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Q-STARLIKE FUNCTIONS OF ORDER α AND TYPE β

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Abstract. In the present investigation, we introduce two new subclasses $\mathcal{S}_q^*(\alpha,\beta)$ and $\mathcal{T}_q^*(\alpha,\beta)$ for q-starlike functions of order α and type β. We establish Several inclusion relationships and study various characteristic properties for the class $\mathcal{T}_q^*(\alpha,\beta)$. Further application of fractional q-calculus are illustrated.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the family of normalized functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$. A function f in $\mathcal{A}$ is said to be univalent in $\mathbb{U}$ if f is one to one. As usual, we denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ consisting of univalent functions in $\mathbb{U}$. A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*(\alpha)$ of starlike functions of order α in $\mathbb{U}$ if it satisfies

$$Re\left\{\frac{zf^{'}(z)}{f(z)}\right\} > \alpha \qquad (0 \leq \alpha < 1, z \in \mathbb{U}).$$

On the other hand a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if it satisfies

$$Re\left\{1 + \frac{zf^{''}(z)}{f^{'}(z)}\right\} > \alpha \qquad (0 \leq \alpha < 1, z \in \mathbb{U}).$$

In particular, we set $\mathcal{S}^*(0) \equiv \mathcal{S}^*$ for a class of starlike functions and $\mathcal{C}(0) \equiv \mathcal{C}$ for a class of convex functions. Let g and f be two analytic functions in $\mathbb{U}$, then function

g is said to be subordinate to f if there exists an analytic function w in the unit disk $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ such that $g(z) = f(w(z))$ $(z \in \mathbb{U})$. We denote this subordination by $g \prec f$. In particular, if the function f is univalent in $\mathbb{U}$ the above subordination is equivalent to $g(0) = f(0)$ *and* $f(\mathbb{U}) \subset g(\mathbb{U})$.

For $0 < q < 1$, the q-derivative of a function f is defined by (see [3, 5, 6, 7])

$$D_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{(q-1)z} & (z \neq 0) \\ \\ f'(0) & (\mathrm{z} = 0), \end{cases} \tag{2}$$

provided that $f'(0)$ exists.

The q-integral of a function f is defined by

$$\int_0^z f(t) d_q t = z(1-q) \sum_{n=0}^{\infty} q^n f(zq^n). \tag{3}$$

From (2), it can be easily obtain that

$$D_q\Big(\sum_{n=1}^{\infty} a_n z^n\Big) = \sum_{n=1}^{\infty} [n]_q a_n z^n,$$

where

$$[n]_q = \frac{1-q^n}{1-q}.$$

As q$\to 1^-$, $[n]_q \to n$ and $\lim_{q \to 1^-} D_q f(z) = f'(z)$.

By making use of the q-derivative of a function $f \in \mathcal{A}$, Agrawal and Sahoo [1] introduced a class $\mathcal{S}_q^*(\alpha)$ of q-starlike functions of order α in the following manner:

A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*(\alpha)$ if it satiesfies

$$\Big| \frac{\frac{z}{f(z)} D_q f(z) - \alpha}{1-\alpha} - \frac{1}{1-q} \Big| \leq \frac{1}{1-q} \qquad (0 \leq \alpha < 1, z \in \mathbb{U}). \tag{4}$$

Equivalent form of the condition (4) is

$$\Big| \frac{\frac{z}{f(z)} D_q f(z) - 1}{(1+q)\big\{\frac{z}{f(z)} D_q f(z) - \alpha\big\} - \big\{\frac{z}{f(z)} D_q f(z) - 1\big\}} \Big| < 1 \quad (0 \leq \alpha < 1, z \in \mathbb{U}).$$

In particular, when $\alpha = 0$, the class $\mathcal{S}_q^*(\alpha)$ coincides with the class $\mathcal{S}_q^* = \mathcal{S}_q^*(0)$, which was introduced by Ismail et *al*. [4] in 1990. Moreover, the class $\mathcal{C}_q(\alpha)$ of q-convex functions can be defined in the form of Alexander type relation between $\mathcal{C}_q(\alpha)$ and $\mathcal{S}_q^*(\alpha)$ as:

$$f \in \mathcal{C}_q(\alpha) \Leftrightarrow zD_qf \in \mathcal{S}_q^*(\alpha).$$

