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INCLUSION RELATIONSHIPS AND SOME INTEGRAL-PRESERVING PROPERTIES OF CERTAIN CLASSES OF MEROMORPHIC P-VALENT FUNCTIONS

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Abstract. We introduce some integral operators defined on the space of p-valent meromorphic functions in the class Σ_p. By using these integral operators, we define several subclasses of p-valent meromorphic functions and investigate various inclusion relationship and integral-preserving properties.

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1. Introduction

Let Σ_p denotes the class of functions f given by

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k-p} z^{k-p} \qquad (p \in \mathbb{N} = \{1, 2, 3, ...\}) \tag{1}$$

which are analytic and p-valent in the punctured unit disc

$$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\}.$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma S_p^*(\alpha)$ of meromorphic p-valent starlike functions of order α in $\mathbb{U}^*$ if

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) < -\alpha, \qquad (z \in \mathbb{U}^*; 0 \leq \alpha < p), \tag{2}$$

also, a function $f \in \Sigma_p$ is said to be in the class $\Sigma C_p(\alpha)$ of meromorphic p-valent convex functions of order α in $\mathbb{U}^*$ if

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) < -\alpha, \qquad (z \in \mathbb{U}^*; 0 \leq \alpha < p). \tag{3}$$

It is easy to observe from (2) and (3) that

$$f \in \Sigma C_p(\alpha) \Leftrightarrow -\frac{zf'}{p} \in \Sigma S_p^*(\alpha). \tag{4}$$

A function $f \in \Sigma_p$ is said to be in the class $\Sigma K_p(\beta, \alpha)$ of meromorphic p-valent close-to-convex functions of order β and type α in $\mathbb{U}^*$ if there exist a function $g \in \Sigma S_p^*(\alpha)$ such that

$$\operatorname{Re}\left(\frac{zf'(z)}{g(z)}\right) < -\beta, \qquad (z \in \mathbb{U}^*; 0 \leq \alpha, \beta < p), \tag{5}$$

furthermore, a function $f \in \Sigma_p$ is said to be in the class $\Sigma K_p^*(\beta, \alpha)$ of meromorphic p-valent quasi-convex functions of order β and type α in $\mathbb{U}^*$ if there exist a function $g \in \Sigma C_p(\alpha)$ such that

$$\operatorname{Re}\left(\frac{\left(zf'(z)\right)'}{g'(z)}\right) < -\beta, \qquad (z \in \mathbb{U}^*; 0 \leq \alpha, \beta < p). \tag{6}$$

It is easy to observe from (5) and (6) that

$$f \in \Sigma K_p^*(\beta, \alpha) \Leftrightarrow -\frac{zf'}{p} \in \Sigma K_p(\beta, \alpha). \tag{7}$$

Definition 1. *Let $0 \leq \mu \leq 1$; $0 \leq \gamma \leq 1$; $p \in \mathbb{N}$ and $f \in \Sigma_p$, we introduce the p-valent Rafid operator $S_{\mu,p}^{\gamma} : \Sigma_p \to \Sigma_p$ which is defined by*

$$S_{\mu,p}^{\gamma} f(z) = \frac{1}{(1-\mu)^{\gamma+1}\,\Gamma(\gamma+1)} \int_0^{\infty} t^{\gamma+p} e^{\left(-\frac{t}{1-\mu}\right)} f(zt)dt \tag{8}$$

then,

$$S_{\mu,p}^{\gamma} f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} L(\gamma, \mu, k) a_{k-p} z^{k-p} \tag{9}$$

where,

$$L(\gamma, \mu, k) = (1-\mu)^k (\gamma+1)_k$$

and $(\nu)_k$ denotes the Pochhammer symbol given by

$$(\nu)_k = \frac{\Gamma(\nu+k)}{\Gamma(\nu)} = \begin{cases} 1 & if \quad k = 0, \\ \nu(\nu+1)...(\nu+k-1) & if \quad k \in \mathbb{N}. \end{cases} \tag{10}$$

Remark 1. *Putting $p = 1$ in (8) we have the Rafid operator S_{μ}^{γ} which is introduced by Rosy and Varma [4].*

