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MONOTONE SUZUKI-MEAN NONEXPANSIVE MAPPINGS WITH APPLICATIONS

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Abstract. In this paper, we introduce a new class of monotone generalized nonexpansive mappings and we establish some weak and strong convergence theorem for a newly proposed iterative process in the frame work of an ordered Banach space. This class of mappings is wider than the class of nonexpansive mappings, mean nonexpansive mappings and mappings satisfying condition (C). In addition, we establish that our newly proposed iterative process is faster than some existing iterative process in the literature. Finally, we provide an application to the space of $L_1([0,1])$ and to nonlinear integral equations. The results obtained in this paper improve, extend and unify some related results in the literature.

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1. Introduction

Banach contraction principle can be seen as the pivot of the theory of fixed points and its applications. The theory of fixed points plays an important role in nonlinear functional analysis and it is very useful for showing the existence and uniqueness theorems for nonlinear differential and integral equations. The importance of Banach contraction principle [4] cannot be over emphasized in the study of fixed point theory and its applications. The Banach contraction principle has been extended and generalized by researchers in this area. Researchers in this area, generalize the well celebrated Banach contraction principle by considering a class of nonlinear mappings and spaces which are more general than the class of a contraction mappings and metric spaces (see [2, 11] and the references therein). One of the generalization of a contraction mapping in the sense of Banach is the well-known nonexpansive mapping. In 1965, Browder [9], Gohde [15] and Kirk [17] gave some existence results for nonexpansive mappings. Thereafter, the generalization of nonexpansive mappings have been greatly explored by researchers in this field (see, [23, 26, 28] and the reference therein).

In 1975, Zhang [28] introduced and studied the class of mean nonexpansive mappings in Banach spaces, he proved the unique existence of fixed points for this class of mappings in Banach spaces with normal structure. We recall that, a mapping $T : C \to C$ is said to be mean nonexpasive if there exist $a, b \geq 0$ with $a + b \leq 1$ such that

$$\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|, \tag{1}$$

for all $x, y \in C$.
In 2007, Wu [27] proved that if $a + b < 1$, then the mean nonexpansive mapping T has a unique fixed point. Zuo in [30] proved that a mean nonexpansive mapping has approximate fixed point sequence, and under some suitable conditions he got some existence and uniqueness theorems of fixed points of the mean nonexpansive mapping.

In 2008, Suzuki [26] introduced the concept of mappings satisfying condition (C) which is also known as Suzuki generalized nonexpansive mapping and he proved some fixed point theorems for such class of mappings.

Definition 1. *Let C be a nonempty subset of a Banach space X, a mapping $T : C \to C$ is said to satisfy condition (C) on C if for all $x, y \in C$,*

$$\frac{1}{2}\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

In 2010, Nakprasit [20] gave an example of a mapping that is mean nonexpansive but not Suzuki generalized nonexpansive and also gave an example of a mapping that is Suzuki generalized nonexpansive but not mean nonexpansive. He also established that an increasing mean nonexpansive mappings implies Suzuki generalized nonexpansive mappings.

Remark 1. *We note from the results obtained in [20] that the class of mean nonexpanisve mappings and the class of Suzuki generalized nonexpansive mappings are two different classes of mappings. It is therefore natural to ask whether we can define a class of mappings that will generalize these classes of mappings (thereby bridging the gap between these two classes of mappings)?*

Zhou and Cui in [29] studied the existence of fixed points for mean nonexpansive mappings in CAT(0) spaces. They also obtain the demiclosed principle for mean nonexpansive mappings in CAT(0) spaces. In addition, they proved a Δ-convergence theorem and a strong convergence theorem of the Ishikawa iteration process for mean nonexpansive mappings under the proper restrictions in CAT(0) spaces. For some recent generalization of mean nonexpansive mappings, the reader should see ([10, 19] and the reference therein).

After Browder [9] established that the class of nonexpansive self mappings on a closed and bounded subset of a uniformly convex Banach space has a fixed point, researchers in this area have developed different iterative processes to approximate fixed points of nonexpansive mappings and a host of other nonlinear mappings. In general developing a faster and more efficient iterative algorithms for approximating fixed points of nonlinear mappings is still an active area of research. The following are some well known iterative algorithm in literature for approximating fixed points of nonlinear mappings. The Mann iterative process [18] is one of the oldest and fundamental iterative process used to approximate the fixed point of nonlinear mappings, which is defined by

$$\begin{cases} x_0 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n \geq 0, \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is a sequence in $(0,1)$, C is a nonempty subset of a Banach space and T is any nonlinear mapping on C.

The Ishikawa iteration [12] is another iterative process that is used to approximate fixed point of nonlinear mappings, this iteration is defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)x_n + \beta_n T y_n, \qquad n \geq 0, \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, C is a nonempty subset of a Banach space and T is any nonlinear mapping on C. It was establish in [12, 24] that the Ishikawa iteration improves the rate of convergence of Mann iteration process for an increasing function.

In [3] Agrawal et al. modified the Ishikawa iterative process and established that their newly proposed iterative process converges faster than the Mann iteration for some contractions. The iterative process is defined by

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ x_{n+1} = (1-\beta_n)T x_n + \beta_n T y_n, \qquad n \geq 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$, C is a nonempty subset of a Banach space and T is any nonlinear mapping on C.

In 2000, Noor in [21] introduced a new iteration process, which is defined as follows: Given a convex subset C of a normed space E and a nonlinear mapping $T : C \to C$. For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\alpha_n)x_n + \alpha_n T x_n, \\ y_n = (1-\beta_n)x_n + \beta_n T z_n \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n T y_n, \quad n \geq 0, \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0, 1)$. They proved that this iterative process converges faster than Mann [18], Ishikawa [12].

Remark 2. *Since it is more desirable to construct iterative processes that are more efficient and have higher rate of convergence, it is therefore natural to ask if we construct a more efficient iterative process for approximating the fixed points of nonlinear mappings.*

It is well-known that nonexpansive mappings are continuous on their domain and the continuity nature of this class of mappings make it less important in theoretical and application wise. On the other hand it has been established that mean nonexpansive mappings and Suzuki generalized nonexpansive mappings need not to be continuous on their domain. As such, these classes of mappings have great importance in theoretical and application wise compare to nonexpansive mappings. To the best of our knowledge, there is no discussion so far concerning the extension or generalization of the concept of mean nonexpansive mappings using the idea of Suzuki [26]. Motivated by the research work described above and the research work in this direction, our purpose in this paper is to introduce a new class of monotone Suzuki-mean nonexpansive mapping which is wider than the class of monotone nonexpansive mappings, the class of monotone mean nonexpansive mappings, monotone mappings satisfying condition (C) and a host of other mappings in the literature. In addition, we introduce a new three steps iteration process for approximating a fixed point of a this new class of mappings in the frame work of an ordered Banach spaces. Using our iteration process, we state and prove some convergence results for approximating fixed point of our new class of mappings. As an application, we apply our result to the space of $L_1([0, 1])$ and to nonlinear integral equations.

2. Preliminaries

Let X be a Banach space with norm $\|\cdot\|$ and the partial order $\leq$. A Banach space X with dimension greater than or equal to 2. The function $\delta_X(\epsilon) : (0, 2] \to [0, 1]$

defined by

$$\delta_X(\epsilon) = \inf\left\{1 - \|\frac{1}{2}(x+y)\| : \|x\| = 1; \|y\| = 1, \epsilon = \|x-y\|\right\}$$

is called the modulus of convexity of X. If $\delta_X(\epsilon) > 0$ for all $\epsilon \in (0,2]$, then X is called uniformly convex.

