Acta Universitatis Apulensis No. 65/2021
ISSN: 1582-5329 pp. 51-66
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.65.05

EXISTENCE OF HYPER GENERALIZED WEAKLY SYMMETRIC LORENTZIAN PARA-SASAKIAN MANIFOLD

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Abstract. In the present paper we have investigated that a hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold is an η-Einstein manifold and some relations between the associated 1-forms.The existence of hyper generalized weakly symmetric Lorentzian para-Sasakian manifold is ensured by an example.

2010 *Mathematics Subject Classification*: 53C15, 53C25

Keywords: Hyper generalized weakly symmetric, LP-Sasakian manifold, η-Einstein, Codazzi type Ricci tensor, Recurrent Ricci tensor.

1. Introduction

In 1989, Matsumoto [11] introduced the notion of Lorentzian para-Sasakian manifolds. In 1992, Mihai and Rosca ([13]) defined some notions independently. Matsumoto, Mihai and Rosaca [12] gave a five dimensional example of such manifold.

An n-dimensional Riemannian (or semi-Riemannian manifold) is said to be hyper generalized weakly symmetric [1] if its Riemannian curvature tensor R admits the relation

$$\begin{aligned}
& (\nabla_X R)(Y,U,V,\ Z) \\
= \ & A_*(X)R(Y,U,V,\ Z) + B_*(Y)R(X,U,V,\ Z) \\
& +B_*(U)R(Y,X,V,\ Z) + D_*(V)R(Y,U,X,\ Z) \\
& +D_*(Z)R(Y,U,V,\ X) + \alpha_*(X)(g\wedge S)(Y,U,V,Z) \\
& +\beta_*(Y)(g\wedge S)(X,U,V,Z) + \beta_*(U)(g\wedge S)(Y,X,V,Z) \\
& +\gamma_*(V)\ (g\wedge S)(Y,U,X,Z) + \gamma_*(Z)\ (g\wedge S)(Y,U,V,X) \qquad (1)
\end{aligned}$$

where

$$\begin{aligned}
(g\wedge S)(Y,U,V,Z) \ = \ & g(Y,Z)S(U,V) + g(U,V)S(Y,Z) \\
& -g(Y,V)S(U,Z) - g(U,Z)S(Y,V) \qquad (2)
\end{aligned}$$

and A_*, B_*, D_*, α_*, β_* and γ_* are non-zero 1-forms defined by $A_*(X) = g(X, \sigma)$, $B_*(X) = g(X, \varrho)$, $D_*(X) = g(X, \pi)$, $\alpha_*(X) = g(X, \theta)$, $\beta_*(X) = g(X, \tau)$ and $\gamma_*(X) = g(X, \upsilon)$. The beauty of such manifold is that it has the flavour of

(i) locally symmetric space [10] (for $A_* = B_* = D_* = \alpha_* = \beta_* = \gamma_* = 0$,)

(ii) recurrent space [15] (for $A_* \neq 0$, $B_* = D_* = \alpha_* = \beta_* = \gamma_* = 0$,)

(iii) hyper recurrent space [**?**] (for $A_* \neq 0, \alpha_* \neq 0$, $B_* = D_* = \beta_* = \gamma_* = 0$,)

(iv) pseudo symmetric space [8] (for $A_* = B_* = D_* = A^*_* \neq 0$ and $\alpha_* = \beta_* = \gamma_* = 0$),

(v) semi-pseudo symmetric space [**?**] (for $B_* = D_*$ and $A_* = \alpha_* = \beta_* = \gamma_* = 0$),

(vi) hyper semi-pseudo symmetric space (for $A_* = \alpha_* = 0, B_* = D_* \neq 0$ and $\beta_* = \gamma_* \neq 0$),

(vii) hyper pseudo symmetric space (for $A_* = 2B_* = 2D_*$, $\alpha_* = 2\beta_* = 2\gamma_*$),

(viii) almost pseudo symmetric space [9] (for $A_* = B_* + H_1, H_1 = B_* = D_* \neq 0$ $\alpha_* = \beta_* = \gamma_* = 0$),

(ix) almost hyper pseudo symmetric space (for $A_* = B_* + H_1, H_1 = B_* = D_* \neq 0$, $\alpha_* = \beta_* + H_2, H_2 = \beta_* = \gamma_* \neq 0$) and

(x) weakly symmetric space [14] (for $\alpha_* = \beta_* = \gamma_* = 0$).

Recently, the authors ([6]) studied hyper generalized pseudo Q-symmetric space in the Riemannian manifolds and its various properties.

We represent our paper as follows: Section-2 is concerned with some known results of Lorentzian para-Sasakian manifold. In section-3 we have investigated that a hyper generalized weakly symmetric Lorentzian para-Sasakian manifold is an η-Einstein manifold and getting some interesting curvature conditions. Finally, we constructed an example of hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold.

2. Properties of Lorentzian para-sasakian manifold

Let M be an n-dimensional differential manifold endowed with a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η, and a Lorentzian metric g of type $(0,2)$ such that for each point $a \in M$, the tensor $g_a : T_aM \times T_aM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-,+,+,...,+)$, where T_aM denotes the tangent space of M at a and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2 = I + \eta \otimes \xi, \tag{3}$$

$$\eta(\xi) = -1, \tag{4}$$

$$g(X, \xi) = \eta(X), \tag{5}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{6}$$

for all vector fields X, Y on M^n. Then, the structure (ϕ, ξ, η, g) is called Lorentzian almost paracontact structure and the manifold with the structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold. In the Lorentzian almost paracontact manifold M, the following relations hold ([?], [2], [3], [4], [5])

$$\phi\xi = 0, \ \eta \circ \phi = 0, \tag{7}$$

$$g(\phi X, Y) = g(X, \phi Y). \tag{8}$$

A Lorentzian almost paracontact manifold M endowed with the structure (ϕ, ξ, η, g) is called an Lorentzian para-Sasakian manifold if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X, \tag{9}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g. In an LP-Sasakian manifold M with the structure (ϕ, ξ, η, g), it is easily seen that

$$\nabla_X \xi = \phi X, \tag{10}$$

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{11}$$

$$S(X, \xi) = (n-1)\eta(X), \tag{12}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{13}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{14}$$

for all vector fields X, Y on M^n.

3. Hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold

A Lorentzian para-Sasakian manifold is said to be hyper generalized weakly symmetric if it admits the curvature condition

$$\begin{aligned}
&(\nabla_X R)(Y, U, V, \ Z) \\
= \ & A_*(X)R(Y, U, V, \ Z) + B_*(Y)R(X, U, V, \ Z) \\
&+B_*(U)R(Y, X, V, \ Z) + D_*(V)R(Y, U, X, \ Z) \\
&+D_*(Z)R(Y, U, V, \ X) + \alpha_*(X)(g \wedge S)(Y, U, V, Z) \\
&+\beta_*(Y)(g \wedge S)(X, U, V, Z) + \beta_*(U)(g \wedge S)(Y, X, V, Z) \\
&+\gamma_*(V) \ (g \wedge S)(Y, U, X, Z) + \gamma_*(Z) \ (g \wedge S)(Y, U, V, X).
\end{aligned} \tag{15}$$

First using (2) in (15) and then contracting with U and V in (15) we obtain

$$\begin{aligned} & (\nabla_X S)(Y,Z) \\ = \; & A_*(X)S(Y,Z) + B_*(Y)S(X,Z) + D_*(Z)S(X,Y) + B_*(R(X,Y)Z) + D_*(R(X,Z)Y) \\ & +\alpha_*(X)[2S(Y,Z) + rg(Y,Z)] + \beta_*(Y)[2S(X,Z) + rg(X,Z)] + \gamma_*(Z)[2S(Y,X) + rg(Y,X)] \\ & +\beta_*(LX)g(Y,Z) + \beta_*(X)S(Y,Z) - \beta_*(Y)S(X,Z) - \beta_*(LY)g(Z,X) \\ & +\gamma_*(LX)g(Y,Z) + \gamma_*(X)S(Y,Z) - \gamma_*(LZ)g(Y,X) - \gamma_*(Z)S(X,Y) \end{aligned} \tag{16}$$

which yields

$$\begin{aligned} & (\nabla_X S)(Y,\xi) \\ = \; & A_*(X)(n-1)\eta(Y) + B_*(Y)(n-1)\eta(X) + D_*(\xi)S(X,Y) + B_*(X)\eta(Y) - B_*(Y)\eta(X) \\ & +D_*(X)\eta(Y) - g(X,Y)D_*(\xi) + \alpha_*(X)\eta(Y)[2(n-1)+r] + \beta_*(Y)\eta(X)[2(n-1)+r] \\ & +\gamma_*(\xi)[2S(Y,X) + rg(Y,X)] + \beta_*(LX)\eta(Y) + (n-1)[\beta_*(X)\eta(Y) - \beta_*(Y)\eta(X)] \\ & -\beta_*(LY)\eta(X) + \gamma_*(LX)\eta(Y) + (n-1)\gamma_*(X)\eta(Y) \\ & -\gamma_*(L\xi)g(X,Y) - \gamma_*(\xi)S(X,Y) \end{aligned} \tag{17}$$

for $Z = \xi$. And putting $Y = \xi$ in (16) then replacing Z by Y we get

$$\begin{aligned} & (\nabla_X S)(\xi,Y) \\ = \; & A_*(X)(n-1)\eta(Y) + B_*(\xi)S(X,Y) + D_*(Y)(n-1)\eta(X) + B_*(X)\eta(Y) - B_*(\xi)g(X,Y) \\ & +D_*(X)\eta(Y) - D_*(Y)\eta(X) + \alpha_*(X)\eta(Y)[2(n-1)+r] + \beta_*(\xi)[2S(Y,X) + rg(Y,X)] \\ & +\gamma_*(Y)\eta(X)[2(n-1)+r] + \beta_*(LX)\eta(Y) + (n-1)\beta_*(X)\eta(Y) - \beta_*(\xi)S(X,Y) \\ & -\beta_*(L\xi)g(X,Y) + \gamma_*(LX)\eta(Y) + (n-1)\gamma_*(X)\eta(Y) \\ & -\gamma_*(LY)\eta(X) - (n-1)\gamma_*(Y)\eta(X). \end{aligned} \tag{18}$$

Now from (17) and (18) we have

$$\begin{aligned} & [B_*(\xi) - D_*(\xi) + \beta_*(\xi) - \gamma_*(\xi)]S(X,Y) \\ = \; & [B_*(\xi) + \beta_*(L\xi) + r\gamma_*(\xi) - r\beta_*(\xi) - D_*(\xi) - \gamma_*(L\xi)]g(X,Y) \\ & +\eta(X)[\gamma_*(LY) + (n-2)B_*(Y) + \beta_*(Y)\{(n-1)+r\} \\ & -\beta_*(LY) - (n-2)D_*(Y) - \gamma_*(Y)\{(n-1)+r\}]. \end{aligned} \tag{19}$$

Setting $X = \xi$ in (19) we obtain

$$\begin{aligned} & \gamma_*(LY) + (n-2)B_*(Y) + \beta_*(Y)\{(n-1)+r\} - \beta_*(LY) - (n-2)D_*(Y) - \gamma_*(Y)\{(n-1)+r\} \\ = \; & [B_*(\xi) + \beta_*(L\xi) + r\gamma_*(\xi) - r\beta_*(\xi) - D_*(\xi) - \gamma_*(L\xi) \\ & -(n-1)\{B_*(\xi) - D_*(\xi) + \beta_*(\xi) - \gamma_*(\xi)\}]\eta(Y). \end{aligned} \tag{20}$$

Using (20) in (19) we obtain

$$\begin{aligned}
& [B_*(\xi) - D_*(\xi) + \beta_*(\xi) - \gamma_*(\xi)]S(X,Y) \\
= \; & [B_*(\xi) + \beta_*(L\xi) + r\gamma_*(\xi) - r\beta_*(\xi) - D_*(\xi) - \gamma_*(L\xi)]g(X,Y) \\
& + [B_*(\xi) + \beta_*(L\xi) + r\gamma_*(\xi) - r\beta_*(\xi) - D_*(\xi) - \gamma_*(L\xi) \\
& - (n-1)\{B_*(\xi) - D_*(\xi) + \beta_*(\xi) - \gamma_*(\xi)\}]\eta(X)\eta(Y). \qquad (21)
\end{aligned}$$

This leads to the following:

