Acta Universitatis Apulensis No. 61/2020
ISSN: 1582-5329 pp. 1-8
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2019.61.01

COMMON FIXED POINTS FOR TWO PAIRS OF ABSORBING MAPPINGS IN PARTIAL METRIC SPACES

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Abstract. In this paper, a general fixed point theorem for two pairs of absorbing mappings satisfying a common coincidence range property in partial metric spaces is proved. As application, we obtain a result for a sequence of mappings in partial metric spaces.

2010 *Mathematics Subject Classification*: 54H25, 47H10.

Keywords: partial metric space, fixed point, pointwise absorbing mappings, common coincidence range property, implicit relation.

1. Introduction

In 1994, Matthews [9] introduced the concept of partial metric space as a part of the study of denotational semantics of dataflow networks and proved a Banach contraction principle in such spaces. Many authors studied fixed points for mappings satisfying some types of contractive conditions in partial metric spaces.

Definition 1 ([9]). *Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}_+$ is said to be a partial metric on X if for all $x, y, z \in X$, the following conditions hold:*
$(P_1) : p(x, x) = p(x, y) = p(y, y)$ *if and only if* $x = y$,
$(P_2) : p(x, x) \leq p(x, y)$,
$(P_3) : p(x, y) = p(y, x)$,
$(P_4) : p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$.
The pair (X, p) is called a partial metric space.

If $p(x, y) = 0$, then (P_1) and (P_2) implies $x = y$, but the converse does not always hold.

In 2011, Sintunavarat and Kumam [16] introduced the notion of common limit range property for a pair of mappings.

Quite recently, Imdad et al. [8] introduced the notion of joint common limit range property for two pairs of mappings.

Other results for pairs of mappings satisfying common limit range property in metric spaces are obtained in [6] - [8] and in other papers.

In all these papers and others of this topic, some convergent sequences in X are used.

We introduce a new type of range property without sequences.

Definition 2. *Let (X,p) be a partial metric space and A,S,T be self mappings on X. The pair (A,S) is said to have a coincidence range property with respect to T, denoted $CP_{(A,S)T}$ - property, if there exists $x \in X$ such that $z = Ax = Sx$ with $z \in T(X)$ and $p(z,z) = 0$.*

Example 1. *Let $X = [0,4]$ and*

$$p(x,y) = \begin{cases} |x-y| & \text{if } x,y \in [0,2] \\ \max\{x,y\} & \text{if } x,y \in (2,4]. \end{cases}$$

Then (X,p) is a partial metric space.

Let the following mappings be on X:

$$Ax = \begin{cases} 2-x, & x \in [0,1] \\ \dfrac{2-x}{2}, & x \in (1,2] \\ 0, & x \in (2,4] \end{cases}, \quad Sx = \begin{cases} \dfrac{3-x}{2}, & x \in [0,1] \\ \dfrac{x}{2}, & x \in (1,4] \end{cases}, \quad Tx = x.$$

Then $Ax = Sx$ for $x = 1$ and $z = 1 = A1 = S1$. Then $z \in T(X)$ and $p(z,z) = p(1,1) = 0$.

The notion of absorbing mappings is introduced and studied in [2], [3] and in other papers.

Definition 3 ([2], [3])**.** *Let A and S be self mappings of a metric space (X,d). A is said to be S absorbing if there exists $R > 0$ such that*

$$d(Sx, SAx) \leq Rd(Sx, Ax) \text{ for all } x \in X.$$

Similarly, S is said to be A absorbing if there exists $R > 0$ such that

$$d(Ax, ASx) \leq Rd(Sx, Ax) \text{ for all } x \in X.$$

Definition 4 ([2], [3])**.** *A is said to be pointwise S absorbing if for given $x \in X$, there exists $R > 0$ such that*

$$d(Sx, SAx) \leq Rd(Sx, Ax).$$

Similarly, S is said to be pointwise A absorbing if for given $x \in X$, there exists $R > 0$ such that

$$d(Ax, ASx) \leq Rd(Ax, Sx).$$

Remark 1. *In a partial metric space we have similar definitions with Definitions 3 and 4 with p instead of d.*

2. Implicit relations

Some fixed point theorems and common fixed point theorems have been unified considering a general condition by an implicit function in [11] - [13] and in other papers.