Definition 2. *A Banach space X is said to be uniformly convex in every direction if for each $\epsilon \in (0,2]$ and $z \in X$ with $\|z\| = 1$, there exists $\delta(\epsilon, z) > 0$ such that*

$$\left\|\frac{x+y}{2}\right\| \leq 1 - \delta(\epsilon, z)$$

for all $x, y \in X$ with $\|x\| \leq 1, \|y\| \leq 1$ and $\|x-y\| \in \{tz : t \in [-2,-\epsilon] \cup [\epsilon, 2]\}$. X is said to be uniformly convex if X is uniformly convex in every direction and $\inf\{\delta(\epsilon, z) : \|z\| = 1\} > 0$.

It is well-known that the class of uniformly convex Banach space is smaller than the class of uniformly convex Banach space in every direction.

Definition 3. *A Banach space X is said to have Opial property [22] if for every weakly convergent sequence $\{x_n\}$ in X with weak limit y, we have*

$$\liminf_{n\to\infty} \|x_n - y\| < \liminf_{n\to\infty} \|x_n - z\| \forall z \in X$$

with $y \neq z$.

Let C be a nonempty subset of a Banach space X and $\{x_n\}$ a bounded sequence in X. For all $x, y \in X$, we define

1. asymptotic radius of $\{x_n\}$ at x by $r(x, \{x_n\}) = \limsup_{n\to\infty} \|x_n - x\|$;
2. asymptotic radius of $\{x_n\}$ relative to C by $r(C, \{x_n\}) = \inf\{r,(x, \{x_n\}) : x \in C\}$;
3. asymptotic center of $\{x_n\}$ relative to C by $A(C, \{x_n\}) = \{r(x, \{x_n\}) = r(C, \{x_n\}) : x \in C\}$.

We note that $A(C, \{x_n\})$ is not empty and more so, if X is uniformly convex, then $A(C, \{x_n\})$ has exactly one point (see [14]).

Throughout this paper, we suppose that order intervals are closed and convex subset of an ordered Banach space $(X, \leq)$. We denote these as follows

$$[a, \to) := \{x \in X; a \leq x\}$$

and

$$[\leftarrow, b) := \{x \in X; x \leq b\}$$

for any $a, b \in X$.

Definition 4. *Let C be a subset of a normed space X. A mapping $T : C \to C$ is said to satisfy condition (I) if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ such that $f(0) = 0$ and $f(t) > 0\ \forall\ t \in (0, \infty)$ and that $\|x - Tx\| \geq f(d(x, F(T)))$ for all $x \in C$, where $d(x, F(T))$ denotes distance from x to $F(T)$.*

Definition 5. *Let $(X, \leq)$ be a partially ordered Banach and $T : X \to X$ be a mapping. The mapping T is said to be monotone if for all $x, y \in X$,*

$$x \leq y \Rightarrow Tx \leq Ty.$$

Definition 6. *[7] Let C be a nonempty subset of a Banach space X. A function $\phi : C \to [0, \infty)$ is said to be a type function, if there exists a bounded sequence $\{x_n\}$ in X such that $\phi(x) = \limsup_{n\to\infty} \|x_n - x\|$, for any $x \in C$.*

Lemma 1. *[7] Let C be a weakly compact nonempty convex subset of a uniformly convex in every direction Banach space X and $\phi : C \to [0, \infty)$ a type function. Then there exists a unique minimum point $v \in C$ such that $\phi(v) = \inf\{\phi(x) : x \in C\}$.*

Lemma 2. *[25] Let X be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\to\infty} \|x_n\| \leq c, \limsup_{n\to\infty} \|y_n\| \leq c$ and $\lim_{n\to\infty} \|t_n x_n + (1 - t_n)y_n\| = c$ holds for some $c \geq 0$. Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

3. Some basic properties for Monotone Suzki-Mean Nonexpansive Mappings

In this section, we introduce the notion of monotone Suzuki-mean nonexpansive and establish some basic properties regarding this class of mappings.

Definition 7. *Let C be a nonempty subset of an ordered Banach space $(X, \leq)$ and let $T : C \to C$ be a monotone mapping. Then T will be called a monotone Suzuki-mean nonexpansive mapping, if there exist $a, b \in [0, 1)$, with $a + b \leq 1$, such that*

$$\frac{1}{2}\|Tx - x\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\| \tag{6}$$

for all $x, y \in C$ with $x \leq y$,

Proposition 3.

1. Every monotone nonexpansive mapping is a monotone Suzuki-mean nonexpansive mapping.

2. Every monotone mapping satisfying condition (C) is a monotone Suzuki-mean nonexpansive mapping.

3. Every monotone mean nonexpansive mapping is a monotone Suzuki-mean nonexpansive mapping.

The following example shows that the converse of these statements are not true.

Example 1. *Let $C = \{(0,0),(0,1),(1,0),(0,2),(2,0)\}$ be a subset of $\mathbb{R}^2$ with dictionary order. Define a norm $\|\cdot\|$ on C by $\|(x_1,x_2)\| = |x_1| + |x_2|$. Then $(C, \|\cdot\|)$ is a Banach space. Define a mapping $T : C \to C$ by*

$$T : \begin{pmatrix} (0,0), & (0,1), & (1,0), & (0,2), & (2,0) \\ (0,0), & (0,0), & (0,0), & (0,0), & (0,2) \end{pmatrix}.$$

It is easy to see that T is monotone and for $a = b = \frac{1}{2}$,

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|,$$

but, we note that for $x = y \neq (0,0)$, and $x = (2,0), y = (1,0)$, we have that

$$\frac{1}{2}\|x - Tx\| > \|x - y\|, \tag{7}$$

as such, we have nothing to show. For example let $x = (1,0)$ and $y = (1,0,)$ we have that

$$\frac{1}{2}\|x - Tx\| = \frac{1}{2}|(1,0) - (0,0)| = \frac{1}{2} > 0 = \|x - y\|, \tag{8}$$

as such, we have nothing to show.

However, for $x = (1,0)$ and $y = (2,0)$, we have

$$\frac{1}{2}\|x - Tx\| = \frac{1}{2}|(1,0) - (0,0)| = \frac{1}{2} \leq 1 = \|x - y\|,$$

but

$$\|Tx - Ty\| = |(0,0) - (0,2)| = 2 > 1 = \|x - y\|.$$

Thus, T is not a monotone Suzuki generalized nonexpansive mapping. It also easy to see that T is not nonexpansive.

To show that T is not a monotone mean nonexpansive. We suppose that T is a monotone mean nonexpansive mapping, so therefore, there exists nonnegative real numbers a and b, such that $a+b \leq 1$ and $\|Tx-Ty\| \leq a\|x-y\| + b\|x-Ty\|$ for all $x, y \in C$. Now suppose $x=(0,0)$ and $y=(0,1)$, we then have that

$$\begin{aligned}\|Tx-Ty\| &= 0 \\ &\leq a\|x-y\| + b\|x-Ty\| \\ &= a.\end{aligned}$$

So $a \leq 1$ and $b=0$. So therefore, T is a nonexpansive mapping, which is a contradiction.

Proposition 4. Let C be a nonempty closed convex subset of an ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping with a fixed point $x \in C$ and $y \leq x$. Then T is monotone quasi-nonexapansive.