Theorem 1. *A hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold is an η-Einstein manifold provided $B_*(\xi) + \beta_*(\xi) \neq D_*(\xi) + \gamma_*(\xi)$.*

Taking the relations (10), (11) and (12) into account we obtain from the definition that

$$(\nabla_X S)(Y,\xi) = (n-1)g(X,\phi Y) - S(Y,\phi X). \qquad (22)$$

Again, in view of the relation (22) and (17) we get

$$\begin{aligned}
& (n-1)g(X,\phi Y) - S(Y,\phi X) \\
= \; & A_*(X)(n-1)\eta(Y) + B_*(Y)(n-1)\eta(X) + D_*(\xi)S(X,Y) + B_*(X)\eta(Y) - B_*(Y)\eta(X) \\
& + D_*(X)\eta(Y) - g(X,Y)D_*(\xi) + \alpha_*(X)\eta(Y)[2(n-1)+r] + \beta_*(Y)\eta(X)[2(n-1)+r] \\
& + \gamma_*(\xi)[2S(Y,X) + rg(Y,X)] + \beta_*(LX)\eta(Y) + (n-1)[\beta_*(X)\eta(Y) - \beta_*(Y)\eta(X)] \\
& - \beta_*(LY)\eta(X) + \gamma_*(LX)\eta(Y) + (n-1)\gamma_*(X)\eta(Y) \\
& - \gamma_*(L\xi)g(X,Y) - \gamma_*(\xi)S(X,Y). \qquad (23)
\end{aligned}$$

Next, setting $X = Y = \xi$ in (23) and with the help of (4), (12), we have

$$(n-1)[A_*(\xi) + B_*(\xi) + D_*(\xi)] + [2(n-1)+r][\alpha_*(\xi) + \beta_*(\xi) + \gamma_*(\xi)] = 0. \qquad (24)$$

This leads to the following:

Theorem 2. *In Lorentzian Para-Sasakian manifold with hyper generalized weakly symmetric curvature condition, the relation (24) hold good.*

Corollary 3. *In each of pseudo symmetric, semi-pseudo symmetric and recurrent Lorentzian Para-Sasakian manifold the vector field associate to the 1-form is perpendicular to the characeristic vector field ξ.*

In consequence of (4), (5), (7), (12), (13) and (14), the equation (16) turns into

$$\begin{aligned}
& (\nabla_X S)(\xi, Z) \\
= \; & A_*(X)(n-1)\eta(Z) + B_*(\xi)S(X,Z) + D_*(Z)(n-1)\eta(X) + B_*(X)\eta(Z) - B_*(\xi)g(X,Z) \\
& + D_*(X)\eta(Z) - \eta(X)D_*(Z) + \alpha_*(X)\eta(Z)[2(n-1)+r] + \beta_*(\xi)[2S(Z,X) + rg(Z,X)] \\
& + \gamma_*(Z)\eta(X)[2(n-1)+r] + \beta_*(LX)\eta(Z) + (n-1)\eta(Z)[\beta_*(X) + \gamma_*(X)] - \beta_*(\xi)S(X,Z) \\
& - \beta_*(L\xi)g(X,Z) + \gamma_*(LX)\eta(Z) - \gamma_*(LZ)\eta(X) - (n-1)\gamma_*(Z)\eta(X) \qquad (25)
\end{aligned}$$

for $Y = \xi$.

By virtue of (22),the equation (25) turns into

$$\begin{aligned}
& (n-1)g(X,\phi Z) - S(Z,\phi X) \\
= \; & A_*(X)(n-1)\eta(Z) + B_*(\xi)S(X,Z) + D_*(Z)(n-1)\eta(X) + B_*(X)\eta(Z) - B_*(\xi)g(X,Z) \\
& + D_*(X)\eta(Z) - \eta(X)D_*(Z) + \alpha_*(X)\eta(Z)[2(n-1)+r] + \beta_*(\xi)[2S(Z,X) + rg(Z,X)] \\
& + \gamma_*(Z)\eta(X)[2(n-1)+r] + \beta_*(LX)\eta(Z) + (n-1)\eta(Z)[\beta_*(X) + \gamma_*(X)] - \beta_*(\xi)S(X,Z) \\
& - \beta_*(L\xi)g(X,Z) + \gamma_*(LX)\eta(Z) - \gamma_*(LZ)\eta(X) - (n-1)\gamma_*(Z)\eta(X). \qquad (26)
\end{aligned}$$

Putting $Z = \xi$ in (26) and with the help of (10), (11) and (12) we have

$$\begin{aligned}
0 \; = \; & -A_*(X)(n-1) + (n-1)B_*(\xi)\eta(X) + D_*(\xi)(n-1)\eta(X) - B_*(X) - B_*(\xi)\eta(X) \\
& - D_*(X) - \eta(X)D_*(\xi) - \alpha_*(X)[2(n-1)+r] + \beta_*(\xi)\eta(X)[2(n-1)+r] \\
& + \gamma_*(\xi)\eta(X)[2(n-1)+r] - \beta_*(LX) - (n-1)[\beta_*(X) + \gamma_*(X)] - (n-1)\beta_*(\xi)\eta(X) \\
& - \beta_*(L\xi)\eta(X) - \gamma_*(LX) - \gamma_*(L\xi)\eta(X) - (n-1)\gamma_*(\xi)\eta(X). \qquad (27)
\end{aligned}$$

Putting $X = \xi$ in (26) and using (10), (11) & (12), we have

$$\begin{aligned}
0 \; = \; & A_*(\xi)(n-1)\eta(Z) + B_*(\xi)(n-1)\eta(Z) - D_*(Z)(n-1) + D_*(\xi)\eta(Z) + D_*(Z) \\
& + \alpha_*(\xi)\eta(Z)[2(n-1)+r] + \beta_*(\xi)\eta(Z)[2(n-1)+r] - \gamma_*(Z)[2(n-1)+r] \\
& + \beta_*(L\xi)\eta(Z) + (n-1)\eta(Z)[\beta_*(\xi) + \gamma_*(\xi)] - \beta_*(\xi)(n-1)\eta(Z) \\
& - \beta_*(L\xi)\eta(Z) + \gamma_*(L\xi)\eta(Z) + \gamma_*(LZ) + (n-1)\gamma_*(Z). \qquad (28)
\end{aligned}$$

