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FEKETE-SZEGÖ RESULTS FOR CERTAIN CLASS OF UNIVALENT FUNCTIONS USING $Q-$DERIVATIVE OPERATOR

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Abstract. In the present paper, we introduce the class $S^{*}_{\lambda,b}(q,\phi)$ of univalent function $f(z)$ for which $1+\frac{1}{b}\left[\frac{zD_qf(z)}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right]\prec\phi(z)$
$(b\in\mathbb{C}^{*}=\mathbb{C}\backslash\{0\},0\leq\lambda<1,0<q<1)$. Sharp bounds for the Fekete-Szegö functional $\left|a_3-\mu a_2^2\right|$ are obtained..

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1. Introduction

Denote by $\mathbb{A}$ the class of analytic univalent analytic functions of the form:

$$f(z)=z+\sum_{k=2}^{\infty}a_kz^k\ \ (z\in\mathbb{U}=\{z:z\in\mathbb{C}\text{ and }|z|<1\}). \tag{1.1}$$

A function $f(z)\in\mathbb{A}$ is said to be in the class $S^{*}(\alpha)$ of starlike functions of order α (see [14]) if :

$$\operatorname{Re}\left\{\frac{zf^{'}(z)}{f(z)}\right\}>\alpha\quad(0\leq\alpha<1).$$

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, the function $f(z)$ is subordinate to $g(z)$ $(f(z)\prec g(z))$ in $\mathbb{U}$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$, $f(z)=g(\omega(z))$ $(z\in\mathbb{U})$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [4] and also [11]):

$$f(z)\prec g(z)\Longleftrightarrow f(0)=g(0)\text{ and }f(\mathbb{U})\subset g(\mathbb{U}).$$

Let $\phi(z)$ be an analytic function with positive real part on $\mathbb{U}$ with $\phi(0)=1$, $\phi^{'}(0)>0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and is symmetric with respect to the real axis.

For function $f(z) \in \mathbb{A}$, Ma and Minda [10] introduced the class $S^*(\phi)$ as follows:

$$\frac{zf'(z)}{f(z)} \prec \phi(z).$$

For a function $f(z) \in \mathbb{A}$ given by (1.1) and $0 < q < 1$, the $q-$derivative of a function $f(z)$ is defined by ([6], [7], [15], [16] and [2]).

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)\,z}, z \neq 0. \tag{1.2}$$

$D_q f(0) = f'(0)$ and $D_q^2 f(z) = D_q\left(D_q f(z)\right)$. From (1.2), we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q\, a_k z^{k-1},$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As $q \to 1^-, [k]_q \to k$, so $\lim_{q\to 1^-} D_q f(z) = f'(z)$.

Making use of the $q-$derivative D_q,we introduce the class $S^*_{\lambda,b}(q,\phi)$ as follows:

Definition 1. A function $f(z) \in \mathbb{A}$ is said to be in the class $S^*_{\lambda,b}(q,\phi)$, if and only if

$$1 + \frac{1}{b}\left[\frac{zD_q f(z)}{(1-\lambda)f(z) + \lambda z D_q f(z)} - 1\right] \prec \phi(z) \;\; \left(0 \leq \lambda < 1, b \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}, 0 < q < 1\right).$$

We note that:

(i) $S^*_{0,b}(q,\phi) = S^*_b(q,\phi)$ (see [15]);

(ii) $\lim_{q\to 1^-} S^*_{0,b}(q,\phi) = S^*_b(\phi)$(see [13]) ;

(iii) $\lim_{q\to 1^-} S^*_{0,b}\left(q, \frac{1+Az}{1+Bz}\right) = S^*_b(A,B)\,(-1 \leq B < A \leq 1)$(see [13]);

(iv) $\lim_{q\to 1^-} S^*_{0,b}\left(q, \frac{1+z}{1-z}\right) = S^*(b)$(see [12] and also [3]);

(v) $\lim_{q\to 1^-} S^*_{0,b}\left(q, \frac{1+(1-2\rho)z}{1-z}\right) = S^*_b(\rho)\,(0 \leq \rho < 1)$ (see [5]);

(vi)$\lim_{q\to 1^-} S^*_{0,1}(q,\phi) = S^*(\phi)$(see [10]);

(vii) $\lim_{q\to 1^-} S^*_{0,(1-\delta)e^{-i\rho}\cos\rho}\left(q, \frac{1+z}{1-z}\right) = S^*(\rho,\delta)\left(|\rho| < \frac{\pi}{2}, 0 \leq \delta < 1\right)$ (see [9], [8]);

(viii) $\lim_{q\to 1^-} S^*_{0,be^{-i\rho}\cos\rho}\left(q, \frac{1+z}{1-z}\right) = S^\rho(b)\left(|\rho| < \frac{\pi}{2}\right)$ (see [1]).