Proof. Let $x \in F(T)$ and $y \in C$.

$$\frac{1}{2}\|x-Tx\| = 0 \leq \|x-y\|,$$

so, by definition of monotone Suzuki-mean nonexpansive, we have

$$\begin{aligned}\|x-Ty\| = \|Tx-Ty\| &\leq a\|x-y\| + b\|x-Ty\| \\ \Rightarrow (1-b)\|x-Ty\| &\leq (1-b)\|x-y\| \\ \Rightarrow \|x-Ty\| &\leq \|x-y\|.\end{aligned}$$

Hence, T is monotone quasi-nonexpansive.

Lemma 5. *Let C be a nonempty closed convex subset of an ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Then $F(T)$ is closed. Moreover, if X is strictly convex and C is convex, then $F(T)$ is also convex.*

Proof. The proof follows similar argument as in Lemma 4 in [26].

Lemma 6. *Let C be a nonempty closed convex subset of an ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Then for all $x, y \in C$ with $x \leq y$:*

1. $\|T^2x - Tx\| \leq \|Tx - x\|$,

2. *either* $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$ *or* $\frac{1}{2}\|Tx - T^2x\| \leq \|Tx - y\|$,
3. *either* $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|$ *or* $\|^2x - Ty\| \leq a\|Tx - y\| + b\|Tx - Ty\|$.

Proof. 1. For all $x \in C$, we have that $\frac{1}{2}\|Tx - x\| \leq \|Tx - x\|$, which implies that

$$\|T^2x - Tx\| = \|T(Tx) - Tx\| \leq a\|Tx - x\| + b\|Tx - Tx\| \leq \|Tx - x\|.$$

2. Suppose, on the contrary $\frac{1}{2}\|x - Tx\| > \|x - y\|$ or $\frac{1}{2}\|T^2x - Tx\| > \|Tx - y\|$, for some $x, y \in C$. Now, using (1), observe that

$$\begin{aligned}\|x - Tx\| &\leq \|x - y\| + \|y - Tx\| \\ &< \frac{1}{2}\|x - Tx\| + \frac{1}{2}\|Tx - T^2x\| \\ &\leq \frac{1}{2}\|x - Tx\| + \frac{1}{2}\|x - Tx\| \\ &= \|x - Tx\|,\end{aligned}$$

which is a contradiction. Thus, we obtain the desired result.

3. The proof of (3) follows from (2). Thus, we omit it.

Lemma 7. *Let C be a nonempty closed convex subset of an ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Then for all $x, y \in C$ with $x \leq y$,*

$$\|x - Ty\| \leq \frac{(2 + a + b)}{(1 - b)}\|x - Tx\| + \|x - y\|.$$

Proof. From Lemma 6, we have for all $x, y \in C$, that $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|$ or $\|T^2x - Ty\| \leq a\|Tx - y\| + b\|Tx - Ty\|$.

Considering $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|$, we obtain that

$$\begin{aligned}\|x - Ty\| &\leq \|x - Tx\| + \|Tx - Ty\| \\ &\leq \|x - Tx\| + a\|x - y\| + b\|x - Ty\| \\ &\leq \|x - Tx\| + (1 - b)\|x - y\| + b\|x - Ty\| \\ \Rightarrow \|x - Ty\| &\leq \frac{1}{(1 - b)}\|x - Tx\| + \|x - y\| \leq \frac{(2 + a + b)}{(1 - b)}\|x - Tx\| + \|x - y\|.\end{aligned}$$

Also, considering $\|T^2x - Ty\| \le a\|Tx - y\| + b\|Tx - Ty\|$, using (1) of Lemma 6, we obtain that

$$\begin{aligned}
\|x - Ty\| &\le \|x - Tx\| + \|Tx - T^2x\| + \|T^2x - Ty\| \\
&\le \|x - Tx\| + \|x - Tx\| + a\|Tx - y\| + b\|Tx - Ty\| \\
&\le 2\|x - Tx\| + a\|Tx - x\| + a\|x - y\| + b\|Tx - x\| + b\|x - Ty\| \\
&\le (2 + a + b)\|x - Tx\| + (1 - b)\|x - y\| + b\|x - Ty\| \\
\Rightarrow \|x - Ty\| &\le \frac{(2 + a + b)}{(1 - b)}\|x - Tx\| + \|x - y\|.
\end{aligned}$$

Thus in both cases, we obtain the desired result.

Theorem 8. *Let C be a nonempty closed convex subset of a uniformly convex ordered Banach space $(X, \le)$. Suppose that $T : C \to C$ is a monotone Suzuki-mean nonexpansive mapping on C. Then $F(T) \neq \emptyset$ if and only if $\{T^n(x)\}$ is a bounded sequence for some $x \in C$ provided that $T^n x \le y$ and $x \le T(x)$ for some $y \in C$.*

Proof. Suppose that $\{T^n(x)\}$ is a bounded sequence for some $x \in C$. Since $x \le T(x)$ and using the monotonicity of T, we have that $Tx \le T^2x \le T^3x \le \cdots$. We define $\{x_n\} = \{T^n(x)\}$ for all $n \in \mathbb{N}$. By the asymptotic center of $\{x_n\}$ with respect to C, we have $A(C, \{x_n\}) = \{v\}$ such that $x_n \le v$ for all $n \in \mathbb{N}$, where v is unique. Since

$$\frac{1}{2}\|Tx_n - x_n\| = \frac{1}{2}\|x_{n+1} - x_n\| \le \|x_{n+1} - x_n\|,$$

we obtain that

$$\begin{aligned}
\|x_{n+2} - x_{n+1}\| &= \|Tx_{n+1} - Tx_n\| \\
&\le a\|x_{n+1} - x_n\| + b\|x_{n+1} - Tx_n\| \\
&= a\|x_{n+1} - x_n\| + b\|x_{n+1} - x_{n+1}\| \\
&\le \|x_{n+1} - x_n\|.
\end{aligned}$$

That is $\|x_{n+2} - x_{n+1}\| \le \|x_{n+1} - x_n\|$.

We claim that $\|x_{n+1} - x_n\| \le 2\|x_n - v\|$ or $\|x_{n+2} - x_{n+1}\| \le 2\|x_{n+1} - v\|$, for all $n \in \mathbb{N}$.
proof of claim: Suppose the contrary that $2\|x_n - v\| < \|x_{n+1} - x_n\|$ or $2\|x_{n+1} - v\| <$

$\|x_{n+2} - x_{n+1}\|$. Now, observe that

$$\begin{aligned}\|x_{n+1} - x_n\| &\leq \|x_{n+1} - y\| + \|y - x_n\| \\ &< \frac{1}{2}\|x_{n+2} - x_{n+1}\| + \frac{1}{2}\|x_{n+1} - x_n\| \\ &\leq \frac{1}{2}\|x_{n+1} - x_n\| + \frac{1}{2}\|x_{n+1} - x_n\| \\ &\leq \|x_{n+1} - x_n\|.\end{aligned}$$

Thus we have a contradiction. Hence for all $n \in \mathbb{N}$, we have that $\frac{1}{2}\|x_{n+1} - x_n\| \leq \|x_n - v\|$ or $\frac{1}{2}\|x_{n+2} - x_{n+1}\| \leq \|x_{n+1} - v\|$.
Now, considering the first case $\frac{1}{2}\|x_{n+1} - x_n\| = \frac{1}{2}\|Tx_n - x_n\| \leq \|x_n - v\|$, by definition, we have that

$$\begin{aligned}\|Tx_n - Tv\| &\leq a\|x_n - v\| + b\|x_n - Tv\| \\ &\leq (1-b)\|x_n - v\| + b\|x_n - Tv\| \\ &\Rightarrow \limsup_{n\to\infty} \|Tx_n - Tv\| \leq (1-b)\limsup_{n\to\infty} \|x_n - v\| + b\limsup_{n\to\infty} \|x_n - Tv\| \\ &\Rightarrow \limsup_{n\to\infty} \|x_n - Tv\| \leq \limsup_{n\to\infty} \|x_n - v\|.\end{aligned}$$

Thus, we have $T(v) \in A(C, \{x_n\})$, so that $Tv = v$. Using similar approach we also obtain that $Tv = v$ for the second case.