Some fixed point theorems for pairs of mappings satisfying implicit relations in partial mertric spaces are proved in [4], [5], [14], [15], [17].

In 2008, Ali and Imdad [1] introduced a new type of implicit relation.

Definition 5 ([1]). *Let $\mathcal{F}$ be the family of lower semi - continuous functions $F : \mathbb{R}_+^6 \to \mathbb{R}$ satisfying the following conditions:*

$(F_1) : F(t, 0, t, 0, 0, t) > 0,\ \forall t > 0$;

$(F_2) : F(t, 0, 0, t, t, 0) > 0,\ \forall t > 0$;

$(F_3) : F(t, t, 0, 0, t, t) > 0,\ \forall t > 0$.

In the following examples, property (F_1) is obviously.

Example 2. $F(t_1, ..., t_6) = t_1 - k \max\{t_2, t_3, ..., t_6\}$, *where* $k \in [0, 1)$.

Example 3. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, t_3, t_4, \dfrac{t_5 + t_6}{2}\right\}$, *where* $k \in [0, 1)$.

Example 4. $F(t_1, ..., t_6) = t_1 - k \max\left\{t_2, \dfrac{t_3 + t_4}{2}, \dfrac{t_5 + t_6}{2}\right\}$, *where* $k \in [0, 1)$.

Example 5. $F(t_1, ..., t_6) = t_1 - at_2 - bt_3 - ct_4 - dt_5 - et_6$, *where* $a, b, c, d, e \geq 0$ *and* $a + b + c + d + e < 1$.

Example 6. $F(t_1, ..., t_6) = t_1 - \alpha \max\{t_2, t_3, t_4\} - (1 - \alpha)(at_5 + bt_6)$, *where* $\alpha \in (0, 1)$, $a, b \geq 0$ *and* $a + b < 1$.

Example 7. $F(t_1, ..., t_6) = t_1 - at_2 - \dfrac{b(t_5 + t_6)}{1 + t_3 + t_4}$, *where* $a, b \geq 0$ *and* $a + 2b < 1$.

Example 8. $F(t_1, ..., t_6) = t_1 - \max\{ct_2, ct_3, ct_4, at_5 + bt_6\}$, *where* $c \in (0, 1)$, $a, b \geq 0$ *and* $a + 2b < 1$.

Example 9. $F(t_1, ..., t_6) = t_1 - at_2 - b\sqrt{t_3 t_4} - c\sqrt{t_5 t_6}$, *where* $a, b, c \geq 0$ *and* $a + b + c < 1$.

For other examples see [1].

The purpose of this paper is to prove a general fixed point theorems for two pairs of absorbing mappings satisfying a common coincidence range property using an implicit relation. As application, a general fixed point theorem for a sequence of mappings is proved.

3. Main results

Theorem 1. *Let A, B, S and T be self mappings of a partial metric space (X, p) such that*

$$F\left(\begin{array}{c} p(Ax, By), p(Sx, Ty), p(Sx, Ax), \\ p(Ty, By), p(Sx, By), p(Ax, Ty) \end{array}\right) \leq 0, \tag{1}$$

for all $x, y \in X$ and some $F \in \mathcal{F}$. If (A, S) and T satisfy $CP_{(A,S)T}$ - property, then $C(B, T) \neq \emptyset$.

Moreover, if A is pointwise S absorbing and B is pointwise T absorbing, then A, B, S and T have a unique common fixed point z with $p(z, z) = 0$.

