''(z)}{q'(z)}\right] \geq 0.$$

Let for $f, g \in \mathcal{A}$

$$\Psi(\alpha, \gamma, g; z) := \alpha\left\{\frac{z(f*g)'(z)}{\Phi(f*g)(z)}\right\} + \gamma\left\{1 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\Phi(f*g)(z))'}{\Phi(f*g)(z)}\right\}.$$

If q satisfies

$$\Psi(\alpha, \gamma, g; z) \prec \alpha q(z) + \frac{\gamma z q'(z)}{q(z)},$$

then

$$\frac{z(f*g)'(z)}{\Phi(f*g)(z)} \prec q(z)$$

and $q(z)$ is the best dominant.

In the present paper, we derive a subordination theorem involving the convolution of analytic functions. As special cases of our main results, we obtain the sufficient conditions for analytic functions to be parabolic $\phi-$like, parabolic starlike, $\phi-$like and starlike.

2. Preliminaries

To prove our main results, we shall use the following lemma of Miller and Mocanu ([3], Theorem 3.4h, p.132).

Lemma 3. *Let q be a univalent in $\mathbb{E}$ and let Θ and Φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\Phi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q(z) = zq'(z)\Phi[q(z)]$, $h(z) = \Theta[q(z)] + Q(z)$ and suppose that either*
(i) h is convex, or
(ii) Q is starlike.

In addition, assume that
(iii) $\Re\dfrac{zh'(z)}{Q(z)} = \Re\left[\dfrac{\Theta'[q(z)]}{\Phi[q(z)]} + \dfrac{zQ'(z)}{Q(z)}\right] > 0.$
If p is analytic in $\mathbb{E}$*, with p(0) = q(0),* $p(\mathbb{E}) \subset \mathbb{D}$ *and*

$$\Theta[p(z)] + zp'(z)\Phi[p(z)] \prec \Theta[q(z)] + zq'(z)\Phi[q(z)] = h(z),$$

then $p \prec q$*, and* q *is the best dominant.*

3. Main Results

In what follows, all the powers taken are principal ones.

Theorem 4. *Let* $q(z) \neq 0$*, be a univalent function in* $\mathbb{E}$ *such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma - 1)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*
(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma - 1)\dfrac{zq'(z)}{q(z)} + \dfrac{\beta(1-\alpha)}{\alpha}(q(z))^{\beta-\gamma} + \gamma\right] > 0.$
If f and $g \in \mathcal{A}$ *satisfy*

$$(1-\alpha)\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^{\beta} + \alpha\left[\frac{z(f*g)'(z)}{\phi(f*g)(z)}\right]^{\gamma}\left[2 + \frac{z(f*g)''(z)}{(f*g)'(z)} - \frac{z(\phi((f*g)(z)))'}{\phi((f*g)(z))}\right]$$
$$\prec (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(1 + \frac{zq'(z)}{q(z)}\right), \tag{6}$$

then

$$\frac{z(f*g)'(z)}{\phi((f*g)(z))} \prec q(z), \quad z \in \mathbb{E},$$

where α, β, γ *are complex numbers such that* $\alpha \neq 0$*, and q(z) is the best dominant.*

Proof. Define the function $p(z)$ by

$$p(z) = \frac{z(f*g)'(z)}{\phi(f*g)(z)}, \quad z \in \mathbb{E}.$$

Then the function $p(z)$ is analytic in $\mathbb{E}$ and $p(0) = 1$. Therefore, from equation (6) we obtain:

$$(1-\alpha)(p(z))^{\beta} + \alpha(p(z))^{\gamma}\left(1 + \frac{zp'(z)}{p(z)}\right) \prec (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(1 + \frac{zq'(z)}{q(z)}\right),$$