Conversely, suppose that $F(T) \neq \emptyset$. Then, there exists say $v \in F(T)$ such that $Tv = v$ and by induction we have that $T^n v = v$ for all $n \in \mathbb{N}$. As such, we have that $\{T^n(v)\}$ is a constant sequence and so bounded.

Theorem 9. *Let C be a nonempty closed convex subset of a uniformly convex ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by (9) is bounded with $x_n \leq y$ for some $y \in C$ and $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F(T) \neq \emptyset$.*

Proof. Suppose that $\{x_n\}$ is a bounded sequence and $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0$. The asymptotic center of $\{x_{n_i}\}$ with respect to C is $A(C, \{x_i\}) = \{v\}$ such that $x_{n_i} \leq v$, such that v is unique. Using the definition of asymptotic center, Lemma 7 and our hypothesis, we have that

$$\begin{aligned}r(Tv) &= \limsup_{i\to\infty} \|x_{n_i} - Tv\| \\ &\leq \limsup_{i\to\infty} \left[\frac{(2+a+b)}{(1-b)}\|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - v\|\right] \\ &= \limsup_{i\to\infty} \|x_{n_i} - v\| \\ &= r(v).\end{aligned}$$

Thus, we have that $T(v) = v$, from the from the uniqueness of v. Hence, $F(T) \neq \emptyset$.

Theorem 10. *Let C be a weakly compact nonempty closed convex subset of a uniformly convex in every direction ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Let $\{x_n\}$ be a sequence defined by (9) and $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$. Then $F(T) \neq \emptyset$.*

Proof. Suppose that $\{x_n\}$ is a bounded sequence and $\limsup_{n\to\infty} \|x_n - Tx_n\| = 0$. Then there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\limsup_{i\to\infty} \|x_{n_i} - Tx_{n_i}\| = 0$. Using the fact that C is weakly compact and the way $\{x_{n_i}\}$ is defined, we have

$$C_\infty = \bigcap_{i=1}^{\infty}[x_{n_i}, \to) \cap C = \bigcap_{i=1}^{\infty}\{x \in C : x_{n_i} \leq x\} \neq \emptyset.$$

It follows that C_∞ is closed convex subset of C. Let $x \in C_\infty$, then $x_{n_i} \leq x$ for all $i \in \mathbb{N}$. Using the fact that T is monotone, we get that

$$x_{n_i} \leq Tx_{n_i} \leq Tx.$$

Thus, we obtain that $T(C_\infty)$ is a subset of C_∞. We define a type function $\phi : C_\infty \to [0, \infty)$ generated by $\{x_{n_i}\}$; that is

$$\phi(x) = \limsup_{i\to\infty} \|x_{n_i} - x\|.$$

From Lemma 1 we can find a unique element say $v \in C_\infty$ such that $\phi(v) = \inf\{\phi(x) : x \in C_\infty\}$. Using the definition of ϕ, we obtain that

$$\phi(Tv) = \limsup_{i\to\infty} \|x_{n_i} - Tv\|.$$

Now, using Lemma 7 and our hypothesis, we have that

$$\begin{aligned}\phi(Tv) &= \limsup_{i\to\infty} \|x_{n_i} - Tv\| \\ &\leq \limsup_{i\to\infty} \left[\frac{(2+a+b)}{(1-b)} \|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - v\| \right] \\ &= \limsup_{i\to\infty} \|x_{n_i} - v\| \\ &= \phi(v).\end{aligned}$$

Thus, we have that $T(v) = v$, from the fact that minimum points are unique. Hence, $F(T) \neq \emptyset$.

4. Convergence Result

In this section, we establish some convergence results for a monotone Suzuki-mean nonexpansive mapping via our newly proposed three steps iterative algorithm. We define our iterative process as follows: For each $x_0 \in C$, the sequence $\{x_n\}$ in C is defined by

$$\begin{cases} z_n = (1-\gamma_n)x_n + \gamma_n Tx_n, \\ y_n = T[(1-\alpha_n)x_n + \alpha_n z_n], \\ x_{n+1} = (1-\beta_n)Tz_n + \beta_n Ty_n, \quad n \geq 0, \end{cases} \tag{9}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$.

Lemma 11. *Let C be a nonempty closed convex subset of an ordered Banach space $(X, \leq)$ and $T : C \to C$ be a monotone mapping and suppose that $x_1 \in C$ be such that $x_1 \leq Tx_1$ ($Tx_1 \leq x_1$). Then, for sequence $\{x_n\}$ defined by (9), we have*

1. *$x_n \leq Tx_n \leq x_{n+1}$ ($x_{n+1} \leq Tx_n \leq x_n$);*
2. *$\{x_n\}$ has at most one weak-cluster point say $x \in C$.*
   Moreover, $x_n \leq x$ for all $n \in \mathcal{N}$ provided that $\{x_n\}$ weakly converges to a point $x \in C$.

Proof. 1. We shall proof (1) using mathematical induction. Note that if $c_1, c_2 \in C$, such that $c_1 \leq c_2$, then $c_1 \leq \lambda c_1 + (1-\lambda)c_2 \leq c_2$. This holds because of the convex property defined on order interval. Using our hypothesis that ($x_1 \leq Tx_1$), we have that

$$\begin{aligned}x_1 \leq (1-\gamma_1)x_1 + \gamma_1 Tx_1 &= z_1 \\ &\leq (1-\gamma_1)Tx_1 + \gamma_1 Tx_1 \\ &= Tx_1.\end{aligned}$$

We obtain that

$$x_1 \leq z_1 \leq Tx_1. \tag{10}$$

Since T is monotone, we have that $Tx_1 \leq Tz_1$. We also obtain

$$\begin{aligned} Tx_1 &= T[(1-\alpha_1)x_1 + \alpha_1 x_1] \\ &\leq T[(1-\alpha_1)x_1 + \alpha_1 z_1] = y_1 \\ &\leq T[(1-\alpha_1)z_1 + \alpha_1 z_1] = Tz_1. \end{aligned}$$

$$Tx_1 \leq y_1 \leq Tz_1. \tag{11}$$

Combining (10) and (11), we have that

$$x_1 \leq z_1 \leq Tx_1 \leq y_1 \leq Tz_1. \tag{12}$$

Also, since T is monotone, we have that $Tz_1 \leq Ty_1$. We also obtain

$$\begin{aligned} Tz_1 &\leq (1-\beta_1)Tz_1 + \beta_1 Tz_1 \\ &\leq (1-\beta_1)Tz_1 + \beta_1 Ty_1 = x_2 \\ &\leq (1-\beta_1)Ty_1 + \beta_1 Ty_1 \\ &= Ty_1. \end{aligned}$$