Proof. Since (A, S) and T satisfy $CP_{(A,S)T}$ - property, there exists $v \in X$ such that $Sv = Av = z$, $z \in T(X)$ and $p(z, z) = 0$. Since $z \in T(X)$, there exists $u \in X$ such that $z = Tu$. By (1) we have

$$F\left(\begin{array}{c} p(Av, Bu), p(Sv, Tu), p(Sv, Av), \\ p(Tu, Bu), p(Sv, Bu), p(Av, Tu) \end{array}\right) \leq 0,$$

$$F(p(z, Bu), 0, 0, p(z, Bu), p(z, Bu), 0) \leq 0,$$

a contradiction of (F_2) if $p(z, Bu) > 0$. Hence, $z = Bu = Tu$ and $C(B, T) \neq \emptyset$. Hence, $z = Sv = Av = Tu = Bu$.

Moreover, if A is pointwise S absorbing, there exists $R_1 > 0$ such that

$$p(Sv, SAv) \leq R_1 p(Sv, Av) = R_1 p(z, z) = 0.$$

Hence, $z = Sv = SAv = Sz$ and z is a fixed point of S. Now we prove that $z = Az$.

By (1) we have

$$F\left(\begin{array}{c} p(Az, Bu), p(Sz, Tu), p(Sz, Az), \\ p(Tu, Bu), p(Sz, Bu), p(Az, Tu) \end{array}\right) \leq 0,$$

$$F(p(Az, z), 0, p(z, Az), 0, 0, p(Az, z)) \leq 0,$$

a contradiction of (F_1) if $p(z, Az) > 0$. Hence, $p(z, Az) = 0$ which implies $z = Az$ and z is a common fixed point of A and S.

If B is pointwise T absorbing, there exists $R_2 > 0$ such that

$$p(Tu, TBu) \leq R_2 p(Tu, Bu) = R_2 p(z, z) = 0,$$

which implies $z = Tu = TBu = Tz$ and z is a fixed point of T.

By (1) we have

$$F\left(\begin{array}{c} p\left(Av,Bz\right),p\left(Sv,Tz\right),p\left(Sv,Av\right), \\ p\left(Tz,Bz\right),p\left(Sv,Bz\right),p\left(Av,Tz\right) \end{array}\right) \leq 0,$$

$$F\left(p\left(z,Bz\right),0,0,p\left(z,Bz\right),p\left(z,Bz\right),0\right) \leq 0,$$

a contradiction of (F_2) if $p\left(z,Bz\right) > 0$. Hence, $p\left(z,Bz\right) = 0$ which implies $z = Bz = Tz$. Hence z is a common fixed point of B and T.

Therefore, z is a common fixed point of A, B, S and T with $p\left(z,z\right) = 0$.

Suppose that A, B, S and T have two common fixed points z_1, z_2 with $p\left(z_i,z_i\right) = 0,\ i = 1,2$. Then by (1) we have

$$F\left(\begin{array}{c} p\left(Az_1,Bz_2\right),p\left(Sz_1,Tz_2\right),p\left(Sz_1,Az_1\right), \\ p\left(Tz_2,Bz_2\right),p\left(Sz_1,Bz_2\right),p\left(Az_1,Tz_2\right) \end{array}\right) \leq 0,$$

$$F\left(p\left(z_1,z_2\right),p\left(z_1,z_2\right),0,0,p\left(z_1,z_2\right),p\left(z_1,z_2\right)\right) \leq 0,$$

a contradiction of (F_3) if $p\left(z_1,z_2\right) > 0$. Hence, $p\left(z_1,z_2\right) = 0$ which implies $z_1 = z_2$.

Example 10. *Let $X = [0,1]$ be a partial metric space with $p\left(x,y\right) = \max\{x,y\}$ and $Ax = 0,\ Sx = \dfrac{x}{x+2},\ Bx = \dfrac{x}{3},\ Tx = x$. Then $Ax = Sx$ implies $x = 0$ and $0 \in T\left(X\right)$, with $p\left(z,z\right) = 0$.*

$$p\left(Sx,SAx\right) = p\left(\frac{x}{x+2},0\right) = \frac{x}{x+2},$$
$$p\left(Sx,Ax\right) = \frac{x}{x+2}.$$

Hence,

$$p\left(Sx,SAx\right) \leq R_1 p\left(Sx,Ax\right) \text{ for } R_1 \geq 1.$$

Hence, A is pointwise S absorbing.

$$p\left(Tx,TBx\right) = \max\left\{x,\frac{x}{3}\right\} = x;\ p\left(Tx,Bx\right) = \max\left\{x,\frac{x}{3}\right\} = x.$$

Hence,

$$p\left(Tx,TBx\right) \leq R_2 p\left(Tx,Bx\right) \text{ for } R_2 \geq 1.$$

Hence, B is pointwise T absorbing.