We note that $\lim_{q\to 1^-} \mathcal{S}_q^*(\alpha) = \mathcal{S}^*(\alpha)$ and $\lim_{q\to 1^-} \mathcal{C}_q(\alpha) = \mathcal{C}(\alpha)$.
Motivated essentially by the work of Juneja and Mogra [8], Owa [12] and recent works on q-derivative especially [1, 2, 10, 11], [14]-[20], we introduce the classes $\mathcal{S}_q^*(\alpha,\beta)$ of q-starlike functions of order α and type β for functions $f \in \mathcal{A}$ and $\mathcal{T}_q^*(\alpha,\beta)$ of q-starlike functions of order α and type β for analytic functions with negative coefficients.

Definition 1. *For $0 \le \alpha < 1$ and $0 < \beta \le 1$, a function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{S}_q^*(\alpha,\beta)$, if it satisfies*

$$\left|\frac{\frac{z}{f(z)}D_qf(z) - 1}{(1+q)\beta\left\{\frac{z}{f(z)}D_qf(z) - \alpha\right\} - \left\{\frac{z}{f(z)}D_qf(z) - 1\right\}}\right| < 1, \quad (z \in \mathbb{U}). \tag{5}$$

Definition 2. *For $0 \le \alpha < 1$ and $0 < \beta \le 1$, an analytic function f of the form*

$$f(z) = a_1z - \sum_{n=2}^{\infty} a_nz^n \qquad (a_1 > 0,\ a_n \ge 0 \ \text{and}\ \ z \in \mathbb{U}) \tag{6}$$

is said to belong to the class $\mathcal{T}_q^*(\alpha,\beta)$, if it satisfies the condition (5)*.*

Here, we note that
(i) $\lim_{q\to 1^-} \mathcal{S}_q^*(\alpha,\beta) = \mathcal{S}^*(\alpha,\beta)$ (see Juneja and Mogra [8])
(ii)$\lim_{q\to 1^-} \mathcal{S}_q^*(\alpha,\frac{1}{2}) = \mathcal{S}^*(\alpha,\frac{1}{2})$ (see McCarty [9])
(iii)$\lim_{q\to 1^-} \mathcal{S}_q^*(\frac{1-\gamma}{1+\gamma},\frac{1+\gamma}{2}) = \mathcal{S}^*(\frac{1-\gamma}{1+\gamma},\frac{1+\gamma}{2})$; $(0 < \gamma \le 1)$ (See Padmanabhan [13])
(iv)$\lim_{q\to 1^-} \mathcal{T}_q^*(\alpha,\beta) = \mathcal{S}_0^*(\alpha,\beta)$ (see Owa [12]).

In the present paper we derive several properties including coefficient estimates, inclusion theorems, distortion theorem, convolution theorem etc. for the functions belong to the class $\mathcal{T}_q^*(\alpha,\beta)$. Applications of fractional q-calculus associated with the class $\mathcal{T}_q^*(\alpha,\beta)$ have also been obtained.

2. Main results

Theorem 1. *Let $0 \le \alpha < 1$ and $0 < \beta \le \frac{1}{1+q}$. Then a function f of the form* (6) *belongs to the class $\mathcal{T}_q^*(\alpha,\beta)$ if and only if*

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha\beta\right]a_n \le (1+q)\beta(1-\alpha)a_1. \tag{7}$$

The result is sharp for the function

$$f(z) = a_1 z - \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta} z^2. \tag{8}$$

Proof. Suppose that $f \in \mathcal{T}_q^*(\alpha,\beta)$. Making the use of series expansion of f in the inequality (5), we obtain

$$\left|\frac{\frac{z}{f(z)}D_q f(z)-1}{(1+q)\beta\left\{\frac{z}{f(z)}D_q f(z)-\alpha\right\}-\left\{\frac{z}{f(z)}D_q f(z)-1\right\}}\right|$$

$$= \left|\frac{\sum_{n=2}^{\infty}([n]_q-1)a_n z^n}{(1+q)\beta(1-\alpha)a_1 z-\sum_{n=2}^{\infty}\{1-(1+q)\alpha\beta+[n]_q-(1+q)\beta[n]_q\}a_n z^n}\right|$$

$$< 1 \tag{9}$$

Since $|Re(z)| \leq |z|$ for any z, choosing z to be real and letting $z \to 1^-$ through real values, (9) yields

$$\sum_{n=2}^{\infty}([n]_q-1)a_n \leq (1+q)\beta(1-\alpha)a_1 - \sum_{n=2}^{\infty}\{1+[n]_q-(1+q)\alpha\beta-(1+q)\beta[n]_q\}a_n \tag{10}$$

which leads us immediately to the desired inequality (7).