We then have that

$$Tz_1 \leq x_2 \leq Ty_1. \tag{13}$$

Combining (10), (11) and (13), we have that

$$x_1 \leq z_1 \leq Tx_1 \leq y_1 \leq Tz_1 \leq x_2 \leq Ty_1. \tag{14}$$

Hence (1) is true for $n = 1$.
We suppose that $x_n \leq Tx_n$ for $n > 1$. That is

$$x_n \leq Tx_n \leq x_{n+1}. \tag{15}$$

Now, using similar approach as for the case $n = 1$, we obtain that

$$x_n \leq z_n \leq Tx_n. \tag{16}$$

Since T is monotone, we have that $Tx_n \leq Tz_n$. We also obtain

$$\begin{aligned} Tx_n &= T[(1-\alpha_n)x_n + \alpha_n x_n] \\ &\leq T[(1-\alpha_n)x_n + \alpha_1 z_n] = y_n \\ &\leq T[(1-\alpha_1)z_n + \alpha_n z_n] = Tz_n. \end{aligned}$$

$$Tx_n \leq y_n \leq Tz_n. \tag{17}$$

Combining (16) and (17), we have that

$$x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n. \tag{18}$$

Also, since T is monotone, we have that $Tz_n \leq Ty_n$. We also obtain

$$\begin{aligned} Tz_n &\leq (1-\beta_n)Tz_n + \beta_n Tz_n \\ &\leq (1-\beta_n)Tz_n + \beta_n Ty_n = x_{n+1} \\ &\leq (1-\beta_n)Ty_n + \beta_n Ty_n \\ &= Ty_n. \end{aligned}$$

We then have that

$$Tz_n \leq x_{n+1} \leq Ty_n. \tag{19}$$

Combining (16), (17) and (19), we have that

$$x_n \leq z_n \leq Tx_n \leq y_n \leq Tz_n \leq x_{n+1} \leq Ty_n. \tag{20}$$

Since T is monotone, we have that $Ty_n \leq Tx_{n+1}$, using similar approach, we obtain that

$$x_{n+1} \leq z_{n+1} \leq Tx_{n+1} \tag{21}$$

Since T is monotone, we have that $Tx_{n+1} \leq Tz_{n+1}$. We also obtain

$$\begin{aligned} Tx_{n+1} &= T[(1-\alpha_{n+1})x_{n+1} + \alpha_{n+1} x_{n+1}] \\ &\leq T[(1-\alpha_{n+1})x_{n+1} + \alpha_{n+1} z_{n+1}] = y_{n+1} \\ &\leq T[(1-\alpha_{n+1})z_{n+1} + \alpha_{n+1} z_{n+1}] = Tz_{n+1}. \end{aligned}$$

$$Tx_{n+1} \leq y_{n+1} \leq Tz_{n+1}. \tag{22}$$

Combining (21) and (22), we have that

$$x_{n+1} \leq z_{n+1} \leq Tx_{n+1} \leq y_{n+1} \leq Tz_{n+1}. \tag{23}$$

Also, since T is monotone, we have that $Tz_{n+1} \leq Ty_{n+1}$. We also obtain

$$\begin{aligned} Tz_{n+1} &\leq (1-\beta_{n+1})Tz_{n+1} + \beta_{n+1}Tz_{n+1} \\ &\leq (1-\beta_{n+1})Tz_{n+1} + \beta_{n+1}Ty_{n+1} = x_{n+2} \\ &\leq (1-\beta n+1)Ty_{n+1} + \beta_{n+1}Ty_{n+1} \\ &= Ty_{n+1}. \end{aligned}$$

We then have that

$$Tz_{n+1} \leq x_{n+2} \leq Ty_{n+1}. \tag{24}$$

Combining (21), (22) and (24), we have that

$$x_{n+1} \leq z_{n+1} \leq Tx_{n+1} \leq y_{n+1} \leq Tz_{n+1} \leq x_{n+2} \leq Ty_{n+1}. \tag{25}$$

Clearly, we obtain that

$$x_{n+1} \leq Tx_{n+1} \leq x_{n+2}. \tag{26}$$

2. The desired conclusion follows from (1) and Lemma 3.1 in [8].

Lemma 12. *Let C be a nonempty closed and convex subset of a uniformly convex ordered Banach space X and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $x_1 \in C$ such that $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). Assume that $F(T) \neq \emptyset$ and $x_1 \leq x^*$ for all $x^* \in F(T)$. Let $\{x_n\}$ be sequence defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$. Then the following hold:*

(i) $\{x_n\}$ *is bounded.*

(ii) $\lim_{n\to\infty} \|x_n - x^*\|$ *and* $\lim_{n\to\infty} d(x_n, F(T))$ *exists for all* $x^* \in F(T)$*, where* $d(x_n, F(T))$ *denotes the distance from* x *to* $F(T)$.

Proof. Let $x^* \in F(T)$, without loss of generality, we suppose that $x_1 \leq x^*$. Using the fact that T is monotone, we have

$$x_1 \leq Tx_1 \leq Tx^* = x^*. \tag{27}$$

Using the fact that T is monotone, (9) and (27), we have

$$\begin{aligned} z_1 &= (1-\gamma_1)x_1 + \gamma_n T x_1 \leq x^* \\ &\Rightarrow Tz_1 \leq Tx^* = x^*, \end{aligned}$$

also,

$$\begin{aligned} y_1 &= T[(1-\alpha_1)x_1 + \alpha_1 z_1] \leq Tx^* = x^* \\ &\Rightarrow Ty_1 \leq Tx^* = x^* \end{aligned}$$

and

$$x_2 = (1-\beta_1)Tz_1 + \beta_1 Ty_1 \leq x^* \tag{28}$$
$$\Rightarrow Tx_2 \leq Tx^* = x^*. \tag{29}$$

From (35) and (26), for $n = 2$, we get that

$$x_2 \leq Tx_2 \leq x^*. \tag{30}$$

Continuing in this way, we obtain

$$x_n \leq Tx_n \leq x^*. \tag{31}$$

Now, using (9) and Proposition 4, we have

$$\begin{aligned} \|z_n - x^*\| &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|Tx_n - x^*\| \\ &\leq (1-\gamma_n)\|x_n - x^*\| + \gamma_n\|x_n - x^*\| \\ &\leq \|x_n - x^*\|. \end{aligned} \tag{32}$$

Using (9), (32) and Proposition 4, we have

$$\begin{aligned} \|y_n - x^*\| &= \|T[(1-\alpha_n)x_n + \alpha_n z_n] - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|z_n - x^*\| \\ &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{33}$$

Using (9), (33) and Proposition 4, we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|(1-\beta_n)Tz_n + \beta_n Ty_n - x^*\| \\ &\leq (1-\beta_n)\|Tz_n - x^*\| + \beta_n\|Ty_n - x^*\| \\ &\leq (1-\beta_n)\|z_n - x^*\| + \beta_n\|y_n - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|x_n - x^*\| \\ &= \|x_n - x^*\|. \end{aligned} \tag{34}$$

This shows that $\{\|x_n - x^*\|\}$ is bounded and non-decreasing for all $x^* \in F(T)$. Thus, $\lim_{n\to\infty} \|x_n - x^*\|$ exists. For all $x^* \in F(T)$ and $n \in \mathbb{N}$, we have that $d(x_{n+1}, x^*) \leq d(x_n, x^*)$. Taking the infimum over all $x^* \in F(T)$, we get that $dist(x_{n+1}, F(T)) \leq dist(x_n, F(T))$ for all $n \in \mathbb{N}$. Thus, the sequence $\{dist(x_n, F(T))\}$ is bounded and non-decreasing. Thus, $\lim_{n\to\infty} dist(x_n, F(T))$ exists.