Now replacing Z by X in the above equation we have

$$\begin{aligned}
0 \; = \; & A_*(\xi)(n-1)\eta(X) + B_*(\xi)(n-1)\eta(X) - D_*(X)(n-1) + D_*(\xi)\eta(X) + D_*(X) \\
& + \alpha_*(\xi)\eta(X)[2(n-1)+r] + \beta_*(\xi)\eta(X)[2(n-1)+r] - \gamma_*(X)[2(n-1)+r] \\
& + \beta_*(L\xi)\eta(X) + (n-1)\eta(X)[\beta_*(\xi) + \gamma_*(\xi)] - \beta_*(\xi)(n-1)\eta(X) \\
& - \beta_*(L\xi)\eta(X) + \gamma_*(L\xi)\eta(X) + \gamma_*(LX) + (n-1)\gamma_*(X). \qquad (29)
\end{aligned}$$

By virtue of (27), (29) and (24) we get

$$\begin{aligned}
0 \; = \; & -A_*(X)(n-1) - B_*(X) - \alpha_*(X)[2(n-1)+r] - \beta_*(LX) \\
& - D_*(X)(n-1) - \gamma_*(X)[(n-1)+r] - (n-1)[\beta_*(X) + \gamma_*(X)] \\
& + \eta(X)[B_*(\xi) - \beta_*(\xi)\{2(n-1)+r\} + \beta_*(L\xi)]. \qquad (30)
\end{aligned}$$

Now, setting $X = \xi$ in (23) and using (24) we get

$$\begin{aligned}
0 \; = \; & -B_*(Y)(n-1) + B_*(Y) - \beta_*(Y)[2(n-1)+r] + (n-1)\beta_*(Y) + \beta_*(LY) \\
& + \eta(Y)[B_*(\xi) - \beta_*(\xi)\{2(n-1)+r\} + \beta_*(L\xi)]. \qquad (31)
\end{aligned}$$

Replacing Y by X in the above equation we get

$$\begin{aligned} 0 &= -B_*(X)(n-1) + B_*(X) - \beta_*(X)[2(n-1)+r] + (n-1)\beta_*(X) + \beta_*(LX) \\ &\quad +\eta(X)[B_*(\xi) - \beta_*(\xi)\{2(n-1)+r\} + \beta_*(L\xi)]. \end{aligned} \tag{32}$$

Substracting (32) from (30) we obtain

$$\begin{aligned} 0 &= (n-1)[A_*(X) - B_*(X) + D_*(X) + 2\alpha_*(X) + 2\gamma_*(X)] + 2B_*(X) \\ &\quad +r[\alpha_*(X) - \beta_*(X) + \gamma_*(X)] + 2\beta_*(LX). \end{aligned} \tag{33}$$

Theorem 4. *In Lorentzian para-sasakian manifold with hyper generalized weakly symmetric curvature condition, the sum of the* 1*-forms is given by (33).*

Corollary 5. *In a hyper recurrent Lorentzian Para-Sasakian manifold, the vector fields associated to the 1-form are of opposite direction.*

3.1. Codazzi type of Ricci tensor

In this section we enquire codazzi type of Ricci tensor. We recall that the Ricci tensor S is codazzi type if

$$(\nabla_X S)(Y,Z) - (\nabla_Z S)(Y,X) = 0. \tag{34}$$

In view of (16), (34) yields

$$\begin{aligned} 0 &= [A_*(X) - D_*(X) + 2\alpha_*(X) + \beta_*(X)]S(Y,Z) \\ &\quad -[A_*(Z) - D_*(Z) + 2\alpha_*(Z) + \beta_*(Z)]S(Y,X) \\ &\quad +[r\alpha_*(X) + \beta_*(LX) + 2\gamma_*(LX) - r\gamma_*(X)]g(Y,Z) \\ &\quad -[r\alpha_*(Z) + \beta_*(LZ) + 2\gamma_*(LZ) - r\gamma_*(Z)]S(Y,X) \\ &\quad +B_*(R(X,Y)Z) + 2D_*(R(X,Z)Y) - B_*(R(Z,Y)X) \end{aligned} \tag{35}$$

Now, setting $Z = \xi$ in (35), we obtain

$$\begin{aligned} 0 &= (n-1)[A_*(X) - D_*(X) + 2\alpha_*(X) + \beta_*(X)]\eta(Y) \\ &\quad -[A_*(\xi) - D_*(\xi) + 2\alpha_*(\xi) + \beta_*(\xi)]S(Y,X) \\ &\quad +[r\alpha_*(X) + \beta_*(LX) + 2\gamma_*(LX) - r\gamma_*(X)]\eta(Y) \\ &\quad -[r\alpha_*(\xi) + \beta_*(L\xi) + 2\gamma_*(L\xi) - r\gamma_*(\xi)]g(Y,X) \\ &\quad +\eta(Y)B_*(X) - \eta(X)B_*(Y) + 2\eta(Y)D_*(X) \\ &\quad -2[r\alpha_*(\xi) + \beta_*(L\xi) + 2\gamma_*(L\xi) - r\gamma_*(\xi)]g(Y,X). \end{aligned} \tag{36}$$

Next, seting $X = \xi$, $Y = \xi$ and $X = Y = \xi$ successively in (36), we obtain

$$B_*(Y) = B_*(\xi)\eta(Y) \tag{37}$$

and

$$\begin{aligned} & (n-1)A_*(X) - (n-3)D_*(X) + (n-1)[2\alpha_*(X) + \beta_*(X)] \\ & +r\alpha_*(X) + \beta_*(LX) + 2\gamma_*(LX) - r\gamma_*(X) \\ = \; & [-2B_*(\xi) + (n-3)D_*(\xi) - (n-1)\{A_*(\xi) + 2\alpha_*(\xi) + \beta_*(\xi)\} \\ & -\{r\alpha_*(\xi) + \beta_*(L\xi) + 2\gamma_*(L\xi) - r\gamma_*(\xi)\}]\eta(X) \end{aligned} \tag{38}$$

and

$$B_*(\xi) = 0 \tag{39}$$

respectively.
Using (37), (38) and (39) in (36), we get

$$\begin{aligned} & [A_*(\xi) - D_*(\xi) + 2\alpha_*(\xi) + \beta_*(\xi)]S(Y, X) \\ = \; & [-2D_*(\xi) - \{r\alpha_*(\xi) + \beta_*(L\xi) + 2\gamma_*(L\xi) - r\gamma_*(\xi)\}]g(Y, X) \\ & +[(n-3)D_*(\xi) - (n-1)\{A_*(\xi) - D_*(\xi) + 2\alpha_*(\xi) + \beta_*(\xi)\} \\ & -\{r\alpha_*(\xi) + \beta_*(L\xi) + 2\gamma_*(L\xi) - r\gamma_*(\xi)\}]\eta(Y)\eta(X). \end{aligned} \tag{40}$$