In order to prove the converse, we assume that the inequality (7) holds true. We have

$$|zD_q f(z) - f(z)| - |(1+q)\beta\{zD_q f(z) - \alpha f(z)\} - zD_q f(z) + f(z)|$$

$$= \left|\sum_{n=2}^{\infty}(1-[n]_q)a_n z^n\right| - \Big|(1+q)\beta(1-\alpha)a_1 z - \{(1+q)\beta-1\}\sum_{n=2}^{\infty}[n]_q a_n z^n$$
$$- \{1-(1+q)\alpha\beta\}\sum_{n=2}^{\infty} a_n z^n\Big|$$

$$\leq \sum_{n=2}^{\infty}([n]_q-1)a_n|z|^n - (1+q)\beta(1-\alpha)a_1|z|$$
$$+ \{1-(1+q)\beta\}\sum_{n=2}^{\infty}[n]_q a_n|z|^n + \{1-(1+q)\alpha\beta\}\sum_{n=2}^{\infty} a_n|z|^n$$

$$\leq \Big[\sum_{n=2}^{\infty}\big(\{2-(1+q)\beta\}[n]_q-(1+q)\alpha\beta\big)a_n - (1+q)\beta(1-\alpha)a_1\Big]|z|$$

$$\leq 0$$

consequently, by the Maximum Modulus Theorem, $f \in \mathcal{T}_q^*(\alpha,\beta)$.

Finally, by observing that the function f given by (8) is indeed an extremal function for the assertion (7). We complete the proof of Theorem 1.

Theorem 2. *Let* $0 \leq \alpha < 1$ *and* $0 < \beta_1 \leq \beta_2 \leq \frac{1}{1+q}$. *Then we have*

$$\mathcal{T}_q^*(\alpha, \beta_1) \subset \mathcal{T}_q^*(\alpha, \beta_2).$$

Proof. Let a function f of the form (6) belongs to the class $\mathcal{T}_q^*(\alpha, \beta_1)$ and $\beta_2 = \beta_1 + \delta$, then we have

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta_1\}[n]_q - (1+q)\alpha\beta_1\right] a_n \leq (1+q)\beta_1(1-\alpha)a_1$$

which gives us

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta_1(1-\alpha)a_1}{2-(1+q)\beta_1-\alpha\beta_1}.$$

Consequently,

$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta_2\}[n]_q - (1+q)\alpha\beta_2\right] a_n$

$= \sum_{n=2}^{\infty} \left[\{2-(1+q)(\beta_1+\delta)\}[n]_q - (1+q)\alpha(\beta_1+\delta)\right] a_n$

$= \sum_{n=2}^{\infty} \left[\{2-(1+q)\beta_1\}[n]_q - (1+q)\alpha\beta_1\right] a_n - \delta \sum_{n=2}^{\infty} \left[(1+q)[n]_q - (1+q)\alpha\right] a_n$

$\leq (1+q)\beta_1(1-\alpha)a_1 - \delta(1+q)([2]_q - \alpha)\sum_{n=2}^{\infty} a_n$

$\leq (1+q)\beta_1(1-\alpha)a_1 + \frac{(1+q)\delta\beta_1(1-\alpha)([2]_q-\alpha)a_1}{2-(1+q)\beta_1-\alpha\beta_1}$

$< (1+q)\beta_1(1-\alpha)a_1 + \delta(1+q)(1-\alpha)a_1$

$= (1+q)\beta_2(1-\alpha)a_1.$

Thus the proof of Theorem 2 is completed.