Lemma 13. *Let C be a nonempty closed and convex subset of a uniformly convex ordered Banach space X and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $x_1 \in C$ such that $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). Assume that $F(T) \neq \emptyset$ and $x_1 \leq x^*$ for all $x^* \in F(T)$. Let $\{x_n\}$ be sequence defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$. Then $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$.*

Proof. Since $F(T) \neq \emptyset$, then we can find $x^* \in F(T)$. We have established in Lemma 12 that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - x^*\|$ exists. Suppose that $\lim_{n\to\infty} \|x_n - x^*\| = c$. If we take $c = 0$, then we are done. Thus, we consider the case where $c > 0$. From (32), we have $\|z_n - x^*\| \leq \|x_n - x^*\|$, it then follows that

$$\limsup_{n\to\infty} \|z_n - x^*\| \leq c. \tag{35}$$

Also, using Proposition 4, we have $\|Tx_n - x^*\| \leq \|x_n - x^*\|$, it then follows that

$$\limsup_{n\to\infty} \|Tx_n - x^*\| \leq c. \tag{36}$$

Using (33) and (34), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|(1 - \beta_n)Tz_n + \beta_n Ty_n - x^*\| \\
&\leq (1 - \beta_n)\|z_n - x^*\| + \beta_n\|y_n - x^*\| \\
&\leq (1 - \beta_n)\|z_n - x^*\| + \beta_n\|x_n - x^*\|.
\end{aligned}$$

Taking the $\liminf_{n\to\infty}$ of both sides and rearranging the inequalities, we have

$$c \leq \liminf_{n\to\infty} \|z_n - x^*\|. \tag{37}$$

From (35) and (37), we obtain that $\lim_{n\to\infty} \|z_n - x^*\| = c$. That is,

$$\lim_{n\to\infty} \|(1 - \gamma_n)x_n + \gamma_n Tx_n - x^*\| = c.$$

Thus, by Lemma 2, we have

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

Theorem 14. *Let C be a nonempty closed and convex subset of a uniformly convex ordered Banach space X with Opial property and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $x_1 \in C$ such that $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). Assume that $F(T) \neq \emptyset$ and totally ordered with $x_1 \leq x^*$ for all $x^* \in F(T)$. Suppose that $\{x_n\}$ is defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0, 1]$. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. In Lemma 12 and Theorem 13 we established that $\{x_n\}$ is bounded and that $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$. Now, since X is uniformly convex, we can find a subsequence say $\{x_{n_i}\}$ of $\{x_n\}$ that converges weakly to some $x^* \in C$. Using Lemma 11, we obtain that $x_1 \leq x_{n_i} \leq x^*$ for all $j \in \mathbb{N}$ and using Lemma 7, we obtain that

$$\|x_{n_i} - Tx^*\| \leq \frac{(2 + a + b)}{(1 - b)} \|x_{n_i} - Tx_{n_i}\| + \|x_{n_i} - x^*\|.$$

This implies

$$\liminf_{n\to\infty} \|x_{n_i} - Tx^*\| \leq \liminf_{n\to\infty} \|x_{n_i} - x^*\|.$$

By the Opial property, we have that $Tx^* = x^*$. Thus, $x^* \in F(T)$. We now establish that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Let x^* and v be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. Using similar approach as in $x^* \in F(T)$, we can show that $v = Tv$. In what follows, we establish uniqueness. From Lemma 12, we have that $\lim_{n\to\infty} \|x_n - v\|$ exists. Now, suppose that $x^* \neq v$, then by Opial's condition,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \lim_{i\to\infty} \|x_{n_i} - x^*\| \\
&< \lim_{i\to\infty} \|x_{n_i} - v\| \\
&= \lim_{n\to\infty} \|x_n - v\| \\
&= \lim_{j\to\infty} \|x_{n_j} - v\| \\
&< \lim_{j\to\infty} \|x_{n_j} - x^*\| \\
&= \lim_{n\to\infty} \|x_n - x^*\|.
\end{aligned}$$

This is a contradiction, so $x^* = v$. Hence, $\{x_n\}$ converges weakly to $x^* \in F(T)$ and this completes the proof.

Theorem 15. *Let C be a nonempty closed and convex subset of a uniformly convex ordered Banach space X with Opial property and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $x_1 \in C$ such that $x_1 \leq$*

Tx_1 (or $Tx_1 \leq x_1$). Assume that $F(T) \neq \emptyset$ and totally ordered with $x_1 \leq x^*$ for all $x^* \in F(T)$. Suppose that $\{x_n\}$ is defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$. Then $\{x_n\}$ converges strongly to a point of $F(T)$ if and only if $\liminf_{n\to\infty} d(x_n, F(T)) = 0$ where $d(x, F(T)) = \inf\{\|x - x^*\| : x^* \in F(T)\}$.

Proof. Suppose that $\{x_n\}$ converges to a fixed point, say x^* of T. Then $\lim_{n\to\infty} d(x_n, x^*) = 0$, and since $0 \leq d(x_n, F(T)) \leq d(x_n, x^*)$, it follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Therefore, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$.

Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. It follows from Lemma 12 that $\lim_{n\to\infty} \|x_n - x^*\|$ exists and that $\lim_{n\to\infty} d(x_n, F(T))$ exists for all $x^* \in F(T)$. By our hypothesis, $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. Suppose $\{x_{n_k}\}$ is any arbitrary subsequence of $\{x_n\}$ and $\{y_k\}$ is a sequence in $F(T)$ such that for all $n \in \mathbb{N}$,

$$\|x_{n_k} - y_k\| < \frac{1}{2^k}$$

it follows from (34) that $\|x_{n+1} - y_k\| \leq \|x_n - y_k\| < \frac{1}{2^k}$, hence

$$\begin{aligned}
\|y_{k+1} - y_k\| &\leq \|y_{k+1} - x_{n+1}\| + \|x_{n+1} - y_k\| \\
&< \frac{1}{2^{k+1}} + \frac{1}{2^k} \\
&< \frac{1}{2^{k-1}}.
\end{aligned}$$

Thus, we have that $\{y_k\}$ is a Cauchy sequence in $F(T)$. Also, by Theorem 21, we have that $F(T)$ is closed. Thus $\{y_k\}$ is a convergent sequence in $F(T)$. Now, suppose that $\{y_k\}$ converges to $p \in F(T)$. Therefore, since

$$\|x_{n_k} - p\| \leq \|x_{n_k} - y_k\| + \|y_k - p\| \to 0 \text{ as } k \to \infty,$$

we obtain that $\lim\limits_{k\to\infty} \|x_{n_k} - p\| = 0$ and so $\{x_{n_k}\}$ converges strongly to $p \in F(T)$. Since $\lim\limits_{n\to\infty} \|x_n - p\|$ exists, it follows that $\{x_n\}$ converges strongly to p.