Let us define the functions Θ and Φ as follows:

$$\Theta(w) = (1-\alpha)w^{\beta} + \alpha w^{\gamma}$$

and

$$\Phi(w) = \alpha w^{\gamma-1}.$$

Obviously, both the functions Θ and Φ are analytic in $\mathbb{D} = \mathbb{C} \setminus \{0\}$ and $\Phi(w) \neq 0$ in $\mathbb{D}$. Therefore,

$$Q(z) = \Phi(q(z))zq'(z) = \alpha zq'(z)(q(z))^{\gamma-1}$$

and

$$h(z) = \Theta(q(z)) + Q(z) = (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(1 + \frac{zq'(z)}{q(z)}\right).$$

On differentiating, we obtain $\dfrac{zQ'(z)}{Q(z)} = 1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)}$ and

$$\frac{zh'(z)}{Q(z)} = 1 + \frac{zq''(z)}{q'(z)} + (\gamma-1)\frac{zq'(z)}{q(z)} + \frac{\beta(1-\alpha)}{\alpha}(q(z))^{\beta-\gamma} + \gamma.$$

In view of the given conditions, we see that Q is starlike and $\Re\left(\dfrac{zh'(z)}{Q(z)}\right) > 0$. Therefore, the proof, now follows from Lemma 3.

Selecting $g(z) = \dfrac{z}{1-z}$ in Theorem 4, we get the following result:

Theorem 5. *Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)}\right] > 0$ *and*
(ii) $\Re\left[1 + \dfrac{zq''(z)}{q'(z)} + (\gamma-1)\dfrac{zq'(z)}{q(z)} + \dfrac{\beta(1-\alpha)}{\alpha}(q(z))^{\beta-\gamma} + \gamma\right] > 0.$
If $f \in \mathcal{A}$ satisfy

$$(1-\alpha)\left[\frac{zf'(z)}{\phi(f(z))}\right]^{\beta} + \alpha\left[\frac{zf'(z)}{\phi(f(z))}\right]^{\gamma}\left[2 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right]$$
$$\prec (1-\alpha)(q(z))^{\beta} + \alpha(q(z))^{\gamma}\left(1 + \frac{zq'(z)}{q(z)}\right),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z), \quad z \in \mathbb{E},$$

where α, β, γ are complex numbers such that $\alpha \neq 0$, and q(z) is the best dominant.

Selecting $\phi(w) = w$ in Theorem 5, we get:

Theorem 6. *Let $q(z) \neq 0$, be a univalent function in $\mathbb{E}$ such that*
(i) $\Re\left[1+\dfrac{zq''(z)}{q'(z)}+(\gamma-1)\dfrac{zq'(z)}{q(z)}\right]>0$ *and*
(ii) $\Re\left[1+\dfrac{zq''(z)}{q'(z)}+(\gamma-1)\dfrac{zq'(z)}{q(z)}+\dfrac{\beta(1-\alpha)}{\alpha}q(z)^{\beta-\gamma}+\gamma\right]>0.$
If $f \in \mathcal{A}$ satisfy

$$(1-\alpha)\left[\frac{zf'(z)}{f(z)}\right]^{\beta}+\alpha\left[\frac{zf'(z)}{f(z)}\right]^{\gamma}\left[2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right]$$
$$\prec (1-\alpha)(q(z))^{\beta}+\alpha(q(z))^{\gamma}\left(1+\frac{zq'(z)}{q(z)}\right),$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z), \quad z \in \mathbb{E},$$

where α, β, γ are complex numbers such that $\alpha \neq 0$, and q(z) is the best dominant.

4. Applications

Remark 1. *When we select $\beta = 1, \gamma = 0$ and $q(z) = 1+\dfrac{2}{\pi^2}\left(\log\left(\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\left(\frac{1-\alpha}{\alpha}\right)q(z)=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}-\frac{\frac{4\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2}$$
$$+\frac{(1-\alpha)}{\alpha}\left(1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right).$$

Thus for real number α such that $0<\alpha\leq 1$, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we derive next two results from Theorem 5 and Theorem 6 respectively.