This leads to the following:

Theorem 6. *A hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold is an η-Einstein if it admites Codazzi type Ricci tensor provided $A_*(\xi) + 2\alpha_*(\xi) + \beta_*(\xi) \neq D_*(\xi)$.*

3.2. Recurrent Ricci tensor

If we assume that the hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold admits recurrent Ricci tensor then

$$(\nabla_X S)(Y, Z) = \lambda(X)S(Y, Z). \tag{41}$$

Now with the help of (16), (41) yields

$$\begin{aligned} & \lambda(X)S(Y, Z) \\ = \; & A_*(X)S(Y, Z) + B_*(Y)S(X, Z) + D_*(Z)S(X, Y) + B_*(R(X, Y)Z) + D_*(R(X, Z)Y) \\ & +\alpha_*(X)[2S(Y, Z) + rg(Y, Z)] + \beta_*(Y)[2S(X, Z) + rg(X, Z)] + \gamma_*(Z)[2S(Y, X) + rg(Y, X)] \\ & +\beta_*(LX)g(Y, Z) + \beta_*(X)S(Y, Z) - \beta_*(Y)S(X, Z) - \beta_*(LY)g(Z, X) \\ & +\gamma_*(LX)g(Y, Z) + \gamma_*(X)S(Y, Z) - \gamma_*(LZ)g(Y, X) - \gamma_*(Z)S(X, Y). \end{aligned} \tag{42}$$

Letting $Z = \xi$ in (42) we get

$$\begin{aligned} & \lambda(X)(n-1)\eta(Y) \\ = \; & A_*(X)(n-1)\eta(Y) + B_*(Y)(n-1)\eta(X) + D_*(\xi)S(X,Y) + B_*(X)\eta(Y) - B_*(Y)\eta(X) \\ & + D_*(X)\eta(Y) - g(X,Y)D_*(\xi) + \alpha_*(X)\eta(Y)[2(n-1)+r] + \beta_*(Y)\eta(X)[2(n-1)+r] \\ & + \gamma_*(\xi)[2S(Y,X) + rg(Y,X)] + \beta_*(LX)\eta(Y) + (n-1)[\beta_*(X)\eta(Y) - \beta_*(Y)\eta(X)] \\ & - \beta_*(LY)\eta(X) + \gamma_*(LX)\eta(Y) + (n-1)\gamma_*(X)\eta(Y) \\ & - \gamma_*(L\xi)g(X,Y) - \gamma_*(\xi)S(X,Y). \end{aligned} \tag{43}$$

Now considering $X = Y = \xi$, $X = \xi$ and $Y = \xi$ in (43) successively, we have

$$\begin{aligned} 0 \; = \; & (n+1)D_*(\xi) + 2(n-1)\gamma_*(\xi) + r\gamma_*(\xi) + (n-1)A_*(\xi) - (n-1)\lambda(\xi) \\ & + (n-1)B_*(\xi) + \{r + 2(n-1)\}\{\alpha_*(\xi) + \beta_*(\xi)\} \end{aligned} \tag{44}$$

and

$$\begin{aligned} & (n-2)B_*(Y) + \{r + (n-1)\}\beta_*(Y) - \beta_*(LY) \\ = \; & [(n-1)D_*(\xi) + 2(n-1)\gamma_*(\xi) + r\gamma_*(\xi) + (n-1)A_*(\xi) - (n-1)\lambda(\xi) \\ & + B_*(\xi) + \{r + 2(n-1)\}\alpha_*(\xi) + (n-1)\beta_*(\xi) + \beta_*(L\xi)]\eta(Y) \end{aligned} \tag{45}$$

and

$$\begin{aligned} & A_*(X)(n-1) - \lambda(X)(n-1) + B_*(X) + D_*(X) + \alpha_*(X)\{2(n-1) + r\} \\ & + \beta_*(LX) + (n-1)\beta_*(X) + \gamma_*(LX) + (n-1)\gamma_*(X) \\ = \; & [(n-2)D_*(\xi) + (n-1)\gamma_*(\xi) + r\gamma_*(\xi) - \gamma_*(L\xi) + (n-1)A_*(\xi) \\ & + (n-2)B_*(\xi) + \{r + 2(n-1)\}\beta_*(\xi) - \beta_*(L\xi)]\eta(X) \end{aligned} \tag{46}$$

respectively.
Using (44), (45) and (46) in (43) we obtain

$$\begin{aligned} & [D_*(\xi) + \gamma_*(\xi)]S(X,Y) + [(n-4)D_*(\xi) + (n-1)\beta_*(\xi) - \gamma_*(L\xi) + (n-1)\gamma_*(\xi) + r\gamma_*(\xi)]\eta(X)\eta(Y) \\ = \; & [D_*(\xi) - r\gamma_*(\xi) + \gamma_*(L\xi)]g(X,Y). \end{aligned} \tag{47}$$

This leads to the following:

Theorem 7. *If the Ricci tensor of a hyper generalized weakly symmetric LP-Sasakian manifold is recurrent then the manifold is an η-Einstein provided* $D_*(\xi) \neq -\gamma_*(\xi)$.