Theorem 16. *Let C be a nonempty closed and convex subset of a uniformly convex ordered Banach space X with Opial property and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $x_1 \in C$ such that $x_1 \leq Tx_1$ (or $Tx_1 \leq x_1$). Assume that $F(T) \neq \emptyset$ and totally ordered with $x_1 \leq x^*$ for all $x^* \in F(T)$. Suppose that $\{x_n\}$ is defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$. Let T satisfy condition (I), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From Lemma 12, we have $\lim_{n\to\infty} d(x_n, F(T))$ exists and by Theorem 13, we have $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using the fact that

$$0 \leq \lim_{n\to\infty} f(d(x, F(T)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0 \ \forall x \in C,$$

we have that $\lim_{n\to\infty} f(d(x_n, F(T))) = 0$. Since f is nondecreasing with $f(0) = 0$ and $f(t) > 0$ for $t \in (0, \infty)$, it then follows that $\lim_{n\to\infty} d(x_n, F(T)) = 0$. Hence, by Theorem 15 $\{x_n\}$ converges strongly to $x^* \in F(T)$.

5. Application to $L_1([0,1])$ space and Nonlinear Integral Equation

In this section, we present an application of our result to nonlinear integral equation and $L_1([0,1])$ the Banach space of real valued function defined on $[0,1]$ with absolute value Lebesgue integrable.

5.1. Application to $L_1([0,1])$ space

We recall that $f = 0$ if and only if the set $\{x \in [0,1] : f(x) = 0\}$ has a Lebesgue measure 0, then we say that $f = 0$ is almost everywhere. An element of $L_1([0,1])$ is therefore seen as a class of functions and the norm of any function say $f \in L_1([0,1])$ is defined as

$$\|f\| = \int_0^1 |f(x)|dx.$$

It is also worth mentioning that $f \leq g$ if an only if $f(x) \leq g(x)$ almost everywhere, for any $f, g \in L_1([0,1])$. We also recall that an ordered interval is a subset of the form

$$[f, \rightarrow) := \{g \in L_1([0,1]); f \leq g\}$$

and

$$[\leftarrow, f) := \{g \in L_1([0,1]); g \leq f\}$$

for any $f, g \in L_1([0,1])$. More so, ordered intervals are closed for convergence almost everywhere and convex.

Definition 8. *Let C be a nonempty subset of $L_1([0,1])$ space which is equipped with a vector order $\leq$. A mapping $T : C \to C$ is said to be monotone if for all $f \leq g$ we have that $T(f) \leq T(g)$.*

Lemma 17. *[6] Let $\{f_n\}$ be a sequence of uniformly L^p-bounded function on a measure space and $f_n \to f$ almost everywhere, then*

$$\liminf_{n\to\infty} \|f_n\|_p^p = \liminf_{n\to\infty} \|f_n - f\|_p^p + \|f\|_p^p,$$

for all $0 \leq p < \infty$.

Theorem 18. *Let $C \subset L_1([0,1])$ be nonempty closed convex and compact for the convergence almost everywhere and $T : C \to C$ be a monotone Suzuki-mean nonexpansive mapping. Suppose that there exists $f_1 \in C$ such that $f_1 \leq T(f_1)$ (or $T(f_1) \leq f_1$). Suppose that $\{f_n\}$ is defined by (9), where $\{\beta_n\}, \{\gamma_n\}$ and $\{\alpha_n\}$ are sequences in $[0,1]$. Then the sequence $\{f_n\}$ converges almost everywhere to some $f \in C$ which is the fixed point of T. Moreover, $f_1 \leq f$.*

Proof. From Theorem 13, we obtain that $\{f_n\}$ converges almost everywhere to some $f \in C$, where $f_n \to f$ for all $n \in \mathbb{N}$. Since $\{f_n\}$ id uniformly bounded by Lemma 17, we have that

$$\liminf_{n\to\infty} \|f_n - T(f)\| = \liminf_{n\to\infty} \|f_n - f\| + \|f - T(f)\|$$

and applying Theorem 13, we obtain that

$$\liminf_{n\to\infty} \|f_n - T(f_n)\| = 0.$$

Therefore, we obtain that

$$\liminf_{n\to\infty} \|f_n - T(f)\| = \liminf_{n\to\infty} \|f_n - f\| + \|f - T(f)\|.$$

Using Lemma 6, Lemma 7 and Theorem 13, we have that

$$\begin{aligned}
&\liminf_{n\to\infty} \|f_n - f\| + \|f - T(f)\| \\
&= \liminf_{n\to\infty} \|f_n - T(f)\| \leq \liminf_{n\to\infty} [\|f_n - T(f_n)\| + \|T(f_n) - T(f)\|] \\
&\leq \liminf_{n\to\infty} [\|f_n - T(f_n)\| + a\|f_n - f\| + b\|f_n - T(f)\|] \\
&\leq \liminf_{n\to\infty} [\|f_n - T(f_n)\| + a\|f_n - f\| + b\frac{(2+a+b)}{(1-b)}\|f_n - T(f_n)\| + b\|f_n - f\|] \\
&\leq \liminf_{n\to\infty} [(1 + b\frac{(2+a+b)}{(1-b)})\|f_n - T(f_n)\| + \|f_n - f\|],
\end{aligned}$$

rearranging the inequality, we obtain that

$$\begin{aligned}
\|f - T(f)\| &\leq \liminf_{n\to\infty} [(1 + b\frac{(2+a+b)}{(1-b)})\|f_n - T(f_n)\| + \|f_n - f\| - \|f_n - f\|] \\
&= (1 + b\frac{(2+a+b)}{(1-b)} \liminf_{n\to\infty} \|f_n - T(f_n)\| \\
&= 0.
\end{aligned}$$

We have that $\|f - T(f)\| = 0$, which implies that $T(f) = f$.

5.2. Application to Integral Equation

Considering the following nonlinear integral equation:

$$x(t) = g(t) + \lambda \int_a^b M(t,s)K(t,x(s))ds, \tag{38}$$

where $\lambda \in (0,1], M : [a,b] \times [a,b] \to \mathbb{R}^+$, $K : [a,b] \times \mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$ with ordered relation $\leq$ in X defined as for $x, y \in X, x \leq y$ if and only if $x(s) \leq y(s)$ for all $s \in [a,b]$. We defined $\|\cdot,\cdot\| : X \times X \to [0,\infty)$ by $\|x - y\| = \sup_{s\in[a,b]} |x(s) - y(s)|$.

Theorem 19. *Let $X = C([a,b],\mathbb{R})$ and $T : X \to X$ the operator given by*

$$Tx(t) = g(t) + \lambda \int_a^b M(t,s)K(t,x(s))ds$$

for all $t, s \in [a,b]$, where $M : [a,b]\times[a,b] \to \mathbb{R}^+$, $K : [a,b]\times\mathbb{R} \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are continuous functions. Let $X = C([a,b],\mathbb{R})$ be the space of all continuous real valued functions defined on $[a,b]$. Furthermore, suppose the following condition hold:

1. *there exists a continuous mapping $\mu : X \times X \to [0,\infty)$ such that*

$$|K(s,x(s)) - K(s,y(s))| \leq \mu(x,y)|x(s) - y(s)|$$

   for all $s \in [a,b]$ and $x, y \in X$.