Remark 2. *Using the equation (9), it is easy to see that*

$$S_{\mu,p}^{\gamma}\left(zf^{'}(z)\right) = z\left(S_{\mu,p}^{\gamma}f(z)\right)'$$

and,

$$z\left(S_{\mu,p}^{\gamma}f(z)\right)' = (\gamma+1)\, S_{\mu,p}^{\gamma+1}f(z) - (\gamma+p+1)S_{\mu,p}^{\gamma}f(z)$$

By putting $a_{k-p} = 1$, $\forall k$ in (9), we get

$$\psi_{\mu,p}^{\gamma}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} L(\gamma,\mu,k) z^{k-p} \tag{11}$$

and $\varphi_{\mu,p}^{\gamma,\lambda}(z)$ be defined using the Hadmard product as

$$\varphi_{\mu,p}^{\gamma,\lambda}(z) * \psi_{\mu,p}^{\gamma}(z) = \frac{1}{z^p(1-z)^{\lambda}} \tag{12}$$

therefore,

$$\varphi_{\mu,p}^{\gamma,\lambda}(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{(\lambda)_k}{(1-\mu)^k(\gamma+1)_k} z^{k-p} \tag{13}$$

Definition 2. *For $0 \le \mu \le 1$, $0 \le \gamma \le 1$, $\lambda > 0$ and $p \in \mathbb{N}$, we introduce the integral operator $J_{\mu,p}^{\gamma,\lambda} : \Sigma_p \to \Sigma_p$ which is defined by*

$$J_{\mu,p}^{\gamma,\lambda} f(z) = \varphi_{\mu,p}^{\gamma,\lambda}(z) * f(z) \tag{14}$$

Therefore,

$$J_{\mu,p}^{\gamma,\lambda} f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} \frac{1}{(1-\mu)^k\,(\gamma+1)_k} \frac{(\lambda)_k}{(1)_k} a_{k-p} z^{k-p} \tag{15}$$

Remark 3. *Using equation (15), it is easy to see that*

$$z\left(J_{\mu,p}^{\gamma+1,\lambda} f(z)\right)' = (\gamma+1)\, J_{\mu,p}^{\gamma,\lambda} f(z) - (p+\gamma+1)\, J_{\mu,p}^{\gamma+1,\lambda} f(z), \tag{16}$$

and

$$z\left(J_{\mu,p}^{\gamma,\lambda} f(z)\right)' = \lambda J_{\mu,p}^{\gamma,\lambda+1} f(z) - (p+\lambda)\, J_{\mu,p}^{\gamma,\lambda} f(z). \tag{17}$$

We now define the following subclasses of the meromorphic function class Σ_p by means of the integral operator $J_{\mu,p}^{\gamma,\lambda}$ given by (14).

$$\Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha) = \left\{f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda} f(z) \in \Sigma S_p^*(\alpha)\right\} \tag{18}$$

$$\Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha) = \left\{f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda} f(z) \in \Sigma C_p(\alpha)\right\} \tag{19}$$

$$\Sigma K_{\mu,p}^{\gamma,\lambda}(\beta,\alpha) = \left\{f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda} f(z) \in \Sigma K_p(\beta,\alpha)\right\} \tag{20}$$

$$\Sigma K_{\mu,p}^{*\gamma,\lambda}(\beta,\alpha) = \left\{f : f \in \Sigma_p \text{ and } J_{\mu,p}^{\gamma,\lambda} f(z) \in \Sigma K_p^*(\beta,\alpha)\right\} \tag{21}$$

where

$$z \in \mathbb{U}, 0 \leq \alpha < p, p \in \mathbb{N}.$$

Before we establish our main result, we need the following lemma due to Miller and Mocanu [3].