Theorem 3. *Let* $0 \leq \alpha_1 \leq \alpha_2 < 1$ *and* $0 < \beta \leq \frac{1}{1+q}$. *Then we have*

$$\mathcal{T}_q^*(\alpha_1, \beta) \supset \mathcal{T}_q^*(\alpha_2, \beta).$$

Proof. Let a function f of the form (6) belongs to the class $\mathcal{T}_q^*(\alpha_2, \beta)$ and $\alpha_1 = \alpha_2 - \delta$. Then we have

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha_2\beta\right] a_n \leq (1+q)\beta(1-\alpha_2)a_1$$

which gives us

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta(1-\alpha_2)a_1}{2-(1+q)\beta-\alpha_2\beta} < a_1.$$

Consequentially,

$\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta\}[n]_q-(1+q)\alpha_1\beta\right]a_n$
$=\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta\}[n]_q-(1+q)(\alpha_2-\delta)\beta\right]a_n$
$=\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta\}[n]_q-(1+q)\alpha_2\beta\right]a_n+\delta(1+q)\beta\sum_{n=2}^{\infty}a_n$
$\leq (1+q)\beta(1-\alpha_2)a_1+\delta(1+q)\beta a_1$
$=(1+q)\beta(1-\alpha_1)a_1.$

Thus the proof of Theorem 3 is completed.

Theorem 4. *Let* $0 \leq \alpha_2 \leq \alpha_1 < 1$ *and* $0 < \beta_1 \leq \beta_2 \leq \frac{1}{1+q}$. *Then we have*

$$\mathcal{T}_q^*(\alpha_1,\beta_1) \subset \mathcal{T}_q^*(\alpha_2,\beta_2).$$

Proof. Let a function f of the form (6) belongs to the class $\mathcal{T}_q^*(\alpha_1,\beta_1)$, $\alpha_2=\alpha_1-\delta$ *and* $\beta_2=\beta_1+\epsilon$. Then we have

$$\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta_1\}[n]_q-(1+q)\alpha_1\beta_1\right]a_n \leq (1+q)\beta_1(1-\alpha_1)a_1$$

which gives us

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta_1(1-\alpha_1)a_1}{2-(1+q)\beta_1-\alpha_1\beta_1} < a_1.$$

Consequently,

$\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta_2\}[n]_q-(1+q)\alpha_2\beta_2\right]a_n$

$=\sum_{n=2}^{\infty}\left[\{2-(1+q)(\beta_1+\epsilon)\}[n]_q-(1+q)(\alpha_1-\delta)(\beta_1+\epsilon)\right]a_n$

$=\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta_1\}[n]_q-(1+q)\alpha_1\beta_1\right]a_n$
$\quad -\epsilon(1+q)\sum_{n=2}^{\infty}([n]_q-\alpha_1)a_n+\delta(1+q)(\beta_1+\epsilon)\sum_{n=2}^{\infty}a_n$

$\leq (1+q)\beta_1(1-\alpha_1)a_1-\epsilon(1+q)([2]_q-\alpha_1)\sum_{n=2}^{\infty}a_n+\delta(1+q)(\beta_1+\epsilon)a_1$

$\leq (1+q)\beta_1(1-\alpha_1)a_1+\frac{\epsilon(1+q)\beta_1(1-\alpha_1)([2]_q-\alpha_1)a_1}{2-(1+q)\beta_1-\alpha_1\beta_1}+\delta(1+q)(\beta_1+\epsilon)a_1$

$\leq (1+q)\beta_1(1-\alpha_1)a_1 + \epsilon(1+q)(1-\alpha_1)a_1 + \delta(1+q)(\beta_1+\epsilon)a_1$

$= (1+q)\beta_2(1-\alpha_2)a_1.$
Thus the proof of Theorem 4 is completed.

The Hadamard products: If the functions g and h are of the form

$$g(z) = b_1 z - \sum_{n=2}^{\infty} b_n z^n \qquad (b_1 > 0,\ b_n \geq 0) \tag{11}$$

and

$$h(z) = c_1 z - \sum_{n=2}^{\infty} c_n z^n \qquad (c_1 > 0,\ c_n \geq 0), \tag{12}$$

then the Hadamard product (or Convolution) of the two functions g and h is defined by

$$g * h(z) = b_1 c_1 z - \sum_{n=2}^{\infty} b_n c_n z^n.$$

Theorem 5. *Let* $0 \leq \alpha_1, \alpha_2 < 1,,\ 0 < \beta_1, \beta_2 \leq \frac{1}{1+q}$ *and* $g \in \mathcal{T}_q^*(\alpha_1, \beta_1)$, $h \in \mathcal{T}_q^*(\alpha_2, \beta_2)$, *then* $g * h \in \mathcal{T}_q^*(\alpha, \beta)$, *where* g *and* h *are given by* (11) *and* (12) *respectively and* $\alpha = Min(\alpha_1, \alpha_2)$, $\beta = Max(\beta_1, \beta_2)$.

