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Some results on Green's higher Abel-Jacobi map

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1. Introduction

This paper is devoted to the study of the first higher Abel-Jacobi invariant defined by M. Green in [4] for zero-cycles on a surface. Green's work is a very original attempt to understand, at least over $\mathbb{C}$, the graded pieces of the conjecturally defined filtration on Chow groups

$$\begin{aligned}\mathrm{CH}^p(X)_{\mathbb{Q}} &= F^0\mathrm{CH}^p(X)_{\mathbb{Q}} \supset F^1\mathrm{CH}^p(X)_{\mathbb{Q}}\\ &= \mathrm{CH}^p(X)_{\mathbb{Q}}^{\mathrm{hom}} \supset \ldots \supset F^{p+1}\mathrm{CH}^p(X)_{\mathbb{Q}} = 0.\end{aligned}$$

This filtration should satisfy the following properties:

i) First of all it should be stable under correspondences, so that a correspondence $\Gamma \subset X \times Y$ should induce

$$\Gamma_* : F^k\mathrm{CH}^p(X)_{\mathbb{Q}} \to F^k\mathrm{CH}^{p'}(Y)_{\mathbb{Q}},$$

where $p' = p + \dim Y - \dim \Gamma$, and

ii) the induced map

$$\mathrm{Gr}^k\Gamma_* : \mathrm{Gr}_F^k\mathrm{CH}^p(X)_{\mathbb{Q}} \to \mathrm{Gr}_F^k\mathrm{CH}^{p'}(Y)_{\mathbb{Q}}$$

should vanish when Γ is homologous to zero.

A filtration satisfying this last property has been constructed by Saito [11], but it is not shown that the filtration terminates, that is $F^{p+1}\mathrm{CH}^p(X)_{\mathbb{Q}} = 0$. A definition has also been proposed by J. P. Murre ([8]), under the assumption that a strong Künneth decomposition of the diagonal exists, but it is not proved to satisfy condition ii) above. In fact proving the existence of such a filtration would solve in particular Bloch's conjecture on zero-cycles of surfaces [1].

In any case, the first steps of the filtration are easy to understand, at least for zero-cycles. Namely one should have $F^2\mathrm{CH}_0(X) = \mathrm{CH}_0(X)_{\mathrm{alb}} = \mathrm{Ker}\,\mathrm{alb}$, where $\mathrm{alb} : \mathrm{CH}_0(X)_{\mathrm{hom}} \to \mathrm{Alb}(X)$ is the Albanese map. More generally for

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the subgroup $\mathrm{CH}^p(X)^{\mathrm{alg}}_{\mathbb{Q}} \subset \mathrm{CH}^p(X)_{\mathbb{Q}}$ of cycles algebraically equivalent to zero, one should have over $\mathbb{C}$, $F^2\mathrm{CH}^p(X)^{\mathrm{alg}}_{\mathbb{Q}} = \mathrm{Ker}\,\Phi^p_X$, where

$$\Phi^p_X : \mathrm{CH}^p(X)^{\mathrm{hom}}_{\mathbb{Q}} \to J^{2p-1}_X = H^{2p-1}(X,\mathbb{C})/F^pH^{2p-1}(X) \oplus H^{2p-1}(X,\mathbb{Z})$$

is the Abel-Jacobi map, defined by Griffiths [5].

In [4], M. Green suggested constructing directly (over $\mathbb{C}$), from Hodge theoretic considerations, higher Abel-Jacobi maps

$$\psi^p_m : F^m\mathrm{CH}^p(X) \to J^p_m(X),$$

so that $F^{m+1}\mathrm{CH}^p(X) = \mathrm{Ker}\,\psi^p_m$. Hence one should have an induced injective map

$$\psi^p_m : \mathrm{Gr}^m\mathrm{CH}^p(X) \to J^p_m(X).$$

In the case of zero cycles on a surface, he proposed an explicit construction of

$$\psi^2_2 : \mathrm{Gr}^2\mathrm{CH}^2(S) = \mathrm{CH}_0(S)_{\mathrm{alb}} \to J^2_2(S) \tag{1.1}$$

that we will review below. The purpose of this paper is to answer some questions raised in [4], concerning the behaviour of ψ^2_2. To simplify the notation, we will assume throughout that S is regular, but this assumption does not play any role in the arguments. Our first result is the following, which answers negatively conjecture 3.4 of [4]:

THEOREM 1. *The higher Abel-Jacobi map ψ^2_2 is not, in general, injective.*

The noninjectivity is proved here for an explicit example but the argument should allow us to prove, as we will explain, that ψ^2_2 is never injective for surfaces with $\mathrm{CH}_0(S)_{\mathrm{alb}} \neq 0$.