Lemma 1. *let $\theta(u,v)$ be a complex-valued function such that $\theta : \mathcal{D} \to \mathbb{C}$, $\mathcal{D} \subset \mathbb{C} \times \mathbb{C}$ ($\mathbb{C}$ is the complex plane) and let $u = u_1 + iu_2$ and $v = v_1 + iv_2$. Suppose that $\theta(u,v)$ satisfies the following conditions:*

- *$\theta(u,v)$ is continuous in $\mathcal{D}$;*
- *$(1,0) \in \mathcal{D}$ and $\operatorname{Re}\{\theta(1,0)\} > 0$;*
- *for all $(iu_2, v_1) \in \mathcal{D}$ such that $v_1 \leq \frac{-1}{2}(1+u_2^2)$, $\operatorname{Re}\{\theta(iu_2, v_1)\} \leq 0$.*

Let,

$$q(z) = 1 + q_1 z + q_2 z^2 + ... \tag{22}$$

be an analytic in U such that $(q(z), zq^{'}(z)) \in \mathcal{D}$ $(z \in \mathbb{U})$. If $\operatorname{Re}\{\theta(q(z), zq'(z))\} > 0$, then $\operatorname{Re}\{q(z)\} > 0$.

2. Inclusion Relationships

In this section, we give several inclusion relationships for p-valent meromorphic function classes, which are associated with the integral operator $J_{\mu,p}^{\gamma,\lambda}$.

Theorem 2. *Let $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\lambda > 0$ and $0 \leq \alpha < p, p \in \mathbb{N}$, then*

$$\Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha) \subset \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha) \subset \Sigma S_{\mu,p}^{*\gamma+1,\lambda}(\alpha) \tag{23}$$

Proof. (i) We first show that

$$\Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha) \subset \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha) \tag{24}$$

Let $f(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha)$ and set

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda} f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda} f(z)} = -\alpha - (p-\alpha)q(z) \tag{25}$$

where $q(z)$ is given by (22). By using equation (17), we have

$$\frac{\lambda J_{\mu,p}^{\gamma,\lambda+1} f(z)}{J_{\mu,p}^{\gamma,\lambda} f(z)} = (p+\lambda-\alpha) - (p-\alpha)q(z) \tag{26}$$

Differentiating (25) logarithmically with respect to z, we obtain

$$\begin{aligned} \frac{z(J_{\mu,p}^{\gamma,\lambda+1} f(z))'}{J_{\mu,p}^{\gamma,\lambda+1} f(z)} &= \frac{z(J_{\mu,p}^{\gamma,\lambda} f(z))'}{J_{\mu,p}^{\gamma,\lambda} f(z)} + \frac{z(p-\alpha)q'(z)}{(p-\alpha)q(z) - (p+\lambda-\alpha)} \\ &= -\alpha - (p-\alpha)q(z) + \frac{z(p-\alpha)q'(z)}{(p-\alpha)q(z) - (p+\lambda-\alpha)} \end{aligned}$$

Let now,

$$\theta(u,v) = (p-\alpha)u - \frac{(p-\alpha)v}{(p-\alpha)u - (p+\lambda-\alpha)} \tag{27}$$

where $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$. Then,

- $\theta(u,v)$ is continuous in $\mathcal{D} = \left\{\mathbb{C} \setminus \left(\frac{p+\lambda-\alpha}{p-\alpha}\right)\right\} \times \mathbb{C}$;
- $(1,0) \in \mathcal{D}$ with $\operatorname{Re}\{\theta(1,0)\} = p - \alpha > 0$;
- for all $(iu_2, v_1) \in \mathcal{D}$ such that $v_1 \le \frac{-1}{2}(1+u_2^2)$, we have

$$\begin{aligned} \operatorname{Re}\{\theta(iu_2,v_1)\} &= \operatorname{Re}\left\{(p-\alpha)iu_2 - \frac{(p-\alpha)v_1}{(p-\alpha)iu_2 - (p+\lambda-\alpha)}\right\} \\ &= \operatorname{Re}\left\{\frac{-(p-\alpha)v_1}{(p-\alpha)iu_2 - (p+\lambda-\alpha)} * \frac{-(p-\alpha)iu_2 - (p+\lambda-\alpha)}{-(p-\alpha)iu_2 - (p+\lambda-\alpha)}\right\} \\ &= \frac{(p+\lambda-\alpha)(p-\alpha)v_1}{((p-\alpha)u_2)^2 + (p+\lambda-\alpha)^2} \\ &\le -\frac{(p+\lambda-\alpha)(p-\alpha)(1+u_2^2)}{2\left[((p-\alpha)u_2)^2 + (p+\lambda-\alpha)^2\right]} < 0 \end{aligned}$$