Proof. Since $g \in \mathcal{T}_q^*(\alpha_1, \beta_1)$ *and* $h \in \mathcal{T}_q^*(\alpha_2, \beta_2)$ so Theorem 1 gives us

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta_1\}[n]_q - (1+q)\alpha_1\beta_1\right] b_n \leq (1+q)\beta_1(1-\alpha_1)b_1$$

and

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta_2\}[n]_q - (1+q)\alpha_2\beta_2\right] c_n \leq (1+q)\beta_2(1-\alpha_2)c_1.$$

Hence

$$\sum_{n=2}^{\infty} b_n \leq \frac{\beta_1(1-\alpha_1)b_1}{\{2-(1+q)\beta_1\} - \alpha_1\beta_1} < b_1$$

and

$$\sum_{n=2}^{\infty} c_n \leq \frac{\beta_2(1-\alpha_2)c_1}{\{2-(1+q)\beta_2\} - \alpha_2\beta_2} < c_1.$$

Therefore for $\alpha = Min(\alpha_1, \alpha_2)$ *and* $\beta = Max(\beta_1, \beta_2)$,

$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha\beta\right] b_n c_n$

$\leq Max\{c_1 \sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha\beta\right] b_n,$
$b_1 \sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha\beta\right] c_n\}$

$\leq (1+q)\beta(1-\alpha)b_1 c_1.$
Consequently, $g * h \in \mathcal{T}_q^*(\alpha, \beta)$.

Theorem 6. *Let* $0 \leq \alpha < 1$, $0 < \beta \leq \frac{1}{1+q}$ *and* $f \in \mathcal{T}_q^*(\alpha, \beta)$. *Then*

$$a_1|z| - \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}|z|^2 \leq \ |f(z)| \ \leq a_1|z| + \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}|z|^2 \quad (z \in \mathbb{U}).$$

The result is sharp for the function

$$f(z) = a_1 z - \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}z^2.$$

Proof. Since $f \in \mathcal{T}_q^*(\alpha, \beta)$, then by virtue of Theorem 1, we have

$$\sum_{n=2}^{\infty} \left[\{2-(1+q)\beta\}[n]_q - (1+q)\alpha\beta\right] a_n \leq (1+q)\beta(1-\alpha)a_1$$

which gives

$$\sum_{n=2}^{\infty} a_n \leq \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}.$$

Therefore, we have

$$|f(z)| \geq a_1|z| - |z|^2 \sum_{n=2}^{\infty} a_n \geq a_1|z| - \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}|z|^2 \quad (z \in \mathbb{U}).$$

Similarly, we also get

$$|f(z)| \leq a_1|z| + |z|^2 \sum_{n=2}^{\infty} a_n \leq a_1|z| + \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-\alpha\beta}|z|^2 \quad (z \in \mathbb{U}).$$

Which completes the proof.

Theorem 7. *Let* $0 \leq \alpha < 1$, $0 < \beta \leq \frac{1}{1+q}$ *and* $f \in \mathcal{T}_q^*(\alpha, \beta)$. *Then*

$$a_1 - \frac{(1+q)\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}|z| \leq |D_q f(z)| \leq a_1 + \frac{(1+q)\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}|z| \ \ (z \in \mathbb{U}).$$

The result is sharp for the function

$$f(z) = a_1 z - \frac{\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}z^2.$$

Proof. Since $f \in \mathcal{T}_q^*(\alpha,\beta)$, then by virtue of Theorem 1, we have

$$\sum_{n=2}^{\infty}\left[\{2-(1+q)\beta\}[n]_q-(1+q)\alpha\beta\right]a_n \leq (1+q)\beta(1-\alpha)a_1,$$

which gives

$$\begin{aligned}(1+q)\beta(1-\alpha)a_1 &\geq \sum_{n=2}^{\infty}[2-(1+q)\beta-\frac{(1+q)}{[n]_q}\alpha\beta][n]_q a_n\\ &\geq \sum_{n=2}^{\infty}[2-(1+q)\beta-(1+q)\alpha\beta][n]_q a_n.\end{aligned}$$