Also, we note that:

(i) $\lim_{q\to 1^-} S^*_{\lambda,b}\left(q, \frac{1+z}{1-z}\right) = S^*_{\lambda,b}\,(0 \le \lambda < 1, b \in \mathbb{C}^*)$

$$= \left\{ f(z) \in \mathbb{A} : \operatorname{Re}\left\{ 1 + \frac{1}{b}\left[\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - 1 \right] \right\} > 0 \right\};$$

(ii) $\lim_{q\to 1^-} S^*_{\lambda,b}\,(q, \phi) = S^*_{\lambda,b}\,(\phi)\,(0 \le \lambda < 1, b \in \mathbb{C}^*)$

$$= \left\{ f(z) \in \mathbb{A} : 1 + \frac{1}{b}\left[\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z f'(z)} - 1 \right] \prec \phi(z) \right\};$$

(iii) $\lim_{q\to 1^-} S^*_{0,b}\left(q, \left(\frac{1+z}{1-z}\right)^{\sigma}\right) = S^*_b\,(\sigma)\,(0 < \sigma \le 1)$

$$= \left\{ f(z) \in \mathbb{A} : \left| \arg\left[1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1 \right) \right] \right| < \frac{\pi}{2}\sigma \right\};$$

(iv) $\lim_{q\to 1^-} S^*_{\lambda,(1-\delta)e^{-i\rho}\cos\rho}\left(q, \frac{1+Az}{1+Bz}\right) = S^*_{\lambda}\,(\delta, \rho; A, B)$

$$= \left\{ \begin{array}{c} f(z) \in \mathbb{A} : e^{i\rho}\left[\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z f'(z)} \right] \prec (1-\delta)\,\frac{1+Az}{1+Bz}\cos\rho + \delta\cos\rho + i\sin\rho, \\ \left(|\rho| < \frac{\pi}{2}, 0 \le \delta < 1, 0 \le \lambda < 1; -1 \le B < A \le 1\right) \end{array} \right\};$$

(v) $\lim_{q\to 1^-} S^*_{0,(1-\delta)e^{-i\rho}\cos\rho}\,(q, \phi) = S^*\,(\delta, \rho; \phi)\left(|\rho| < \frac{\pi}{2}, 0 \le \delta < 1\right)$

$$= \left\{ f(z) \in \mathbb{A} : e^{i\rho}\left[\frac{zf'(z)}{f(z)} \right] \prec (1-\delta)\,\phi(z)\cos\rho + \delta\cos\rho + i\sin\rho \right\}.$$

In order to prove our results, we need the following lemmas.

Lemma 1 [10]. If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then

$$\left| c_2 - \mu c_1^2 \right| \le 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the function

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Lemma 2 [10]. If $p(z) = 1 + c_1 z + c_2 z^2 + \ldots$ is an analytic function with a positive real part in $\mathbb{U}$, then

$$\left| c_2 - v c_1^2 \right| \le \left\{ \begin{array}{cc} -4v + 2, & \text{if } v \le 0, \\ 2, & \text{if } 0 \le v \le 1, \\ 4v - 2, & \text{if } v \ge 1, \end{array} \right.$$

when $v<0$ or $v>1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0<v<1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $v=0$, the equality holds if and only if

$$p(z)=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}\quad (0\leq\lambda\leq 1),$$

or one of its rotations. If $v=1$, the equality holds if and only if

$$\frac{1}{p(z)}=\left(\frac{1+\lambda}{2}\right)\frac{1+z}{1-z}+\left(\frac{1-\lambda}{2}\right)\frac{1-z}{1+z}\quad (0\leq\lambda\leq 1),$$

or one of its rotations. Also the above upper bound is sharp, and it can be improved as follows when $0<v<1$:

$$\left|c_2-vc_1^2\right|+v\left|c_1\right|^2\leq 2\quad\left(0<v\leq\frac{1}{2}\right)$$

and

$$\left|c_2-vc_1^2\right|+(1-v)\left|c_1\right|^2\leq 2\quad\left(\frac{1}{2}<v<1\right).$$

2. Main Results

Unless otherwise mentioned, we assume throughout this paper that $\phi(0)=1,\phi^{'}(0)>0, 0\leq\lambda<1, b\in\mathbb{C}^*, 0<q<1$ and $z\in\mathbb{U}$.

