Acta Universitatis Apulensis No. 60/2019
ISSN: 1582-5329 pp. 53-59
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.60.04

COEFFICIENT ESTIMATES FOR CERTAIN SUBCLASS OF BI-BAZILEVIČ FUNCTIONS ASSOCIATED WITH CHEBYSHEV POLYNOMIALS

A.G. AlAmoush

Abstract. In the present article, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belong to bi univalent functions of the Bazilevič type of order α by using the Chebyshev polynomials. Fekete-Szegö inequalities of functions belonging to this subclass are also founded.

2010 *Mathematics Subject Classification*: 30C45.

Keywords: Analytic functions, Univalent and bi-univalent function,; Bazilevič functions, Fekete-Szegö problem, Chebyshev polynomials, Coefficient bounds, Subordination.

1. Introduction

A functions of the form $f(z)$ normalized by the following Taylor Maclaurin series

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit open disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, and belongs to class A.

Let $\mathbb{S}$ be class of all functions in A which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in $\mathbb{U}$. Some of the important and well-investigated subclasses of the univalent function class $\mathbb{S}$ includes the class $S^*(\alpha)(0 \leq \alpha < 1)$ of starlike functions of order α in $\mathbb{U}$ and the class $K(\alpha)(0 \leq \alpha < 1)$ of convex functions of order α. Also, it is known that the class

$$B_1(\mu) = \left\{ f \in \mathbb{A} : \ \Re\left(\frac{z^{1-\mu}(f^{'}(z))}{[f(z)]^{1-\mu}} \right) > 0, \ \mu \geq 0, \ z \in \mathbb{U} \right\} \tag{2}$$

is the class of univalent functions in $\mathbb{U}$(see [1]). So we have

$$B_{\alpha,\mu}=\left\{f\in\mathbb{A}:\ \Re\left(\frac{z^{1-\mu}(f'(z))}{[f(z)]^{1-\mu}}\right)>\alpha,\quad 0\leq\alpha<1,\ \mu\geq 0,\ z\in\mathbb{U}\right\}. \tag{3}$$

For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, written as $f(z)\prec g(x), \quad (z\in\mathbb{U})$, provided that there exists an analytic function(that is, Schwarz function) $w(z)$ defined on $\mathbb{U}$ with

$$w(0)=0 \ \text{ and } \qquad |w(z)|<1 \text{ for all } \ z\in\mathbb{U},$$

such that $f(z)=g(w(z))$ for all $z\in\mathbb{U}$.

Indeed,it is known that

$$f(z)\prec g(z)\ (z\in\mathbb{U})\ \Rightarrow\ f(0)=g(0) \ \text{ and } \ f(\mathbb{U})\subset g(\mathbb{U}).$$

It is well known that every function $f\in\mathbb{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z))=z \quad (z\in\mathbb{U}),$$

and

$$f^{-1}(f(w))=w \quad (|w|<r_0(f);r_0(f)\geq\frac{1}{4}),$$

where

$$f^{-1}(w)=w+a_2w^2+(2a_2^2-3a_3)w^3-(5a_2^3-5a_2a_3+a_4)w^4+\ldots\ . \tag{4}$$

A function $f\in A$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ' denote the class of bi-univalent functions in $\mathbb{U}$ given by (1). The beginning of estimating bounds for the coefficients of classes of bi-univalent functions is in 1967 when Lewin show that $|a_2|<1.51$ (For more details see [2]). Later the papers of Brannan and Taha [3] and Srivastava et al. [4] and other (see [5], [6], [7]) studied the bi-univalent results for may classes. The coefficient estimate problem for each of the following Taylor-Maclaurin coefficients $|a_n|(n\in N\setminus 1,2)$ for each $f\in\Sigma'$ given by (1) is still an open problem.

The significance of Chebyshev polynomial in numerical analysis is increased in both theoretical and practical points of view. Chebyshev polynomials, which is denoted by $T_n(t)$ and $U_n(t)$ (see [8] and [9]). The Chebyshev polynomial of degree n of the second kind, are defined for $t\in[1,1]$ by the following relations

$$U_0(t)=1,U_1(t)=2t,U_2(t)=4t^2-1,U_3(t)=8t^3-4t,...,U_{n+1}(t)=2tU_n(t)-U_{n-1}(t). \tag{5}$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$, is given by:

$$H(z,t) = \frac{1}{1-2zt+z^2} = \sum_{n=2}^{\infty} U_n(t) z^n \ (z \in \mathbb{U}).$$

In this paper, for subclass of Bazilevič type function of order α, we use the Chebyshev polynomials expansions to provide the initial coefficients and the Fekete-Szegö inequality for functions belonging to the class $\Sigma'_\mu(t)$.