Therefore,

$$\sum_{n=2}^{\infty}[n]_q a_n \leq \frac{(1+q)\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}.$$

Hence,

$$|D_qf(z)| \geq a_1-|z|\sum_{n=2}^{\infty}[n]_q a_n \geq a_1-\frac{(1+q)\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}|z|,$$

and

$$|D_qf(z)| \leq a_1+|z|\sum_{n=2}^{\infty}[n]_q a_n \leq a_1+\frac{(1+q)\beta(1-\alpha)a_1}{2-(1+q)\beta-(1+q)\alpha\beta}|z| \quad (z\in\mathbb{U}).$$

Which completes the proof.

In the following theorem, we obtain the radius of q-convexity for the class $\mathcal{T}_q^*(\alpha,\beta)$.

Theorem 8. *Let $0\leq\alpha<1$, $0<\beta\leq\frac{1}{1+q}$ and $f\in\mathcal{T}_q^*(\alpha,\beta)$. Then f is q-convex in the disc*

$$|z|<r=r(\alpha,\beta)=\inf_{n\geq 2}\left[\frac{2-(1+q)\beta-\alpha\beta}{([n]_q)^2\beta(1-\alpha)}\right]^{\frac{1}{n-1}}. \tag{13}$$

Proof. In order to prove the required result, we must show that

$$\left|\frac{D_q(zD_qf(z))}{D_qf(z)}-\frac{1}{1-q}\right| \leq \frac{1}{1-q} \quad (|z|<r(\alpha,\beta)). \tag{14}$$

We have

$$\Big|\frac{D_q(zD_qf(z))}{D_qf(z)}-\frac{1}{1-q}\Big| = \Big|\frac{D_q(a_1z-\sum_{n=2}^{\infty}a_n[n]_qz^n)}{a_1-\sum_{n=2}^{\infty}a_n[n]_qz^{n-1}}-\frac{1}{1-q}\Big| \leq \frac{a_1q+\sum_{n=2}^{\infty}[n]_q\{[n]_q(1-q)-1\}a_n|z|^{n-1}}{(1-q)(a_1-\sum_{n=2}^{\infty}[n]_qa_n|z|^{n-1})}.$$

Hence (14) holds true if

$$\sum_{n=2}^{\infty}\frac{([n]_q)^2}{a_1}a_n|z|^{n-1} \leq 1.$$

In view of Theorem 1, we get

$$([n]_q)^2|z|^{n-1} \leq \frac{2-(1+q)\beta-\alpha\beta}{\beta(1-\alpha)} \qquad (n=2,3,\cdots),$$

which gives us

$$|z| \leq \Big[\frac{2-(1+q)\beta-\alpha\beta}{([n]_q)^2\beta(1-\alpha)}\Big]^{\frac{1}{n-1}} \qquad (n=2,3,\cdots).$$

This completes the proof.

3. Fractional q-Calculus

In the theory of q-calculus (see [3]), the q-shifted factorial is defined for η, q $\in\mathbb{C}$ and n $\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ as a product of n factors by

$$(\eta;q)_n=\begin{cases}1 & \text{n}=0\\(1-\eta)(1-\eta q)\cdots(1-\eta q^{n-1}) & \text{n}\in\mathbb{N}\end{cases} \tag{15}$$

and in terms of the basic analogue of the gamma function

$$(q^\eta;q)_n=\frac{\Gamma_q(\eta+n)(1-q)^n}{\Gamma_q(\eta)} \qquad (n>0),$$

where the q-gamma function is defined by

$$\Gamma_q(x)=\frac{(q;q)_\infty(1-q)^{1-x}}{(q^x;q)_\infty} \qquad (0<q<1).$$

We note that, if $|q| < 1$, the definition of q-shifted factorial (15) remains meaningful for n=∞ as a convergent infinite product given as

$$(\eta; q)_\infty = \prod_{k=0}^{\infty}(1 - \eta q^k).$$

We recall here the following q-analogue definitions given in [3]. The recurrence relation for q-gamma function is given by

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$$

and the q-binomial expansion is given by

$$(x-y)_v = x^v(-y/x; q)_v = x^v \prod_{n=0}^{\infty}\Big[\frac{1-(y/x)q^n}{1-(y/x)q^{v+n}}\Big].$$