4. Example of hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold

Example 1. *(see [7], p-60) Let $M^3(\phi,\xi,\eta,g)$ be an LP-Sasakian manifold (M^3,g) with a ϕ-basis*

$$e = e^z \frac{\partial}{\partial x},\ \phi e = e^{z-\alpha x}\frac{\partial}{\partial x},\ \xi = \frac{\partial}{\partial x},\ \text{where } \alpha \text{ is non-zero constant.}$$

Using Koszul's formula for Lorentzian metric g, we can calculate the Levi-Civita connection as follows

$$\begin{array}{rclccc} \nabla_e\xi &=& \phi e, & \nabla_e\phi e = 0, & \nabla_e e = -\xi, \\ \nabla_{\phi e}\xi &=& e, & \nabla_{\phi e}\phi e = \alpha e^z e, & \nabla_{\phi e}e = \alpha e^z\phi e, \\ \nabla_\xi\xi &=& 0, & \nabla_\xi\phi e = 0, & \nabla_\xi e = 0. \end{array}$$

In view of the above relations, one can easily find out the non-vanishing components of the curvature tensor R, Ricci tensor S, scalar curvature r and generalized quasi-conformal curvature tensor W as follows

$$\begin{array}{lll} R(\phi e,\xi)\xi = -\phi e, & R(e,\xi)\xi = -e, & R(e,\phi e)\phi e = (1-\alpha^2 e^{2z})e, \\ R(\phi e,\xi)\phi e = -\xi, & R(e,\xi)e = -\xi, & R(e,\phi e)e = -(1-\alpha^2 e^{2z})\phi e, \\ S(e,e) = -\alpha^2 e^{2z}, & S(\phi e,\phi e) = -\alpha e^z, & S(\xi,\xi) = -2,\ r = 2(1-\alpha^2 e^{2z}), \end{array}$$

and the components which can be obtained from these by the symmetry properties. Since $\{e, \phi e, \xi\}$ *is a basis, any vector field* $X,\ Y,\ U,\ V \in \chi(M)$ *can be written as*

$$\begin{aligned}
X &= a_1 e + a_2\phi e + a_3\xi,\ Y = b_1 e + b_2\phi e + b_3\xi, \\
U &= c_1 e + c_2\phi e + cz_3\xi, \quad V = d_1 e + d_2\phi e + d_3\xi, \\
\bar{R}(X,Y,U,V) &= (a_1b_2 \ - a_2b_1)(c_1d_2 \ - c_2d_1)(1-\alpha^2 e^{2z}) + (a_1b_3 \ - a_3b_1)(c_1d_3 \\
&\quad - c_3d_1) + (a_2b_3 \ - a_3b_2)(c_2d_3 \ - c_3d_2) \\
&= H_1 \text{ (say)} \\
\bar{R}(e,Y,U,V) &= b_2\ (c_1d_2 \ - c_2d_1)(1-\alpha^2 e^{2z}) + b_3(c_1d_3 \ - c_3d_1) = \Gamma_1 \text{ (say)}, \\
\bar{R}(\phi e,Y,U,V) &= b_3(c_2d_3 \ - c_3d_2) - b_1\ (c_1d_2 \ - c_2d_1)(1-\alpha^2 e^{2z}) = \Gamma_2 \text{ (say)}, \\
\bar{R}(\xi,Y,U,V) &= -b_1(c_1d_3 - c_3d_{1-}) - b_2\ (c_2d_3 - c_3d_2) = \Gamma_3 \text{ (say)}, \\
\bar{R}(X,e,U,V) &= -a_2\ (c_1d_2 \ - c_2d_1)(1-\alpha^2 e^{2z}) - a_3(c_1d_3 \ - c_3d_1) = \Gamma_4 \text{ (say)}, \\
\bar{R}(X,\phi e,U,V) &= a_1\ (c_1d_2 - c_2d_1)(1-\alpha^2 e^{2z}) - a_3(c_2d_3 \ - c_3d_2) = \Gamma_5 \text{ (say)}, \\
\bar{R}(X,\xi,U,V) &= a_1(c_1d_3 - c_3d_1\ ) + a_2\ (c_2d_3 \ - c_3d_2) = \Gamma_6 \text{ (say)}, \\
\bar{R}(X,Y,e,V) &= d_2\ (a_1b_2 - a_2b_1\ )(1-\alpha^2 e^{2z}) + d_3(a_1b_3 \ - a_3b_1) = \Gamma_7 \text{ (say)}, \\
\bar{R}(X,Y,\phi e,V) &= d_3(a_2b_3 \ - a_3b_2) - d_1\ (a_1b_2 \ - a_2b_1)(1-\alpha^2 e^{2z}) = \Gamma_8 \text{ (say)}, \\
\bar{R}(X,Y,\xi,V) &= -d_1(a_1b_3 - a_3b_1) - d_2\ (a_2b_3 - a_3b_2) = \Gamma_9 \text{ (say)}, \\
\bar{R}(X,Y,U,e) &= -c_3(a_1b_3 \ - a_3b_1) - c_2\ (a_1b_2 - a_2b_1\ )(1-\alpha^2 e^{2z}) = \Gamma_{10} \text{ (say)}, \\
\bar{R}(X,Y,U,\phi e) &= -c_3(a_2b_3 - a_3b_2\ ) + c_1\ (a_1b_2 - a_2b_1)(1-\alpha^2 e^{2z}) = \Gamma_{11} \text{ (say)}, \\
\bar{R}(X,Y,U,\xi) &= c_1(a_1b_3 - a_3b_1) + c_2\ (a_2b_3 - a_3b_2\ ) = \Gamma_{12}\ \Omega_9 \text{(say)}
\end{aligned}$$

Also

$$\begin{aligned}
g(X,U) &= (a_1c_1 + a_2c_2 - a_3c_3), g(X,V) = (a_1d_1 \ + a_2d_2 - a_3d_3), \\
g(Y,U) &= (b_1c_1 + b_2c_2 - b_3c_3), g(Y,V) = (b_1d_1 + b_2d_2 - b_3d_3), \\
S(X,U) &= 2a_3c_3 - \alpha^2 e^{2z}(a_1c_1 + a_2c_2) \\
S(X,V) &= 2a_3d_3 - \alpha^2 e^{2z}(a_1d_1 \ + a_2d_2) \\
S(Y,U) &= 2b_3c_3 - \alpha^2 e^{2z}(b_1c_1 + b_2c_2) \\
S(Y,V) &= 2b_3d_3 - \alpha^2 e^{2z}(b_1d_1 \ + b_2d_2),
\end{aligned}$$