2. COEFFICIENT BOUNDS FOR THE FUNCTION CLASS $\Sigma'_\mu(t)$

We begin by introducing the function class $\Sigma'_\mu(t)$ by means of the following definitions.

Definition 1. *For $\mu \geq 0,\ 0 \leq \alpha < 1$, a function $f \in \Sigma'$ given by (1) is said to be in the class $\Sigma'_\mu(t)$, if the following conditions are satisfied:*

$$\Re\left(\frac{z^{1-\mu}(f'(z))}{[f(z)]^{1-\mu}}\right) \prec H(t,z) = \frac{1}{1-2tz+z^2} \tag{6}$$

and

$$\Re\left(\frac{z^{1-\mu}(g'(w))}{[g(w)]^{1-\mu}}\right) \prec H(t,w) = \frac{1}{1-2tw+w^2} \tag{7}$$

where the function $g(w) = f^{-1}(z)$ is given by (4).

We first state and prove the following result.

Theorem 1. *For $\mu \geq 0$ and $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'_\mu(t)$. Then*

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(1+\mu)[1+\mu(1-2t^2)]|}} \tag{8}$$

$$|a_3| \leq \frac{2t}{2+\mu} + \frac{4t^2}{(1+\mu)^2}, \tag{9}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{2+\mu} & , |\eta - 1| \leq \frac{|(1+\mu)[1+\mu(1-2t^2)]|}{4(2+\mu)t^2} \\ \frac{8t^3|\eta-1|}{|(1+\mu)[1+\mu(1-2t^2)]|} & , |\eta - 1| \geq \frac{|(1+\mu)[1+\mu(1-2t^2)]|}{4(2+\mu)t^2} \end{cases}. \tag{10}$$

Proof. Let $f \in \Sigma'$. From (6) and (7), we have

$$\Re\left(\frac{z^{1-\mu}(f'(z))}{[f(z)]^{1-\mu}}\right) = 1 + U_1(t)p(z) + U_2(t)p^2(z) + ... \tag{11}$$

and

$$\Re\left(\frac{z^{1-\mu}(f'(z))}{[f(z)]^{1-\mu}}\right) = 1 + U_1(t)q(w) + U_2(t)q^2(w) + ... \tag{12}$$

for some analytic functions

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + ... \qquad (z \in \mathbb{U})$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + ... \qquad (w \in \mathbb{U}),$$

such that $p(0) = q(0)$, $|p(z)| < 1$, $(z \in \mathbb{U})$, $|q(w)| < 1$, $(w \in \mathbb{U})$. It is well known that if $|p(z)| < 1$ and $|q(w)| < 1$,then

$$|p_i| \leq 1 \quad \text{and} \quad |q_i| \leq 1 \quad \text{for all} \quad i \in \mathbb{R}. \tag{13}$$

Now, equating the Coefficients in (11) and (12), we get

$$(1+\mu)a_2 = U_1(t)p_1 \tag{14}$$

$$\left[\frac{(\mu-1)(\mu+2)}{2}a_2^2 + (\mu+2)a_3\right] = U_1(t)p_2 + U_2(t)p_1^2 \tag{15}$$

$$-(1+\mu)a_2 = U_1(t)q_1 \tag{16}$$

$$\left[2(2+\mu) + \frac{(\mu-1)(\mu+2)}{2}\right]a_2^2 - (\mu+2)a_3 = U_1(t)q_2 + U_2(t)q_1^2. \tag{17}$$

From (14) and (16), we find that

$$p_1 = -q_1 \tag{18}$$

and

$$2(1+\mu)^2 a_2^2 = U_1^2(t)(p_1^2 + q_1^2). \tag{19}$$

Also, by using (15) and (17), we obtain

$$[(\mu+2)(\mu-1) + 2(2+\mu)]\, a_2^2 = U_1(t)(p_2+q_2) + U_2(t)(p_1^2+q_1^2). \tag{20}$$

By using (19) in (20), we get

$$\left[((\mu+2)(\mu-1) + 2(2+\mu) - \frac{2U_2(t)}{U_1^2(t)}(1+\mu)^2\right] a_2^2 = U_1(t)(p_2+q_2). \tag{21}$$

From (5), (13) and (21), we have the desired inequality (8).