2. *there exists $\tau \in (0,1]$, such that*

$$\int_a^b M(t,s)\mu(x,y) \leq \tau.$$

Then the integral equation (38) *has a solution.*

Proof. Without loss of generality, we suppose that $x \leq y$, so that

$$\sup\{|y(s) - x(s)| : s \in [a,b]\} \geq \sup\{|Tx(s) - x(s)| : s \in [a,b]\},$$

which implies that

$$\|y - x\| \geq \|Tx - x\| \geq \frac{1}{2}\|Tx - x\|.$$

Thus, we have that

$$\begin{aligned}|Ty(s) - Tx(s)| &\leq \left(\lambda \int_a^b |M(t,s)[K(t,y(s)) - K(t,x(s))]|ds\right)\\ &\leq \left(\lambda \int_a^b M(t,s)\mu(x,y)|y(s) - x(s)|ds\right)\\ &\leq \left(\sup_{s\in[a,b]} |y(s) - x(s)|\lambda \int_a^b M(t,s)\mu(x,y)ds\right)\\ &\leq \|y - x\|\lambda\tau\\ &\leq \|y - x\|.\end{aligned}$$

Thus, we have that

$$\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|.$$

Clearly, T is monotone Suzuki generalized nonexpansive mapping and by Proposition 3, T is monotone Suzuki-mean nonexpansive all the conditions in Theorem 9 are satisfied, as such T has a fixed point, that is the integral equation (38) has a solution.

6. Numerical Example

In this section, we present a numerical example to show the efficiency of new iteration process.

Example 2. *Define a mapping* $T : [0,1] \to [0,1]$ *as*

$$Tx = \begin{cases} 1 - x & \text{if } x \in [0,1/5), \\ \frac{x+4}{5} & \text{if } x \in [1/5,1]. \end{cases} \tag{39}$$

Then T is a Suzuki mean nonexpansive mapping, but not mean nonexpansive.

Proof. To show that T is not mean nonexpansive. We suppose that T is mean nonexpansive, so therefore, there exists nonnegative real numbers a and b, such that $a + b \leq 1$ and $\|Tx - Ty\| \leq a\|x - y\| + b\|x - Ty\|$ for all $x, y \in [0,1]$. Now suppose $x = 1$ and $y = 0$, we then have that

$$\begin{aligned}\|Tx - Ty\| &= 0\\ &\leq a\|x - y\| + b\|x - Ty\|\\ &= a.\end{aligned}$$

So $a \leq 1$ and $b = 0$. So therefore, T is a nonexpansive mapping, but this contradicts the fact that T is not continuous. Hence T is not mean nonexpansive.

To establish that T is a monotone Suzuki mean nonexpansive mapping, it suffices to show that T is a monotone Suzuki generalized nonexpansive mapping. To do this, we consider the following cases:

Case 1: Let $x \in [0, \frac{1}{5})$, as such we have that $\frac{1}{2}\|x - Tx\| = \frac{1-2x}{2} \in (\frac{3}{10}, \frac{1}{2}]$. By definition, for $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have that $y \geq \frac{1}{2}$, that is $y \in [\frac{1}{2}, 1]$. And so, we obtain that

$$\|Tx - Ty\| = \left|\frac{5x + y - 1}{5}\right| < \frac{1}{5}$$

and

$$\|x - y\| = |x - y| > \left|\frac{1}{5} - \frac{1}{2}\right| = \frac{3}{10}.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.

Case 2: Let $x \in [\frac{1}{5}, 1]$, as such we have that $\frac{1}{2}\|x - Tx\| = \frac{2-2x}{5} \in [0, \frac{4}{5}]$. By definition, for $\frac{1}{2}\|x - Tx\| \leq \|x - y\|$, we must have that $\frac{2-2x}{5} \leq |x - y|$. Due to $|x - y|$, we have two possibilities.

Case 2a: If $x < y$, we have that $\frac{2-2x}{5} < y - x$, as such we must have that $\frac{2+3x}{5} \leq y \Rightarrow y \in [\frac{13}{25}, 1] \subset [\frac{1}{5}, 1]$. And so, we obtain that

$$\|Tx - Ty\| = \left|\frac{x + 4}{5} - \frac{y + 4}{5}\right| = \frac{1}{5}|x - y| \leq \|x - y\|.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.

Case 2b: If $x \geq y$, we have that $\frac{2-2x}{5} \leq x - y$, as such we must have that $y \leq \frac{7x-2}{5} \Rightarrow y \in [\frac{-3}{25}, 1]$. We only need to consider the case in which $y \in [0, 1]$. For $y \leq \frac{7x-2}{5}$, we obtain that $x \geq \frac{5y+2}{7}$, which implies that $x \in [\frac{2}{7}, 1]$, as such we going to consider $x \in [\frac{2}{7}, 1]$ and $y \in [0, 1]$. For $x \in [\frac{2}{7}, 1]$ and $y \in [\frac{1}{5}, 1]$ have been considered in case 2a. So, we consider $x \in [\frac{2}{7}, 1]$ and $y \in [0, \frac{1}{5})$. To start with suppose $x \in [\frac{2}{7}, \frac{2}{5}]$ and $y \in [0, \frac{1}{5})$, we therefore have that

$$\|Tx - Ty\| = \left|\frac{x + 4}{5} - (1 - y)\right| = \left|\frac{x + 5y - 1}{5}\right| \leq \frac{2}{25}$$

and

$$\|x - y\| = |x - y| > \left|\frac{2}{7} - \frac{1}{5}\right| = \frac{3}{35}.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$.
Also for $x \in [\frac{2}{5}, 1]$ and $y \in [0, \frac{1}{5})$, we therefore have that

$$\|Tx - Ty\| = \left|\frac{x+4}{5} - (1-y)\right| = \left|\frac{x+5y-1}{5}\right| \leq \frac{1}{5}$$

and

$$\|x - y\| = |x - y| > \left|\frac{2}{5} - \frac{1}{5}\right| = \frac{1}{5}.$$

Thus, we have that $\frac{1}{2}\|x - Tx\| \leq \|x - y\| \Rightarrow \|Tx - Ty\| \leq \|x - y\|$. Hence T is a Suzuki generalized nonexpansive mapping and thus a generalized mean nonexpansive mapping.

In what follows, we numerically compare our new iteration process with some existing iterative processes. Taking $\alpha_n = \frac{2n}{\sqrt{7n+9}}, \gamma_n = \frac{2}{n+9}, \beta_n = \frac{1}{3n+7}$ *and* $x_0 = 0.9$.

Step	*Our Algorithm*	*Noor Algorithm*
x_0	*0.9*	*0.9*
x_1	*0.9853333*	*0.9182857*
x_2	*0.9977284*	*0.9316476*
x_3	*0.9996355*	*0.9417682*
x_4	*0.9999400*	*0.9496508*
x_5	*0.9999899*	*0.9559300*
x_6	*0.9999983*	*0.9610262*
x_7	*1*	*0.9652278*
x_8	*1*	*0.9687390*

Comparison shows that the iterative processes (9) *converges faster than the iterative processes* (5) *and consequently faster than a host of other iterative processes in the literature.*

7. Conclusion

Throughout this paper, we have discussed some fixed point results for the classs of monotone Suzuki-mean nonexpansive mappings. In addition, we introduce a new iterative algorithm for approximating a fixed point of our newly proposed class of mappings in the frame work of uniformly convex ordered Banach spaces. Furthermore, we apply our fixed point result to nonlinear integral equations and $L_1([0,1])$ spaces. Finally, in Section 6, we establish that our newly proposed iterative process is more efficient and converges faster than the Noor iterative process and consequently faster than a host of other iterative processes in the literature.

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