which shows that $\theta(u,v)$ satisfies the hypotheses of Lemma 1 then $\operatorname{Re} q(z) > 0$. Consequently, we easily obtain the inclusion relationship (24).
(ii) by using the similar argument in proving relation (24) together with (16) and $\theta(u,v)$ is continuous in $\mathcal{D} = \left\{\mathbb{C} \setminus \left(\frac{p+\gamma+1-\alpha}{p-\alpha}\right)\right\} \times \mathbb{C}$, we can prove the right part of Theorem 1 that is

$$\Sigma S^{*\gamma,\lambda}_{\mu,p}(\alpha) \subset \Sigma S^{*\gamma+1,\lambda}_{\mu,p}(\alpha) \tag{28}$$

By combining the inclusion relationships (24) and (28), we complete the proof of Theorem 1.

Theorem 3. *Let* $0 \le \mu \le 1$, $0 \le \gamma \le 1, \lambda > 0$ *and* $0 \le \alpha < p, p \in \mathbb{N}$, *then*

$$\Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma+1,\lambda}_{\mu,p}(\alpha) \tag{29}$$

Proof. Let $f(z) \in \Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha)$. Then, from (18), we have

$$J^{\gamma,\lambda+1}_{\mu,p} f \in \Sigma C_p(\alpha)$$

Furthermore, in view of (4), we find that

$$-\frac{z}{p}\left(J^{\gamma,\lambda+1}_{\mu,p} f\right)' \in \Sigma S^*_p(\alpha)$$

that is,

$$J^{\gamma,\lambda+1}_{\mu,p}\left(-\frac{z}{p} f'\right) \in \Sigma S^*_p(\alpha)$$

Therefore,

$$-\frac{z}{p} f' \in \Sigma S^{*\gamma,\lambda+1}_{\mu,p}$$

In view of Theorem 1, we have

$$-\frac{z}{p} f' \in \Sigma S^{*\gamma,\lambda+1}_{\mu,p} \subset \Sigma S^{*\gamma,\lambda}_{\mu,p}(\alpha)$$

Then, we get that $f \in \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha)$ which implies that,

$$\Sigma C^{\gamma,\lambda+1}_{\mu,p}(\alpha) \subset \Sigma C^{\gamma,\lambda}_{\mu,p}(\alpha)$$

The right part of Theorem 2 can be proved using the same arguments. The proof is thus completed.

Theorem 4. *Let* $0 \le \mu \le 1$, $0 \le \gamma \le 1$, $\lambda > 0$ *and* $0 \le \alpha < p, p \in \mathbb{N}$, *then*

$$\Sigma K^{\gamma,\lambda+1}_{\mu,p}(\beta,\alpha) \subset \Sigma K^{\gamma,\lambda}_{\mu,p}(\beta,\alpha) \subset \Sigma K^{\gamma+1,\lambda}_{\mu,p}(\beta,\alpha) \tag{30}$$

Proof. (i) let us first prove that

$$\Sigma K_{\mu,p}^{\gamma,\lambda+1}(\beta,\alpha) \subset \Sigma K_{\mu,p}^{\gamma,\lambda}(\beta,\alpha) \tag{31}$$

Let $f(z) \in \Sigma K_{\mu,p}^{\gamma,\lambda+1}(\beta,\alpha)$. Then there exists a function $\Omega(z) \in \Sigma S_p^{*}(\alpha)$ such that

$$\mathrm{Re}\left(\frac{z\left(J_{\mu,p}^{\gamma,\lambda+1}f(z)\right)'}{\Omega(z)}\right) < -\beta \qquad (z \in U^*)$$

We set

$$\Omega(z) = J_{\mu,p}^{\gamma,\lambda+1}g(z)$$

So that we have

$$g(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha) \quad \text{and} \quad \mathrm{Re}\left(\frac{z\left(J_{\mu,p}^{\gamma,\lambda+1}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda+1}g(z)}\right) < -\beta \qquad (z \in U^*)$$