Next, by subtracting (17) from (15), we have

$$2(2+\mu)a_3 - 2(2+\mu)a_2^2 = U_1(t)(p_2 - q_2) + U_2(t)(p_1^2 - q_1^2). \tag{22}$$

Further, in view of (18), we obtain

$$a_3 = a_2^2 + \frac{U_1(t)}{2(2+\mu)}(p_2 - q_2). \tag{23}$$

Hence using (19) and applying (5), we get desired inequality (9).
Now, by using (21) and (23) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = (1-\eta)\left[\frac{U_1^3(t)(p_2+q_2)}{[(2+\mu)(1+\mu)+2(2+\mu)]U_1^2(t) - 2(1+\mu)^2U_2(t)}\right] + \frac{U_1(t)(p_2-q_2)}{2(2+\mu)}$$

$$= U_1(t)\left[\left(h(\eta) + \frac{1}{2(2+\mu)}\right)p_2 + \left(h(\eta) - \frac{1}{2(2+\mu)}\right)q_2\right],$$

where

$$h(\eta) = \frac{U_1^2(t)(1-\eta)}{(2+\mu)(1+\mu) + 2(2+\mu)U_1^2(t) - 2(1+\mu)^2U_2(t)}.$$

So, we conclude that

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{2t}{(2+\mu)} & , \ |h(\eta)| \leq \frac{1}{2(2+\mu)} \\ 4t|h(\eta)|\&, \quad |h(\eta)| \geq \frac{1}{2(2+\mu)} \end{cases}$$

.

This proves Theorem 1.

Taking $\mu = 0$ in Theorem 1, we get the following consequence.

Corollary 2. *For $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(t)$. Then*

$$|a_1| \leq 2t\sqrt{2t} \tag{24}$$

$$|a_3| \leq 2t + 4t^2, \tag{25}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} t & , \ |\eta - 1| \leq \frac{1}{8t^2} \\ 8t^3|\eta - 1| & , |\eta - 1| \geq \frac{1}{8t^2} \end{cases}. \tag{26}$$

Taking $\mu = 1$ in Theorem 1, we get the following consequence.

Corollary 3. *For $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(t)$. Then*

$$|a_1| \leq \frac{t\sqrt{2t}}{\sqrt{|1-t^2|}} \tag{27}$$

$$|a_3| \leq \frac{2t}{3} + t^2, \tag{28}$$

and for some $\eta \in R$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{4t}{3} & , \ |\eta - 1| \leq \frac{1-t^2}{3t^2} \\ \frac{2|\eta-1|}{1-t^2} & , \ |\eta - 1| \geq \frac{1-t^2}{3t^2} \end{cases}. \tag{29}$$

Taking $\eta = 1$ in Corollary 3, we get the following consequence

Corollary 4. *For $t \in (1/2, 1)$, let the function $f \in \Sigma'$ given by (1) be in the class $\Sigma'(t)$. Then*

$$|a_3 - a_2^2| \leq \frac{4t}{3}. \tag{30}$$

Acknowledgements. The author is deeply grateful to the reviewers for careful reading of the manuscript and helpful suggestions.

References

[1] I. E. Bazilevič, *On a case of integrability in quadratures of the Loewner-Kufarev equation*, Mat. Sb. 37, 79 (1955), 471-476.

[2] M. Lewin, *On a coefficient problem for bi-univalent functions*, Proc. Amer. Math. Soc. 18, (1967), 63-68.

[3] D. A. Brannan, T. S. Taha, *On some classes of bi-unvalent functions, in Mathematical Analysis and Its Applications* (Kuwait; February 18–21, 1985) (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors), pp. 53–60, KFAS Proceedings Series, Vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988; see also Studia Univ. Babeş-Bolyai Math. 31, 2 (1986), 70-77.

[4] H. M. Srivastava, A. K. Mishra, P. Gochhayat, *Certain subclasses of analytic and bi-univalent functions*, Appl. Math. Lett. 23, 10 (2010), 1188-1192.

[5] A. G. Alamoush, M. Darus *Coefficient bounds for new subclasses of bi-univalent functions using Hadamard product*, Acta Univ. Apul. 3, (2014), 153-161.

[6] A. G. Alamoush, M. Darus, *On coefficient estimates for new generalized subclasses of bi-univalent functions*, AIP Conference Proceedings 1614, 844 (2014).

[7] A. G. Alamoush, M. Darus, *Coefficients estimates for bi-univalent of fox-wright functions*, Far East Jour. Math. Sci. 89, 2 (2014), 249-262.

[8] E. H. Doha, *The first and second kind Chebyshev Coefficients of the moments of the general-order derivative of an infinitely differentiable function*, Intern. J. Comput. Math. 51, (1994), 21-35.

[9] J. C. Mason, *Chebyshev polynomials approximations for the L-membrane eigenvalue problem*, SIAM J. Appl. Math., 15, (1967), 172-186.

Adnan Ghazy AlAmoush
Faculty of Science, Taibah University,
Madinah, Saudi Arabia
email: *agalamoush@taibahu.edu.sa*