On the other hand,

$$\begin{array}{rcl} p\left(Ax,By\right) & = & \max\left\{0,\frac{y}{3}\right\} = \frac{y}{3}, \\ p\left(Ty,By\right) & = & \max\left\{y,\frac{y}{3}\right\} = y. \end{array}$$

Hence,

$$p(Ax, By) \le kp(Ty, By),$$

where $k \in \left[\frac{1}{3}, 1\right)$, *which implies*

$$p(Ax, By) \le k \max\{p(Sx, Ty), p(Sx, Ax), p(Ty, By), p(Sx, By), p(Ax, Ty)\}$$

for all $x, y \in X$ *and* $k \in \left[\frac{1}{3}, 1\right)$.

By Theorem 1 and Example 2, A, B, S *and* T *have a unique common fixed point* $z = 0$ *with* $p(0,0) = 0$.

4. Fixed point for a sequence of mappings

For a function $f : (X, p) \to (X, p)$ we denote

$$pFix(f) = \{x \in X : x = fx \text{ and } p(x, x) = 0\}.$$

Theorem 2. *Let* A, B, S *and* T *be self mappings of a partial metric space* (X, p). *If the inequality (1) holds for all* $x, y \in X$ *and some* $F \in \mathcal{F}$, *then we have:*

$$[pFix(S) \cap pFix(T)] \cap pFix(A) = [pFix(S) \cap pFix(T)] \cap pFix(B).$$

Proof. Let $x \in [pFix(S) \cap pFix(T)] \cap pFix(A)$. Then by (1) we have

$$F\begin{pmatrix} p(Ax, Bx), p(Sx, Tx), p(Sx, Ax), \\ p(Tx, Bx), p(Sx, Bx), p(Ax, Tx) \end{pmatrix} \le 0,$$

$$F(p(x, Bx), p(x, x), p(x, x), p(x, Bx), p(x, Bx), p(x, x)) \le 0,$$

$$F(p(x, Bx), 0, 0, p(x, Bx), p(x, Bx), 0) \le 0,$$

a contradiction of (F_2) if $p(x, Bx) > 0$. Hence $p(x, Bx) = 0$ which implies $x = Bx$. Therefore,

$$[pFix(S) \cap pFix(T)] \cap pFix(A) \subset [pFix(S) \cap pFix(T)] \cap pFix(B).$$

Similarly, by (1) and (F_1), we obtain

$$[pFix(S) \cap pFix(T)] \cap pFix(B) \subset [pFix(S) \cap pFix(T)] \cap pFix(A).$$

By Theorems 1 and 2 we obtain:

Theorem 3. *Let S, T and $\{A_i\}_{i\in\mathbb{N}^*}$ be self mappings of a partial metric space (X, p) such that (A_1, S) and T have $CP_{(A_1,S)T}$ - property and*

$$F\left(\begin{array}{c} p(A_ix, A_{i+1}y), p(Sx, Ty), p(Sx, A_ix), \\ p(Ty, A_{i+1}y), p(Sx, A_{i+1}y), p(Ty, A_ix) \end{array}\right) \leq 0,$$

holds for all $x, y \in X$, $i \in \mathbb{N}^*$ and some $F \in \mathcal{F}$.

If A_1 is S pointwise absorbing and A_2 is pointwise T absorbing, then S, T and $\{A_i\}_{i\in\mathbb{N}^*}$ have a unique common fixed point z with $p(z, z) = 0$.

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