and

$$
\begin{aligned}
& (g \wedge S)(X,Y,U,V) \\
= \;& g(X,V)S(Y,U) + g(Y,U)S(X,V) - g(X,U)S(Y,V) - g(Y,V)S(X,U) \\
= \;& (a_1d_1 + a_2d_2 - a_3d_3)\{2b_3c_3 - \alpha^2 e^{2z}(b_1c_1 + b_2c_2)\} \\
& +(b_1c_1 + b_2c_2 - b_3c_3)\{2a_3d_3 - \alpha^2 e^{2z}(a_1d_1 + a_2d_2)\} \\
& -(a_1c_1 + a_2c_2 - a_3c_3)\{2b_3d_3 - \alpha^2 e^{2z}(b_1d_1 + b_2d_2)\} \\
& -(b_1d_1 + b_2d_2 - b_3d_3)\{2a_3c_3 - \alpha^2 e^{2z}(a_1c_1 + a_2c_2)\} \\
= \;& H_2 \ (say),
\end{aligned}
$$

$$
\begin{aligned}
& (g \wedge S)(e, Y, U, V) \\
= \;& g(e, V)S(Y, U) + g(Y, U)S(e, V) - g(e, U)S(Y, V) - g(Y, V)S(e, U) \\
= \;& 2\alpha^2 e^{2z}(c_1 d_2 - c_2 d_1)b_2 - (2 + \alpha^2 e^{2z})(c_1 d_3 - c_3 d_1)b_3 = \Omega_1(say), \\
& (g \wedge S)(\phi e, Y, U, V) \\
= \;& g(\phi e, V)S(Y, U) + g(Y, U)S(\phi e, V) - g(\phi e, U)S(Y, V) - g(Y, V)S(\phi e, U) \\
= \;& -2\alpha^2 e^{2z}(c_1 d_2 - c_2 d_1)b_1 - (2 + \alpha^2 e^{2z})(c_2 d_3 - c_3 d_2)b_3 = \Omega_2(say), \\
& (g \wedge S)(\xi, Y, U, V) \\
= \;& g(\xi, V)S(Y, U) + g(Y, U)S(\xi, V) - g(\xi, U)S(Y, V) - g(Y, V)S(\xi, U) \\
= \;& -\alpha^2 e^{2z}[d_3(b_1 c_1 + b_2 c_2) + (b_1 d_1 + b_2 d_2)c_3] - 2[c_1(b_2 d_2 - b_3 d_3) - d_1(b_2 c_2 - b_3 c_3)] = \Omega_3(say), \\
& (g \wedge S)(X, e, U, V) \\
= \;& g(X, V)S(e, U) + g(e, U)S(X, V) - g(X, U)S(e, V) - g(e, V)S(X, U) \\
= \;& (2 + \alpha^2 e^{2z})(c_1 d_3 - c_3 d_1)a_3 - 2\alpha^2 e^{2z}(c_1 d_2 - c_2 d_1)a_2 = \Omega_4(say), \\
& (g \wedge S)(X, \phi e, U, V) \\
= \;& g(X, V)S(\phi e, U) + g(\phi e, U)S(X, V) - g(X, U)S(\phi e, V) - g(\phi e, V)S(X, U) \\
= \;& (2 + \alpha^2 e^{2z})(c_2 d_3 - c_3 d_2)a_3 + 2\alpha^2 e^{2z}(c_1 d_2 - c_2 d_1)a_1 = \Omega_5(say), \\
& (g \wedge S)(X, \xi, U, V) \\
= \;& g(X, V)S(\xi, U) + g(\xi, U)S(X, V) - g(X, U)S(\xi, V) - g(\xi, V)S(X, U) \\
= \;& \alpha^2 e^{2z}[d_3(a_1 c_1 + a_2 c_2) + (a_1 d_1 + a_2 d_2)c_3] + 2[c_1(a_2 d_2 - a_3 d_3) - d_1(a_2 c_2 - a_3 c_3)] = \Omega_6(say), \\
& (g \wedge S)(X, Y, e, V) \\
= \;& g(X, V)S(Y, e) + g(Y, e)S(X, V) - g(X, e)S(Y, V) - g(Y, V)S(X, e) \\
= \;& 2\alpha^2 e^{2z}(a_1 b_2 - a_2 b_1)d_2 - (2 + \alpha^2 e^{2z})(a_1 b_3 - a_3 b_1)d_3 = \Omega_7(say), \\
& (g \wedge S)(X, Y, \phi e, V) \\
= \;& g(X, V)S(Y, \phi e) + g(Y, \phi e)S(X, V) - g(X, \phi e)S(Y, V) - g(Y, V)S(X, \phi e) \\
= \;& -2\alpha^2 e^{2z}(a_1 b_2 - a_2 b_1)d_1 - (2 + \alpha^2 e^{2z})(a_2 b_3 - a_3 b_2)d_3 = \Omega_8(say), \\
& (g \wedge S)(X, Y, \xi, V) \\
= \;& g(X, V)S(Y, \xi) + g(Y, \xi)S(X, V) - g(X, \xi)S(Y, V) - g(Y, V)S(X, \xi) \\
= \;& -\alpha^2 e^{2z}[d_3(b_1 a_1 + b_2 a_2) + (b_1 d_1 + b_2 d_2)a_3] - 2[a_1(b_2 d_2 - b_3 d_3) - d_1(b_2 a_2 - b_3 a_3)] = \Omega_9(say), \\
& (g \wedge S)(X, Y, U, e) \\
= \;& g(X, e)S(Y, U) + g(Y, U)S(X, e) - g(X, U)S(Y, e) - g(Y, e)S(X, U) \\
= \;& (2 + \alpha^2 e^{2z})(c_1 b_3 - c_3 b_1)a_3 - 2\alpha^2 e^{2z}(c_1 b_2 - c_2 b_1)a_2 = \Omega_{10}(say),
\end{aligned}
$$

$$\begin{aligned}
& (g \wedge S)(X, Y, U, \phi e) \\
= \;& g(X, \phi e) S(Y, U) + g(Y, U) S(X, \phi e) - g(X, U) S(Y, \phi e) - g(Y, \phi e) S(X, U) \\
= \;& (2 + \alpha^2 e^{2z})(c_2 b_3 - c_3 b_2) a_3 + 2\alpha^2 e^{2z}(c_1 b_2 - c_2 b_1) a_1 = \Omega_{11} \textit{(say)},
\end{aligned}$$

$$\begin{aligned}
& (g \wedge S)(X, Y, U, \xi) \\
= \;& g(X, \xi) S(Y, U) + g(Y, U) S(X, \xi) - g(X, U) S(Y, \xi) - g(Y, \xi) S(X, U) \\
= \;& \alpha^2 e^{2z}[b_3(a_1 c_1 + a_2 c_2) + (a_1 b_1 + a_2 b_2) c_3] \\
& +2[c_1(a_2 b_2 - a_3 b_3) - b_1(a_2 c_2 - a_3 c_3)] \\
= \;& \Omega_{12}(say),
\end{aligned}$$