Theorem 7. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$

$$\prec (1-\alpha)\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]$$

$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Theorem 8. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$

$$\prec (1-\alpha)\left[1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2\right]$$

$$+\frac{\frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}{1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Remark 2. *When we select $\beta = 1, \gamma = 0$ and $q(z) = e^z$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = 1$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left(\frac{1-\alpha}{\alpha}\right)q(z) = 1 + \left(\frac{1-\alpha}{\alpha}\right)e^z.$$

Thus for positive real number α such that $0 < \alpha \leq 1$, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we get the following results from Theorem 5 and Theorem 6 respectively .

Theorem 9. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec (1-\alpha)e^z + \alpha z,$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \quad z \in \mathbb{E}.$$

Theorem 10. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec (1-\alpha)e^z + \alpha z,$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \quad z \in \mathbb{E}.$$

Remark 3. *When we select $\beta = 1, \gamma = 0$ and $q(z) = \dfrac{1+(1-2\delta)z}{1-z}; 0 \leq \delta < 1$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+(1-2\delta)z^2}{(1-z)(1+(1-2\delta)z)}$$

and

$$1+\frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left(\frac{1-\alpha}{\alpha}\right)q(z) = \frac{1+(1-2\delta)z^2}{(1-z)(1+(1-2\delta)z)} + \left(\frac{1-\alpha}{\alpha}\right)\left[\frac{1+(1-2\delta)z}{1-z}\right].$$

Thus for real number α such that $0 < \alpha \leq 1$, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we arrive at the following results from Theorem 5 and Theorem 6 respectively .

Theorem 11. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec (1-\alpha)\left[\frac{1+(1-2\delta)z}{1-z}\right] + \frac{2\alpha z(1-\delta)}{(1-z)[1+(1-2\delta)z]}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\delta)z}{1-z}, \text{ where } 0 \leq \delta < 1.$$

Theorem 12. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec (1-\alpha)\left[\frac{1+(1-2\delta)z}{1-z}\right] + \frac{2\alpha z(1-\delta)}{(1-z)[1+(1-2\delta)z]}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta)z}{1-z}, \textit{ where } 0 \leq \delta < 1.$$

Remark 4. *When we select $\beta = 1, \gamma = 0$ and $q(z) = 1 + az; 0 \leq a < 1$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1+az}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left(\frac{1-\alpha}{\alpha}\right)q(z) = \frac{1}{1+az} + \left(\frac{1-\alpha}{\alpha}\right)(1+az).$$

Thus for real number α such that $0 < \alpha \leq 1$, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we obtain the following results from Theorem 5 and Theorem 6 respectively.

Theorem 13. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec (1-\alpha)(1+az) + \frac{\alpha az}{1+az}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + az, \textit{ where } 0 \leq a < 1.$$

Theorem 14. *Let α be a real number such that $0 < \alpha \leq 1$. If $f \in \mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec (1-\alpha)(1+az) + \frac{\alpha az}{1+az}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + az, \textit{ where } 0 \leq a < 1.$$

Remark 5. *When we select* $\beta = 1, \gamma = 0$ *and* $q(z) = \left(\dfrac{1+z}{1-z}\right)^{\eta}, \quad 0 < \eta \leq 1$ *in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1+z^2}{1-z^2}$$

and

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} + \left(\frac{1-\alpha}{\alpha}\right) q(z) = \frac{1+z^2}{1-z^2} + \left(\frac{1-\alpha}{\alpha}\right)\left(\frac{1+z}{1-z}\right)^{\eta}.$$

Thus for real number α *such that* $0 < \alpha \leq 1$, *we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we have the following results from Theorem 5 and Theorem 6 respectively .*

Theorem 15. *Let* α *be a real number such that* $0 < \alpha \leq 1$. *If* $f \in \mathcal{A}$ *satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2\eta\alpha z}{1-z^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^{\eta}, \text{where } 0 < \eta \leq 1.$$

Theorem 16. *Let* α *be a real number such that* $0 < \alpha \leq 1$. *If* $f \in \mathcal{A}$ *satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec (1-\alpha)\left(\frac{1+z}{1-z}\right)^{\eta} + \frac{2\eta\alpha z}{1-z^2}, z \in \mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta}, \text{where } 0 < \eta \leq 1.$$