Theorem 1. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1\neq 0$. If $f(z)$ given by (1.1) belongs to the class $S_{\lambda,b}^*(q,\phi)$, then

$$\left|a_3-\mu a_2^2\right|\leq\tfrac{|bB_1|}{q(1-\lambda)(1+q)}\max\left\{1,\left|\tfrac{B_2}{B_1}+\left[(1+\lambda q)-\mu(1+q)\right]\frac{bB_1}{q(1-\lambda)}\right|\right\}, B_1\neq 0,\tag{2.1}$$

The result is sharp.

Proof. If $f(z)\in S_{\lambda,b}^*(q,\phi)$, then there is a Schwarz function ω, analytic in $\mathbb{U}$ with $\omega(0)=0$ and $|\omega(z)|<1$ in $\mathbb{U}$ such that

$$1+\frac{1}{b}\left[\frac{zD_qf(z)}{(1-\lambda)f(z)+\lambda zD_qf(z)}-1\right]=\phi(\omega(z)).\tag{2.2}$$

Define the function $p(z)$ by

$$p(z)=\frac{1+\omega(z)}{1-\omega(z)}=1+c_1z+c_2z^2+....\tag{2.3}$$

Since $\omega(z)$ is a Schwarz function, we see that $\mathrm{Re}\{p(z)\} > 0$ and $p(0) = 1$. Therefore,

$$\begin{aligned}\phi(\omega(z)) &= \phi\left(\frac{p(z)-1}{p(z)+1}\right) \\ &= \phi\left\{\frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right) z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) z^3 + ...\right]\right\} \\ &= 1 + \frac{1}{2} c_1 B_1 z + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} c_1^2 B_2\right] z^2 + ... \quad . \end{aligned} \tag{2.4}$$

Now by substituting (2.4) in (2.2), we have

$$\begin{aligned} & 1 + \frac{1}{b}\left[\frac{z D_q f(z)}{(1-\lambda) f(z) + \lambda z D_q f(z)} - 1\right] \\ = & \ 1 + \frac{1}{2} c_1 B_1 z + \left[\frac{1}{2} B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} c_1^2 B_2\right] z^2 + ... \quad . \end{aligned}$$

So, we obtain

$$q(1-\lambda) a_2 = \frac{1}{2} b c_1 B_1,$$

$$\begin{aligned} & q(1-\lambda)(q+1) a_3 + q(1-\lambda)(1+\lambda q) a_2^2 \\ = & \ \frac{1}{2} b B_1 \left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4} b B_2 c_1^2, \end{aligned}$$

or, equivalenty,

$$a_2 = \frac{b c_1 B_1}{2q(1-\lambda)},$$

$$a_3 = \frac{b B_1}{2q(1-\lambda)(q+1)} \left\{c_2 - \frac{1}{2}\left[1 - \tfrac{B_2}{B_1} - \frac{(1+\lambda q) b B_1}{q(1-\lambda)}\right] c_1^2\right\}.$$

Therefore,

$$a_3 - \mu a_2^2 = \frac{b B_1}{2q(1-\lambda)(q+1)} \left[c_2 - v c_1^2\right], \tag{2.5}$$

where

$$v = \frac{1}{2}\left\{1 - \tfrac{B_2}{B_1} - \left[\frac{(1+\lambda q)}{q(1-\lambda)} - \frac{\mu(q+1)}{q(1-\lambda)}\right] b B_1\right\}. \tag{2.6}$$

Our result now follows by using Lemma 1. The result is sharp for the functions

$$1 + \frac{1}{b}\left[\frac{z D_q f(z)}{(1-\lambda) f(z) + \lambda z D_q f(z)} - 1\right] = \phi(z^2)$$

and

$$1+\frac{1}{b}\left[\frac{zD_qf\left(z\right)}{\left(1-\lambda\right)f\left(z\right)+\lambda zD_qf\left(z\right)}-1\right]=\phi(z).$$

This completes the proof of Theorem 1. ∎

Remark 1.