By setting that,

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}g(z)} = -\beta - (p-\beta)q(z) \tag{32}$$

where $q(z)$ is given by (22). Then, By using the identity (17), we obtain

$$\begin{aligned}
\frac{z\left(J_{\mu,p}^{\gamma,\lambda+1}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda+1}g(z)} &= \frac{J_{\mu,p}^{\gamma,\lambda+1}\left(zf'(z)\right)}{J_{\mu,p}^{\gamma,\lambda+1}g(z)} \\
&= \frac{z\left(J_{\mu,p}^{\gamma,\lambda}\left(zf'(z)\right)\right)' + (p+\lambda)\,J_{\mu,p}^{\gamma,\lambda}\left(zf'(z)\right)}{z\left(J_{\mu,p}^{\gamma,\lambda}g(z)\right)' + (p+\lambda)\,J_{\mu,p}^{\gamma,\lambda}g(z)} \\
&= \frac{\dfrac{z\left(J_{\mu,p}^{\gamma,\lambda}\left(zf'(z)\right)\right)'}{J_{\mu,p}^{\gamma,\lambda}g(z)} + (p+\lambda)\,\dfrac{J_{\mu,p}^{\gamma,\lambda}\left(zf'(z)\right)}{J_{\mu,p}^{\gamma,\lambda}g(z)}}{\dfrac{z\left(J_{\mu,p}^{\gamma,\lambda}g(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}g(z)} + (p+\lambda)}
\end{aligned}$$

Since $g(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda+1}(\alpha)$, by Theorem 1, we can setting

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda}g(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}g(z)} = -\alpha - (p-\alpha)H(z) \tag{33}$$

where $H(z) = g_1(x, y) + ig_2(x, y)$ and $\text{Re}\{H(z)\} = g_1(x, y) > 0 \quad (z \in U^*)$. Then,

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda+1} f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda+1} g(z)} = \frac{\dfrac{z\left(J_{\mu,p}^{\gamma,\lambda}\left(zf'(z)\right)\right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} + (p+\lambda)\left[-\beta - (p-\beta)q(z)\right]}{-\alpha - (p-\alpha)H(z) + (p+\lambda)} \tag{34}$$

Thus we have from (32) that

$$z\left(J_{\mu,p}^{\gamma,\lambda} f(z)\right)' = -J_{\mu,p}^{\gamma,\lambda} g(z)\left[\beta + (p-\beta)q(z)\right] \tag{35}$$

Differentiating both sides of (35) with respect to z, we obtain

$$\begin{aligned} \frac{z\left(J_{\mu,p}^{\gamma,\lambda} zf'(z)\right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} &= -\left[\beta + (p-\beta)q(z)\right]\frac{z\left(J_{\mu,p}^{\gamma,\lambda} g(z)\right)'}{J_{\mu,p}^{\gamma,\lambda} g(z)} - (p-\beta)zq'(z) \\ &= -(p-\beta)zq'(z) + \left[\beta + (p-\beta)q(z)\right]\left[\alpha + (p-\alpha)H(z)\right] \end{aligned} \tag{36}$$

Now, substituting from (36) into (34), we have

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda+1} f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda+1} g(z)} = -\beta - (p-\beta)q(z) + \frac{(p-\beta)zq'(z)}{(p-\alpha)H(z) + \alpha - (p+\lambda)}$$

Taking $u = q(z) = u_1 + iu_2$ and $v = zq'(z) = v_1 + iv_2$, we define the function $\Phi(u, v)$ by

$$\Phi(u, v) = (p-\beta)u - \frac{(p-\beta)v}{(p-\alpha)H(z) + \alpha - (p+\lambda)} \tag{37}$$

where $(u, v) \in \mathcal{D} = \{(\mathbb{C} \setminus D^*) \times \mathbb{C}\}$ and

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } \text{Re}\{H(z)\} = g_1(x, y) \geq 1 + \frac{\lambda}{p-\alpha}\right\}$$

Then, it follows from (37) that,

- $\Phi(u, v)$ is continuous in $\mathcal{D} = (\mathbb{C} \setminus D^*) \times \mathbb{C}$;
- $(1, 0) \in \mathcal{D}$ with $\text{Re}\{\Phi(1, 0)\} = p - \beta > 0$;