Remark 6. *When we select* $\beta = 1, \gamma = 0$ *and* $q(z) = \dfrac{\alpha'(1-z)}{\alpha' - z}, \ \alpha' > 1$ *in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} = \frac{1}{1-z} + \frac{z}{\alpha' - z}$$

and

$$1+\frac{zq''(z)}{q'(z)}-\frac{zq'(z)}{q(z)}+\left(\frac{1-\alpha}{\alpha}\right)q(z)=\frac{1}{1-z}+\frac{z}{\alpha'-z}+\left(\frac{1-\alpha}{\alpha}\right)\left[\frac{\alpha'(1-z)}{\alpha'-z}\right].$$

Thus for real number α such that $0<\alpha\leq 1$, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we get the following results from Theorem 5 and Theorem 6 respectively .

Theorem 17. *Let α be a real number such that $0<\alpha\leq 1$. If $f\in\mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{\phi(f(z))}+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec(1-\alpha)\frac{\alpha'(1-z)}{\alpha'-z}+\frac{(1-\alpha')\alpha z}{(1-z)(\alpha'-z)},z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{\phi(f(z))}\prec\frac{\alpha'(1-z)}{\alpha'-z},\text{ where }\alpha'>1.$$

Theorem 18. *Let α be a real number such that $0<\alpha\leq 1$. If $f\in\mathcal{A}$ satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)$$
$$\prec(1-\alpha)\frac{\alpha'(1-z)}{\alpha'-z}+\frac{(1-\alpha')\alpha z}{(1-z)(\alpha'-z)},z\in\mathbb{E},$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{\alpha'(1-z)}{\alpha'-z},\text{ where }\alpha'>1.$$

Remark 7. *When we select $\beta=1,\gamma=1$ and $q(z)=1+\frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1+\frac{zq''(z)}{q'(z)}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}$$

and

$$1+\frac{zq''(z)}{q'(z)}+\frac{1}{\alpha}=\frac{1+z}{2(1-z)}+\frac{\sqrt{z}}{(1-z)\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)}+\frac{1}{\alpha}.$$

Thus for positive real number α, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we obtain the following results from Theorem 5 and Theorem 6 respectively.

Theorem 19. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$

$$\prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2 + \frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Theorem 20. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$

$$\prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2$$

$$+ \frac{4\alpha\sqrt{z}}{\pi^2(1-z)}\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right),$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{\pi^2}\left(\log\left(\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)\right)^2, \quad z \in \mathbb{E}.$$

Remark 8. *When we select $\beta = 1, \gamma = 1$ and $q(z) = \dfrac{1+(1-2\delta)z}{1-z}$, for $0 \leq \delta <$ 1in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1+z}{1-z}$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{1+z}{1-z} + \frac{1}{\alpha}.$$

Thus for positive real number α, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we arrive at the following results from Theorem 5 and Theorem 6 respectively .

Theorem 21. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$

$$\prec \frac{1+(1-2\delta)z}{1-z} + \frac{2\alpha z(1-\delta)}{(1-z)^2},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{1+(1-2\delta)z}{1-z}, \quad z \in \mathbb{E}, \textit{where } 0 \le \delta < 1.$$

Theorem 22. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$
$$\prec \frac{1+(1-2\delta)z}{1-z} + \frac{2\alpha z(1-\delta)}{(1-z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{1+(1-2\delta)z}{1-z}, \quad z \in \mathbb{E}, \textit{where } 0 \le \delta < 1.$$

Remark 9. *When we select $\beta = 1, \gamma = 1$ and $q(z) = e^z$, in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = 1 + z$$

and

$$1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = 1 + z + \frac{1}{\alpha}.$$

Thus for positive real number α, we notice that q(z) satisfies the condition (i) and (ii) in Theorem 5 and Theorem 6. Therefore, we have the following results from Theorem 5 and Theorem 6 respectively.