(i) Putting $\lambda=0$ in Theorem 1, we get the result obtained by Seoudy and Aouf [15, Theorem 1];

(ii) Putting $q\to 1^{-}, b=\left(1-\delta\right)e^{-i\rho}\cos\rho\left(\left|\rho\right|<\frac{\pi}{2}, 0\leq\delta<1\right)$ and $\phi(z)=\frac{1+z}{1-z}$ in Theorem 1, we get the result obtained by Keogh and Markes [8, Theorem 1];

(iii) Putting $q\to 1^{-}, b=1$ and $\lambda=0$ in Theorem 1, we get the result obtained by Ma and Minda [10].

Also, we note that

Putting $q\to 1^{-}$ in Theorem 1, we obtain the following result.

Corollary 1. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1\neq 0$. If $f(z)$ given by (1.1) belongs to the class $S^{*}_{\lambda,b}\left(\phi\right)$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\left|bB_1\right|}{2(1-\lambda)}\max\left\{1,\left|\frac{B_2}{B_1}+\frac{\left(1+\lambda-2\mu\right)}{\left(1-\lambda\right)}bB_1\right|\right\}, B_1\neq 0.$$

The result is sharp.

Putting $q\to 1^{-}$ and $\lambda=0$ in Theorem 1, we obtain the following result which improves the result of Ravichandran et al. [13, Theorem 4.1].

Corollary 2. Let $\phi(z)=1+B_1z+B_2z^2+...$ with $B_1\neq 0$. If $f(z)$ given by (1.1) belongs to the class $S^{*}_{b}\left(\phi\right)$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\left|bB_1\right|}{2}\max\left\{1,\left|\frac{B_2}{B_1}+\left(1-2\mu\right)bB_1\right|\right\}, B_1\neq 0.$$

The result is sharp.

Putting $q\to 1^{-}$ and $\phi(z)=\frac{1+z}{1-z}$ in Theorem 1, we obtain the following result.

Corollary 3. Let $f(z)$ given by (1.1) belongs to the class $S^{*}_{\lambda,b}$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\left|b\right|}{\left(1-\lambda\right)}\max\left\{1,\left|1+\frac{2\left(1+\lambda-2\mu\right)}{\left(1-\lambda\right)}b\right|\right\}.$$

The result is sharp.

Putting $q\to 1^{-}, b=\left(1-\delta\right)e^{-i\rho}\cos\rho\left(\left|\rho\right|<\frac{\pi}{2}, 0\leq\delta<1\right)$ and $\phi(z)=\frac{1+Az}{1+Bz}\left(-1\leq B<A\leq 1\right)$ in Theorem 1, we obtain the following result.

Corollary 4. Let $f(z)$ given by (1.1) belongs to the class $S^{*}_{\lambda}\left(\delta,\rho;A,B\right)$, then

$$\left|a_3-\mu a_2^2\right|\leq\frac{\left(A-B\right)\left(1-\delta\right)\cos\rho}{2(1-\lambda)}\max\left\{1,\left|-B+\frac{\left(A-B\right)\left(1+\lambda\right)}{\left(1-\lambda\right)}-\frac{2\rho\left(A-B\right)\left(1-\delta\right)e^{-i\rho}\cos\rho}{\left(1-\lambda\right)}\right|\right\}.$$

The result is sharp.

Putting $q \to 1^-, b = (1-\delta)\, e^{-i\rho} \cos\rho \left(|\rho| < \frac{\pi}{2}, 0 \le \delta < 1\right)$ and $\lambda = 0$ in Theorem 1, we obtain the following result.

Corollary 5. Let $f(z)$ given by (1.1) belongs to the class $S^*(\delta, \rho; \phi)$, then

$$\left|a_3 - \mu a_2^2\right| \le \tfrac{(1-\delta) B_1 \cos\rho}{2} \max\left\{1, \left|\frac{B_2}{B_1} + (1-\delta)\, B_1 \cos\rho e^{-i\rho} - 2\rho B_1 (1-\delta) \cos\rho e^{-i\rho}\right|\right\}.$$

The result is sharp.