- for all $(iu_2, v_1) \in \mathcal{D}$ such that $v_1 \leq \frac{-1}{2}(1+u_2^2)$, we have

$$\begin{aligned}
\operatorname{Re}\{\Phi(iu_2, v_1)\} &= \operatorname{Re}\left\{(p-\beta)iu_2 - \frac{(p-\beta)v_1}{(p-\alpha)H(z)+\alpha-(p+\lambda)}\right\} \\
&= \operatorname{Re}\left\{\frac{-(p-\beta)v_1}{i(p-\alpha)g_2(x,y)+[(p-\alpha)g_1(x,y)+\alpha-(p+\lambda)]}\right\} \\
&= \frac{(p-\beta)\left[(p+\lambda)-\alpha-(p-\alpha)g_1(x,y)\right]v_1}{\left[(p-\alpha)g_2(x,y)\right]^2+\left[(p-\alpha)g_1(x,y)+\alpha-(p+\lambda)\right]^2} \\
&\leq -\frac{1}{2}\frac{(p-\beta)\left[(p+\lambda)-\alpha-(p-\alpha)g_1(x,y)\right](1+u_2^2)}{\left[(p-\alpha)g_2(x,y)\right]^2+\left[(p-\alpha)g_1(x,y)+\alpha-(p+\lambda)\right]^2} < 0
\end{aligned}$$

which proves that $\Phi(u,v)$ satisfies the hypotheses of Lemma 1 , then $\operatorname{Re}(q(z)) > 0$. Thus, in the light of (32), we easily deduce the inclusion relationship (31).
(ii) By using the similar argument in proving relation (31) together with (16) and

$$D^* = \left\{z : z \in \mathbb{C} \text{ and } \operatorname{Re}\{H(z)\} = g_1(x,y) \geq \frac{\gamma+1-\alpha}{p-\alpha}\right\}$$

we can prove the right part of Theorem 3. that is

$$\Sigma K_{\mu,p}^{\gamma,\lambda}(\beta,\alpha) \subset \Sigma K_{\mu,p}^{\gamma+1,\lambda}(\beta,\alpha) \tag{38}$$

By combining the inclusion relationships (31) and (38), we complete the proof of Theorem 3.

Theorem 5. *Let* $0 \leq \mu \leq 1$, $0 \leq \gamma \leq 1$, $\lambda > 0$ *and* $0 \leq \alpha < p, p \in \mathbb{N}$, *then*

$$\Sigma K_{\mu,p}^{*\gamma,\lambda+1}(\beta,\alpha) \subset \Sigma K_{\mu,p}^{*\gamma,\lambda}(\beta,\alpha) \subset \Sigma K_{\mu,p}^{*\gamma+1,\lambda}(\beta,\alpha) \tag{39}$$

Proof. Just as we derived Theorem 2 as a consequence of Theorem 1 by using the equivalence (4), we can also use the equivalence (7) to prove this Theorem as a consequence of Theorem 3.

3. A set of integral-preserving properties

In this section, we present some integral-preserving properties of the meromorphic function classes introduced here. We first recall a familiar integral operator $\mathcal{L}_{c,p}$ which introduced by Bernardi [2] defined by

$$\mathcal{L}_{c,p}f(z) = \frac{c}{z^{c+p}}\int_0^z t^{c+p-1}f(t)dt \quad (f \in \Sigma_p; c > 0; p \in \mathbb{N}) \tag{40}$$

which satisfies the following relationship:

$$z\left(J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)\right)' = cJ_{\mu,p}^{\gamma,\lambda}f(z) - (p+c)J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z) \tag{41}$$

In order to obtain the integral-preserving properties involving the integral operator $\mathcal{L}_{c,p}$, we also need the following lemma which is popularly known as Jack's lemma [1].

Lemma 6. *Let $\omega(z)$ be a non-constant function analytic in U with $\omega(0) = 0$. If $|\omega(z)|$ attains its maximum value on the circle $|z| = r < 1$ at z_0, then*

$$z_0\omega^{'}(z_0) = \zeta\omega(z_0) \tag{42}$$

where ζ is a real number and $\zeta \geq 1$.

Unless otherwise mentioned, we assume in the reminder of this section that $c, \lambda > 0; 0 \leq \mu, \gamma \leq 1; \zeta \geq 1$ and $0 \leq \alpha, \beta < p, p \in \mathbb{N}$.