Theorem 23. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1+\frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right) \prec e^z(1+\alpha z),$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec e^z, \quad z \in \mathbb{E}.$$

Theorem 24. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1+\frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec e^z(1+\alpha z),$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \quad z \in \mathbb{E}.$$

Remark 10. *When we select* $\beta = 1, \gamma = 1$ *and* $q(z) = \dfrac{\alpha'(1-z)}{\alpha' - z}, 1 \leq \alpha' < \dfrac{3}{2}$ *in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{\alpha' + z}{\alpha' - z},$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{\alpha' + z}{\alpha' - z} + \frac{1}{\alpha}.$$

For a positive real number α, *we see that q(z) satisfies the conditions (i) and (ii) of Theorem 5 and Theorem 6. Therefore , we get the following results from Theorem 5 and Theorem 6 respectively.*

Theorem 25. *Let* α *be a positive real number. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$

$$\prec \frac{\alpha'(1-z)}{\alpha' - z} + \alpha\frac{(\alpha' - \alpha'^2)z}{(\alpha' - z)^2},$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \frac{\alpha'(1-z)}{\alpha' - z}, \quad z \in \mathbb{E}, \text{where } 1 \leq \alpha' < \frac{3}{2}.$$

Theorem 26. *Let* α *be a positive real number. If* $f \in \mathcal{A}$ *satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)$$

$$\prec \frac{\alpha'(1-z)}{\alpha' - z} + \alpha\frac{(\alpha' - \alpha'^2)z}{(\alpha' - z)^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\alpha'(1-z)}{\alpha' - z}, \quad z \in \mathbb{E}, \text{where } 1 \leq \alpha' < \frac{3}{2}.$$

Remark 11. *When we select* $\beta = 1, \gamma = 1$ *and* $q(z) = 1 + az, 0 \leq a < 1$ *in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = 1,$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = 1 + \frac{1}{\alpha}.$$

For a positive real number α, *we see that q(z) satisfies the conditions (i) and (ii) of Theorem 5 and Theorem 6. Therefore , we derive next two results from Theorem 5 and Theorem 6 respectively.*

Theorem 27. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right) \prec 1 + az + \alpha az,$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec 1 + az, \quad z \in \mathbb{E}, \text{where } 0 \leq a < 1.$$

Theorem 28. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec 1 + az + \alpha az,$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + az, \quad z \in \mathbb{E}, \text{where } 0 \leq a < 1.$$

Remark 12. *When we select $\beta = 1, \gamma = 1$ and $q(z) = \left(\frac{1+z}{1-z}\right)^\eta, 0 < \eta \leq 1$ in Theorem 5 and Theorem 6, a little calculation yields that*

$$1 + \frac{zq''(z)}{q'(z)} = \frac{1 + z^2 + 2\eta z}{1 - z^2},$$

$$1 + \frac{zq''(z)}{q'(z)} + \frac{1}{\alpha} = \frac{1 + z^2 + 2\eta z}{1 - z^2} + \frac{1}{\alpha}.$$

For a positive real number α, we see that q(z) satisfies the conditions (i) and (ii) of Theorem 5 and Theorem 6. Therefore , we obtain the following results from Theorem 5 and Theorem 6 respectively.

Theorem 29. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{\phi(f(z))} + \alpha\left(\frac{zf'(z)}{\phi(f(z))}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(f(z))'}{\phi(f(z))}\right)$$
$$\prec \left(\frac{1+z}{1-z}\right)^\eta \left[1 + \frac{2\alpha\eta z}{1 - z^2}\right],$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec \left(\frac{1+z}{1-z}\right)^\eta, \quad z \in \mathbb{E}, \text{where } 0 < \eta \leq 1.$$

Theorem 30. *Let α be a positive real number. If $f \in \mathcal{A}$ satisfies*

$$\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf'(z)}{f(z)}\right)\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\eta}\left[1 + \frac{2\alpha\eta z}{1-z^2}\right],$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\eta}, \quad z \in \mathbb{E}, \textit{where } 0 < \eta \leq 1.$$

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