Theorem 2. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + ...\ (B_i > 0, i = 1,2; b > 0)$. If $f(z)$ given by (1.1) belongs to the class $S^*_{\lambda,b}(q,\phi)$, then

$$\sigma_1 = \frac{q(B_2 - B_1)(1-\lambda) + (1+\lambda q)\, b B_1^2}{b B_1^2 (1+q)}, \tag{2.7}$$

$$\sigma_2 = \frac{q(B_2 + B_1)(1-\lambda) + (1+\lambda q)\, b B_1^2}{b B_1^2 (1+q)}, \tag{2.8}$$

$$\sigma_3 = \frac{q B_2 (1-\lambda) + (1+\lambda q)\, b B_1^2}{b B_1^2 (1+q)}. \tag{2.9}$$

If $f(z)$ given by (1.1) belongs to the class $S^*_{\lambda,b}(q,\phi)$, then

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{b}{q(1-\lambda)(1+q)} \left\{B_2 + \frac{b B_1^2}{q(1-\lambda)} \left[(1+\lambda q) - \mu(1+q)\right]\right\} & \mu \le \sigma_1, \\ \frac{b B_1}{q(1-\lambda)(1+q)} & \sigma_1 \le \mu \le \sigma_2, \\ \frac{b}{q(1-\lambda)(1+q)} \left\{-B_2 - \frac{b B_1^2}{q(1-\lambda)} \left[(1+\lambda q) - \mu(1+q)\right]\right\} & \mu \ge \sigma_2. \end{cases}$$

Further, If $\sigma_1 \le \mu \le \sigma_3$, then

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| + \frac{q(1-\lambda)}{(1+q)\, b B_1^2} \left\{B_1 - B_2 - \frac{b B_1^2}{q(1-\lambda)} \left[(1+\lambda q) - \mu(1+q)\right]\right\} |a_2|^2 \\ \le\ & \tfrac{b B_1}{q(1-\lambda)(1+q)}. \end{aligned}$$

If $\sigma_3 \le \mu \le \sigma_2$, then

$$\begin{aligned} &\left|a_3 - \mu a_2^2\right| + \frac{q(1-\lambda)}{(1+q)\, b B_1^2} \left\{B_1 + B_2 + \frac{b B_1^2}{q(1-\lambda)} \left[(1+\lambda q) - \mu(1+q)\right]\right\} |a_2|^2 \\ \le\ & \tfrac{b B_1}{q(1-\lambda)(1+q)}. \end{aligned}$$

The result is sharp.

Proof. The results of Theorem 2 follows by applying Lemma 2 to (2.5). To show that the bounds are sharp, we define the functions $\chi_{\phi n}\,(n=2,3,4,...)$, $\digamma_{\epsilon}$ and ξ_{ϵ} $(0\leq\epsilon\leq1)$, respectively, by

$$1+\frac{1}{b}\left[\tfrac{zD_q\chi_{\phi n}(z)}{(1-\lambda)\chi_{\phi n}(z)+\lambda zD_q\chi_{\phi n}(z)}-1\right]=\phi\left(z^{n-1}\right),$$

$$\chi_{\phi n}\left(0\right)=0=\chi_{\phi n}^{'}\left(0\right)-1,$$

$$1+\frac{1}{b}\left[\tfrac{zD_q\digamma_{\epsilon}(z)}{(1-\lambda)\digamma_{\epsilon}(z)+\lambda zD_q\digamma_{\epsilon}(z)}-1\right]=\phi\left(\frac{z\left(z+\varepsilon\right)}{1+\varepsilon z}\right),$$

$$\digamma_{\epsilon}\left(0\right)=0=\digamma_{\epsilon}^{'}\left(0\right)-1$$

and

$$1+\frac{1}{b}\left[\tfrac{zD_q\xi_{\epsilon}(z)}{(1-\lambda)\xi_{\epsilon}(z)+\lambda zD_q\xi_{\epsilon}(z)}-1\right]=\phi\left(-\frac{1+\varepsilon z}{z\left(z+\varepsilon\right)}\right),$$

$$\xi_{\epsilon}\left(0\right)=0=\xi_{\epsilon}^{'}\left(0\right)-1.$$

Clearly, the functions $\chi_{\phi n},\digamma_{\epsilon}$ and $\xi_{\epsilon}\in S_{\lambda,b}^{*}\left(q,\phi\right)$. If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if $f(z)$ is $\chi_{\phi 2}$, or one of its rotations. When $\sigma_1<\mu<\sigma_2$, the equality holds if and only if $f(z)$ is $\chi_{\phi 3}$, or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if $f(z)$ is $\digamma_{\epsilon}$, or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if $f(z)$ is ξ_{ϵ}, or one of its rotations. This completes the proof of Theorem 2. ∎

Remark 2.

(i) Putting $\lambda=0$ in Theorem 2, we get the result obtained by Seoudy and Aouf [15, Theorem 3];

(ii) Putting $q\rightarrow1^{-},b=1$ and $\lambda=0$ in Theorem 2, we get the result obtained by Ma and Minda [10].