Theorem 7. *If $f(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$, Then*

$$\mathcal{L}_{c,p}f(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha). \tag{43}$$

Proof. Suppose that $f(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$ and let

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)} = -\frac{p+(p-2\alpha)\omega(z)}{1-\omega(z)} \tag{44}$$

where $\omega(0) = 0$.Then, by using (41) and (44), we have

$$\frac{J_{\mu,p}^{\gamma,\lambda}f(z)}{J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)} = \frac{c-(2p+c-2\alpha)\omega(z)}{c\,(1-\omega(z))} \tag{45}$$

which, upon logarithmic differentiation, we get

$$\begin{aligned}\frac{z\left(J_{\mu,p}^{\gamma,\lambda}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}f(z)} &= \frac{z\left(J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}\mathcal{L}_{c,p}f(z)} - \frac{(2p+c-2\alpha)z\omega'(z)}{c-(2p+c-2\alpha)\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} \\ &= -\frac{p+(p-2\alpha)\omega(z)}{1-\omega(z)} - \frac{(2p+c-2\alpha)z\omega'(z)}{c-(2p+c-2\alpha)\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)}\end{aligned} \tag{46}$$

so that,

$$\frac{z\left(J_{\mu,p}^{\gamma,\lambda}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}f(z)} + \alpha = (\alpha-p)\,\frac{1+\omega(z)}{1-\omega(z)} - \frac{(2p+c-2\alpha)z\omega'(z)}{c-(2p+c-2\alpha)\omega(z)} + \frac{z\omega'(z)}{1-\omega(z)} \tag{47}$$

Now, assuming that $\underset{|z|\leq|z_0|}{max} |\omega(z)| = |\omega(z_0)| = 1, (z \in U^*)$ and applying Jack's lemma, we obtain

$$z_0\omega'(z_0) = \zeta\omega(z_0)$$

If we set $\omega(z_0) = e^{i\theta}$, $(\cos\theta < 0)$ in (47) and observe that

$$\text{Re}\left\{(\alpha - p)\,\frac{1+\omega(z)}{1-\omega(z)}\right\} = 0$$

then, we obtain

$$\begin{aligned}\text{Re}\left\{\frac{z\left(J_{\mu,p}^{\gamma,\lambda}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}f(z)} + \alpha\right\} &= \text{Re}\left\{\frac{z_0\omega'(z_0)}{1-\omega(z_0)} - \frac{(2p+c-2\alpha)z_0\omega'(z_0)}{c-(2p+c-2\alpha)\omega(z_0)}\right\}\\ &= \text{Re}\left\{\frac{\zeta e^{i\theta}}{1-e^{i\theta}} - \frac{(2p+c-2\alpha)\zeta e^{i\theta}}{c-(2p+c-2\alpha)e^{i\theta}}\right\}\\ &= \frac{2\zeta(c+p-\alpha)(p-\alpha)}{c^2+(2p+c-2\alpha)^2-2c(2p+c-2\alpha)\cos\theta}\\ &> 0\end{aligned}$$

which obviously contradicts the hypothesis $f(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$. Consequently, we can deduce that $|\omega(z)| < 1$ $(z \in U^*)$, which, in view of (44), proves the integral-preserving property asserted by Theorem 5.

Theorem 8. *If* $f(z) \in \Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha)$, *Then*

$$\mathcal{L}_{c,p}f(z) \in \Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha).$$

Proof. Suppose that $f(z) \in \Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha)$, then

$$zf'(z) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$$

by applying Theorem 5, we have

$$\mathcal{L}_{c,p}\left(zf'(z)\right) \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$$

and so,

$$z\left(\mathcal{L}_{c,p}f(z)\right)' \in \Sigma S_{\mu,p}^{*\gamma,\lambda}(\alpha)$$

which is equivalent to,

$$\mathcal{L}_{c,p}f(z) \in \Sigma C_{\mu,p}^{\gamma,\lambda}(\alpha)$$

The proof is completed.