Also it may be noted that the q-Gauss hypergeometric function is defined by

$${}_2\Phi_1[\eta, \zeta; \xi; q, z] = \sum_{n=0}^{\infty}\frac{(\eta; q)_n(\zeta; q)_n}{(\xi; q)_n(q; q)_n}z^n \qquad (|q| < 1, |z| < 1)$$

and as a special case of the above series for $\zeta = \xi$, we get ${}_1\Phi_0[\eta, -; q, z]$.
In the following, we define the fractional q-calculus operators of a complex-valued function $f(z)$, which were recently studied by Purohit and Raina [14].

Definition 3. ***(Fractional*** q***-integral operator)*** *The fractional* q*-integral operator* $I^{\delta}_{q,z}$ *of a function* f *of order* δ *is defined by*

$$I^{\delta}_{q,z} \equiv D^{-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(\delta)}\int_0^z (z-tq)_{\delta-1}f(t)d_qt \qquad (\delta > 0), \tag{16}$$

where f *is analytic in a simply connected region of the* z*- plane containing the origin and the* q*-binomial function* $(z-tq)_{\delta-1}$ *is given by*

$$(z-tq)_{\delta-1} = z^{\delta-1}{}_1\Phi_0[q^{-\delta+1}; -; q, tq^{\delta}/z].$$

The series ${}_1\Phi_0[\delta; -; q, z]$ is single valued when $|arg(z)| < \pi$ and $|z| < 1$ (see for details [3], pp. 104-106). Therefore, the function $(z-tq)_{\delta-1}$ in (16) is single valued when $|arg(-tq^{\delta}/z)| < \pi$, $|tq^{\delta}/z| < 1$ and $|arg(z)| < \pi$.

Definition 4. ***(Fractional q-derivative operator)*** *The fractional q-derivative operator $D^{\delta}_{q,z}$ of a function f of order δ is defined by*

$$D^{\delta}_{q,z}f(z) \equiv D_{q,z}I^{1-\delta}_{q,z}f(z) = \frac{1}{\Gamma_q(1-\delta)}D_{q,z}\int_0^z (z-tq)_{-\delta}\, f(t)d_qt \qquad (0 \leq \delta < 1), \tag{17}$$

where f is suitably constrained and multiplicity of $(z-tq)_{-\delta}$ is removed as in Definition 3.

Here we note that for $\delta > 0$ and $n > -1$,

$$I^{\delta}_{q,z}z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n+\delta+1)}z^{n+\delta}.$$

Also for $\delta \geq 0$ and $n > -1$,

$$D^{\delta}_{q,z}z^n = \frac{\Gamma_q(n+1)}{\Gamma_q(n-\delta+1)}z^{n-\delta}.$$

Theorem 9. *Let $0 \leq \alpha < 1$, $0 < \beta \leq \frac{1}{1+q}$, $0 \leq \delta < 1$ and a function f of the form (6) belongs to the class $\mathcal{T}^*_q(\alpha,\beta)$. Then*

$$|D^{-\delta}_{q,z}f(z)| \geq \frac{a_1|z|^{1+\delta}}{\Gamma_q(2+\delta)}\Big\{1 - \frac{\beta(1-\alpha)(1-q^2)}{\big\{2-(1+q)\beta-\alpha\beta\big\}(1-q^{2+\delta})}|z|\Big\} \quad (z \in \mathbb{U}) \tag{18}$$

and

$$|D^{-\delta}_{q,z}f(z)| \leq \frac{a_1|z|^{1+\delta}}{\Gamma_q(2+\delta)}\Big\{1 + \frac{\beta(1-\alpha)(1-q^2)}{\big\{2-(1+q)\beta-\alpha\beta\big\}(1-q^{2+\delta})}|z|\Big\} \quad (z \in \mathbb{U}). \tag{19}$$