and the components which can be obtained from these by the symmetry properties. The covariant derivatives of the curvature tensor are as follows

$$\begin{aligned}
(\nabla_{e_1} \bar{R})(X, Y, U, V) &= a_1 \Gamma_3 - a_3 \Gamma_2 + b_1 \Gamma_6 - b_3 \Gamma_5 \\
& \quad + c_1 \Gamma_9 - c_3 \Gamma_8 + d_1 \Gamma_{12} - d_3 \Gamma_{11} \\
&= H_3 \text{ (say)}, \\
(\nabla_{\phi e} \bar{R})(X, Y, U, V) &= -\alpha e^z (a_1 \Gamma_2 + a_2 \Gamma_1) - a_3 \Gamma_1 - \alpha e^z (b_1 \Gamma_5 + b_2 \Gamma_4) - b_3 \Gamma_4 \\
& \quad -\alpha e^z (c_1 \Gamma_8 + c_2 \Gamma_7) - c_3 \Gamma_7 - \alpha e^z (d_1 \Gamma_{11} + d_2 \Gamma_{10}) - d_3 \Gamma_{10} \\
&= H_4 \text{ (say)}, \\
(\nabla_\xi \bar{R})(X, Y, U, V) &= 0.
\end{aligned}$$

For the following choice of the 1-forms

$$\begin{aligned}
A_1(e) &= \frac{H_3}{H_1}, \\
A_2(e) &= \frac{1}{H_2}\left[\left(\frac{c_3 \Omega_7 + d_3 \Omega_{10}}{c_3 \Omega_8 + d_3 \Omega_{11}}\right) - \left(\frac{a_3 \Omega_1 + b_3 \Omega_4}{a_3 \Omega_2 + b_3 \Omega_5}\right)\right], \\
A_1(\phi e) &= \frac{1}{H_1}\left[\left(\frac{c_3 \Gamma_8 + d_3 \Gamma_{11}}{c_3 \Gamma_7 + d_3 \Gamma_{10}}\right) - \left(\frac{a_3 \Gamma_2 + b_3 \Gamma_5}{a_3 \Gamma_1 + b_3 \Gamma_4}\right)\right], \\
A_2(\phi e) &= \frac{H_4}{H_2}, \\
A_1(\xi) &= \frac{1}{H_1}\left[\left(\frac{c_3 \Gamma_9 + d_3 \Gamma_{12}}{c_3 \Gamma_7 + d_3 \Gamma_{10}}\right) - \left(\frac{a_3 \Gamma_3 + b_3 \Gamma_6}{a_3 \Gamma_1 + b_3 \Gamma_4}\right)\right], \\
A_2(\xi) &= \frac{1}{H_2}\left[\left(\frac{c_3 \Omega_9 + d_3 \Omega_{12}}{c_3 \Omega_8 + d_3 \Omega_{11}}\right) - \left(\frac{a_3 \Omega_3 + b_3 \Omega_6}{a_3 \Omega_2 + b_3 \Omega_5}\right)\right], \\
B_1(\xi) &= \frac{1}{(a_3 \Gamma_1 + b_3 \Gamma_4)}, \; B_2(\xi) = \frac{1}{(a_3 \Omega_2 + b_3 \Omega_5)}, \\
D_1(\xi) &= -\frac{1}{(c_3 \Gamma_7 + d_3 \Gamma_{10})}, \; D_2(\xi) = -\frac{1}{(c_3 \Omega_8 + d_3 \Omega_{11})},
\end{aligned}$$

one can easily verify the relations

$$
\begin{aligned}
(\nabla_e \bar{R})(X,Y,U,V) &= A_1(e)\bar{R}(X,Y,U,V)\\
&+B_1(X)\bar{R}(e,Y,U,V)+B_1(Y)\bar{R}(X,e,U,V)\\
&+D_1(U)\bar{R}(X,Y,e,V)+D_1(V)\bar{R}(X,Y,U,e)\\
&+\ A_2(e)\ (g\wedge S)(X,Y,U,V)\\
&+B_2(X)(g\wedge S)(e,Y,U,V)+B_2(Y)(g\wedge S)(X,e,U,V)\\
&+D_2(U)(g\wedge S)(X,Y,e,V)+D_2(V)(g\wedge S)(X,Y,U,e)\\
(\nabla_{\phi e} \bar{R})(X,Y,U,V) &= A_1(\phi e)\bar{R}(X,Y,U,V)+B_1(X)\bar{R}(\phi e,Y,U,V)+B_1(Y)\bar{R}(X,\phi e,U,V)\\
&+D_1(U)\bar{R}(X,Y,\phi e,V)+D_1(V)\bar{R}(X,Y,U,\phi e)+\ A_2(\phi e)\ (g\wedge S)(X,Y,U,V)\\
&+B_2(X)(g\wedge S)(\phi e,Y,U,V)+B_2(Y)(g\wedge S)(X,\phi e,U,V)\\
&+D_2(U)(g\wedge S)(X,Y,\phi e,V)+D_2(V)(g\wedge S)(X,Y,U,\phi e)\\
(\nabla_{\xi} \bar{R})(X,Y,U,V) &= A_1(\xi)\bar{R}(X,Y,U,V)+B_1(X)\bar{R}(\xi,Y,U,V)\\
&+B_1(Y)\bar{R}(X,\xi,U,V)+D_1(U)\bar{R}(X,Y,\xi,V)\\
&+D_1(V)\bar{R}(X,Y,U,\xi)+\ A_2(\xi)\ (g\wedge S)(X,Y,U,V)\\
&+B_2(X)(g\wedge S)(\xi,Y,U,V)+B_2(Y)(g\wedge S)(X,\xi,U,V)\\
&+D_2(U)(g\wedge S)(X,Y,\xi,V)+D_2(V)(g\wedge S)(X,Y,U,\xi)
\end{aligned}
$$

From the above, we can state that the manifold (M^3, g) under consideration is a hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold.

Theorem 8. *There exists a hyper generalized weakly symmetric Lorentzian Para-Sasakian manifold* (M^3, g) .

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