Theorem 9. *If* $f(z) \in \Sigma K_{\mu,p}^{\gamma;\lambda}(\beta,\alpha)$, *Then*

$$\mathcal{L}_{c,p} f(z) \in \Sigma K_{\mu,p}^{\gamma;\lambda}(\beta,\alpha).$$

Proof. Suppose that $f(z) \in \Sigma K_{\mu,p}^{\gamma;\lambda}(\beta,\alpha)$. Then, there exist a function $g(z) \in \Sigma S_{\mu,p}^{*\gamma;\lambda}(\alpha)$ such that

$$\mathrm{Re}\left\{ \frac{z\left(J_{\mu,p}^{\gamma;\lambda} f(z)\right)'}{J_{\mu,p}^{\gamma;\lambda} g(z)} \right\} < -\alpha$$

Let us now setting,

$$\frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} f(z)\right)'}{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)} + \beta = -(p-\beta)q(z) \tag{48}$$

where $q(z)$ is given by (22), we find from (41) that

$$\begin{aligned} \frac{z\left(J_{\mu,p}^{\gamma;\lambda} f(z)\right)'}{J_{\mu,p}^{\gamma;\lambda} g(z)} &= \frac{J_{\mu,p}^{\gamma;\lambda}\left(zf'(z)\right)}{J_{\mu,p}^{\gamma;\lambda} g(z)} \\ &= \frac{\frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p}\left(zf'(z)\right)\right)'}{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)} + (p+c)\frac{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p}\left(zf'(z)\right)}{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)}}{\frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)\right)'}{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)} + (p+c)} \end{aligned} \tag{49}$$

Since $g(z) \in \Sigma S_{\mu,p}^{*\gamma;\lambda}(\alpha)$, then according to Theorem 5 we have $\mathcal{L}_{c,p} g(z) \in \Sigma S_{\mu,p}^{*\gamma;\lambda}(\alpha)$. Then, we can set

$$\frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)\right)'}{J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)} + \alpha = -(p-\alpha)Q(z) \tag{50}$$

where $\mathrm{Re}\{Q(z)\} > 0$. Equation (48) can be written as,

$$J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p}\left(zf'(z)\right) = \left(-\beta - (p-\beta)q(z)\right) J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z) \tag{51}$$

By differntioniating both sides of (48) with respect to z, we get

$$\begin{aligned} \frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p}(zf'(z))\right)'}{\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)\right)} &= -zq'(z)(p-\beta) + \left(-\beta - (p-\beta)q(z)\right) \frac{z\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)\right)'}{\left(J_{\mu,p}^{\gamma;\lambda} \mathcal{L}_{c,p} g(z)\right)} \\ &= -zq'(z)(p-\beta) + \left(\beta + (p-\beta)q(z)\right)\left(\alpha + (p-\alpha)Q(z)\right) \end{aligned} \tag{52}$$

Then, by substituting (48), (50) and (52) into (49), we have

$$\frac{z\left(J_{\mu,p}^{\gamma;\lambda}f(z)\right)'}{J_{\mu,p}^{\gamma,\lambda}g(z)}+\beta=-(p-\beta)q(z)-\frac{zq'(z)(p-\beta)}{(p+c-\alpha)-(p-\alpha)Q(z)} \tag{53}$$

Then, by setting $u=q(z)=u_1+iu_2$ and $v=zq'(z)=v_1+iv_2$, we can define the function $\Omega(u,v)$ by

$$\Omega(u,v)=-(p-\beta)u-\frac{(p-\beta)v}{(p+c-\alpha)-(p-\alpha)Q(z)} \tag{54}$$

where $(u,v)\in\mathcal{D}\subset\mathbb{C}\times\mathbb{C}$. The remainder of our proof of Theorem 7 is similar to that of Theorem 3, so we choose to omit the analogous details involved.

Theorem 10. *If* $f(z)\in\Sigma K_{\mu,p}^{*,\gamma,\lambda}(\beta,\alpha)$, *Then*

$$\mathcal{L}_{c,p}f(z)\in\Sigma K_{\mu,p}^{*,\gamma,\lambda}(\beta,\alpha).$$

Proof. Just as we derived Theorem 6 from Theorem 5. Easily, we can deduce Theorem 8 from Theorem 7. So we choose to omit the proof.

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