Proof. In order to prove these inequalities, we may write

$$\begin{aligned} F(z) &= \Gamma_q(2+\delta)z^{-\delta}D^{-\delta}_{q,z}f(z) \\ &= a_1z - \sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2+\delta)}{\Gamma_q(n+\delta+1)}a_nz^n, \\ &= a_1z - \sum_{n=2}^{\infty}\phi(n,\delta)a_nz^n, \end{aligned}$$

where $\phi(n,\delta) = \frac{\Gamma_q(n+1)\Gamma_q(2+\delta)}{\Gamma_q(n+\delta+1)}$, $n \geq 2$ is decreasing in n. By making use of q-gamma properties, we get

$$0 < \phi(n,\delta) \leq \phi(2,\delta) = \frac{1-q^2}{1-q^{2+\delta}},$$

and by Theorem 1

$$\begin{aligned}\Gamma_q(2+\delta)|z^{-\delta}||D_{q,z}^{-\delta}f(z)| &\geq |z| - \phi(n,\delta)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geq |z| - \frac{\beta(1-\alpha)a_1(1-q^2)}{\left\{2-(1+q)\beta-\alpha\beta\right\}(1-q^{2+\delta})}|z|^2. \qquad (20)\end{aligned}$$

Similarly, we have

$$\begin{aligned}\Gamma_q(2+\delta)|z^{-\delta}||D_{q,z}^{-\delta}f(z)| &\leq |z| + \phi(n,\delta)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leq |z| + \frac{\beta(1-\alpha)a_1(1-q^2)}{\left\{2-(1+q)\beta-\alpha\beta\right\}(1-q^{2+\delta})}|z|^2. \qquad (21)\end{aligned}$$

From (20) and (21), we obtain the inequalities (18) and (19).

Theorem 10. *Let* $0 \leq \alpha < 1$, $0 < \beta \leq \frac{1}{1+q}$, $0 \leq \delta < 1$ *and a function* f *of the form* (6) *belongs to the class* $\mathcal{T}_q^*(\alpha,\beta)$. *Then*

$$|D_{q,z}^{\delta}f(z)| \geq \frac{a_1|z|^{1-\delta}}{\Gamma_q(2-\delta)}\left\{1 - \frac{\beta(1-\alpha)(1-q^2)}{\left\{2-(1+q)\beta-\alpha\beta\right\}(1-q^{2-\delta})}|z|\right\} \quad (z\in\mathbb{U}) \qquad (22)$$

and

$$|D_{q,z}^{\delta}f(z)| \leq \frac{a_1|z|^{1-\delta}}{\Gamma_q(2-\delta)}\left\{1 + \frac{\beta(1-\alpha)(1-q^2)}{\left\{2-(1+q)\beta-\alpha\beta\right\}(1-q^{2-\delta})}|z|\right\} \quad (z\in\mathbb{U}). \qquad (23)$$

Proof. In order to prove these inequalities, we may write

$$\begin{aligned}G(z) &= \Gamma_q(2-\delta)z^{\delta}D_{q,z}^{\delta}f(z) \\ &= a_1z - \sum_{n=2}^{\infty}\frac{\Gamma_q(n+1)\Gamma_q(2-\delta)}{\Gamma_q(n-\delta+1)}a_nz^n, \\ &= a_1z - \sum_{n=2}^{\infty}\psi(n,\delta)a_nz^n,\end{aligned}$$

where $\psi(n,\delta) = \frac{\Gamma_q(n+1)\Gamma_q(2-\delta)}{\Gamma_q(n-\delta+1)}$, $n \geq 2$ is decreasing in n. By making use of q-gamma properties, we get

$$0 < \psi(n,\delta) \leq \psi(2,\delta) = \frac{1-q^2}{1-q^{2-\delta}},$$

and by Theorem 1

$$\begin{aligned}\Gamma_q(2-\delta)|z^\delta||D^\delta_{q,z}f(z)| &\geq |z| - \psi(n,\delta)|z|^2\sum_{n=2}^{\infty}a_n \\ &\geq |z| - \frac{\beta(1-\alpha)a_1(1-q^2)}{\big\{2-(1+q)\beta-\alpha\beta\big\}(1-q^{2-\delta})}|z|^2. \qquad (24)\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\Gamma_q(2-\delta)|z^\delta||D^\delta_{q,z}f(z)| &\leq |z| + \psi(n,\delta)|z|^2\sum_{n=2}^{\infty}a_n \\ &\leq |z| + \frac{\beta(1-\alpha)a_1(1-q^2)}{\big\{2-(1+q)\beta-\alpha\beta\big\}(1-q^{2-\delta})}|z|^2. \qquad (25)\end{aligned}$$

Thus, we get the desired results.

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