Putting $q\rightarrow1^{-}$ in Theorem 2, we obtain the following result.

Corollary 6. Let $\phi(z)=1+B_1z+B_2z^2+...\ (B_i>0,i=1,2;b>0)$. If $f(z)$ given by (1.1) belongs to the class $S_{\lambda,b}^{*}\left(\phi\right)$, then

$$\sigma_4=\frac{\left(B_2-B_1\right)\left(1-\lambda\right)+\left(1+\lambda\right)bB_1^2}{2bB_1^2},$$

$$\sigma_5=\frac{\left(B_2+B_1\right)\left(1-\lambda\right)+\left(1+\lambda\right)bB_1^2}{2bB_1^2},$$

$$\sigma_6=\frac{B_2\left(1-\lambda\right)+\left(1+\lambda\right)bB_1^2}{2bB_1^2}.$$

If $f(z)$ given by (1.1) belongs to the class $S^*_{\lambda,b}(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{b}{2(1-\lambda)}\left[B_2+\frac{bB_1^2(1+\lambda-2\mu)}{(1-\lambda)}\right] & \mu \leq \sigma_4, \\ \frac{bB_1}{2(1-\lambda)} & \sigma_4 \leq \mu \leq \sigma_5, \\ \frac{b}{2(1-\lambda)}\left[-B_2-\frac{bB_1^2(1+\lambda-2\mu)}{(1-\lambda)}\right] & \mu \geq \sigma_5. \end{cases}$$

Further, If $\sigma_4 \leq \mu \leq \sigma_6$, then

$$\begin{aligned} & \left|a_3-\mu a_2^2\right| + \frac{(1-\lambda)}{2bB_1^2}\left[B_1-B_2-\frac{bB_1^2(1+\lambda-2\mu)}{(1-\lambda)}\right]|a_2|^2 \\ \leq & \ \frac{bB_1}{2(1-\lambda)}. \end{aligned}$$

If $\sigma_6 \leq \mu \leq \sigma_5$, then

$$\begin{aligned} & \left|a_3-\mu a_2^2\right| + \frac{(1-\lambda)}{2bB_1^2}\left[B_1+B_2+\frac{bB_1^2(1+\lambda-2\mu)}{(1-\lambda)}\right]|a_2|^2 \\ \leq & \ \frac{bB_1}{2(1-\lambda)}. \end{aligned}$$

The result is sharp.

Putting $q \to 1^-$ and $\lambda = 0$ in Theorem 2, we obtain the following result.

Corollary 7. Let $\phi(z) = 1 + B_1 z + B_2 z^2 + \ldots$ $(B_i > 0, i = 1, 2; b > 0)$. If $f(z)$ given by (1.1) belongs to the class $S^*_{\lambda,b}(\phi)$, then

$$\sigma_7 = \frac{B_2 - B_1 + bB_1^2}{2bB_1^2},$$

$$\sigma_8 = \frac{B_2 + B_1 + bB_1^2}{2bB_1^2},$$

$$\sigma_9 = \frac{B_2 + bB_1^2}{2bB_1^2}.$$

If $f(z)$ given by (1.1) belongs to the class $S^*_{\lambda,b}(\phi)$, then

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \frac{b}{2}\left[B_2+bB_1^2(1-2\mu)\right] & \mu \leq \sigma_7, \\ \frac{bB_1}{2} & \sigma_7 \leq \mu \leq \sigma_8, \\ \frac{b}{2}\left[-B_2-bB_1^2(1-2\mu)\right] & \mu \geq \sigma_8. \end{cases}$$

Further, If $\sigma_7 \leq \mu \leq \sigma_9$, then

$$\begin{aligned} & \left|a_3-\mu a_2^2\right| + \frac{1}{2bB_1^2}\left[B_1-B_2-bB_1^2(1-2\mu)\right]|a_2|^2 \\ \leq & \ \frac{bB_1}{2}. \end{aligned}$$

If $\sigma_9 \leq \mu \leq \sigma_8$, then

$$\begin{aligned} & \left|a_3 - \mu a_2^2\right| + \frac{1}{2bB_1^2}\left[B_1 + B_2 + bB_1^2\left(1 - 2\mu\right)\right]\left|a_2\right|^2 \\ \leq \ & \tfrac{bB_1}{2}. \end{aligned}$$

The result is sharp.

Remark 3. Specializing the parameters λ, b, ϕ and q in the above results, we obtain results corresponding to different classes given in the introduction.

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