Our second result solves, in particular, conjecture 3.6 of [4]:

THEOREM 2. *The map ψ^2_2 is nontrivial modulo torsion (and has an infinite dimensional image), when $h^{2,0}(S) \neq 0$.*

As an intermediate step, we explain how Mumford's pull-back $Z^*(\omega)$ of a holomorphic two-form ω of a surface S on a variety W parametrizing 0-cycles $(Z_w)_{w\in W}$ of S (cf. [7]) can be computed when one has a family $\mathcal{C} \to W$ of curves of S parametrized by W, such that for each $w \in W$, the 0-cycle Z_w is supported on C_w. There are then two associated Abel-Jacobi invariants $e_{C_w,S}$ and f_{Z_w,C_w} to be defined below, which play a key role in the construction of $\psi^2_2(Z_w)$, and we show that $Z^*(\omega)$ can be computed from the wedge product $de \wedge df$. We then use this result to show that in fact $Z^*\omega$ depends only on the map $\psi^2_2 \circ Z_* : W \to J^2_2(S)$.

Thus our results show that the first new higher Abel-Jacobi map defined by Mark Green is not strong enough to capture the whole of $\mathrm{CH}_0(S)_{\mathrm{alb}}$ as it should conjecturally do, but that it is strong enough to determine Mumford's invariants, which were used to show that $\mathrm{CH}_0(S)_{\mathrm{alb}}$ is infinite dimensional, when $h^{2,0}(S) \neq 0$. The question of whether it is possible to refine it so as to get the desired injectivity of 1.1 is still open.

The paper is organized as follows: The result above concerning the pull-back of holomorphic two forms (Proposition 2) provides the contents of Section 3. Theorem 1 is proved in Section 2, and Theorem 2 is proved in Section 4.

We conclude this introduction with a brief description of ψ_2^2, which will serve also as an introduction for the notation used throughout the paper.

Let S be a regular surface, and let C be a smooth (not necessarily connected) curve; let $\psi : C \to S$ be a morphism generically one-to-one on its image. We can find an immersion $\phi : C \hookrightarrow \tilde{S}$, and a birational morphism $\tau : \tilde{S} \to S$ such that $\psi = \tau \circ \phi$.

Now let Z be a 0-cycle of C, of degree 0 on each component of C. We construct two Abel-Jacobi invariants $e_{C,S}$ and $f_{Z,C}$ as follows:

The mixed Hodge structure on $H^2(\tilde{S}, C)$ is given by the Hodge filtration $F^{\cdot}$ on $H^2(\tilde{S}, C)$, which fits in the exact sequence

$$0 \to H^1(C, \mathbb{Z}) \to H^2(\tilde{S}, C, \mathbb{Z}) \to \mathrm{Ker}(H^2(\tilde{S}, \mathbb{Z}) \to H^2(C, \mathbb{Z})) \to 0. \tag{1.2}$$

The filtration $F^{\cdot}$ restricts to the Hodge filtration on $H^1(C)$ and projects to the Hodge filtration on $\mathrm{Ker}(H^2(\tilde{S}, \mathbb{Z}) \to H^2(C, \mathbb{Z}))$.

Define $H^2(S, \mathbb{Z})_{\mathrm{tr}}$ as the quotient $H^2(S, \mathbb{Z})/NS(S)$. Then its dual $H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{}$ is the orthogonal of $NS(S)$ in $H^2(S, \mathbb{Z})$. There is an inclusion of Hodge structures

$$\tau^* : H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{} \hookrightarrow \mathrm{Ker}(H^2(\tilde{S}, \mathbb{Z}) \to H^2(C, \mathbb{Z})).$$

Restricting the extension (1.2) to $H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{}$, we get an exact sequence of mixed Hodge structures

$$0 \to H^1(C, \mathbb{Z}) \to H^2(\tilde{S}, C, \mathbb{Z})_{\mathrm{tr}} \to H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{} \to 0.$$

The extension class of this exact sequence is an element $e_{C,S}$ of the complex torus (cf. [3]),

$$\begin{aligned} J(C \times S)_{\mathrm{tr}} \quad := \quad & H^1(C, \mathbb{C}) \otimes H^2(S, \mathbb{C})_{\mathrm{tr}} / [F^2(H^1(C) \otimes H^2(S)_{\mathrm{tr}}) \\ & \oplus H^1(C, \mathbb{Z}) \otimes H^2(S, \mathbb{Z})_{\mathrm{tr}}]. \end{aligned}$$

It is not difficult to prove that it can be also computed as the natural projection of the Abel-Jacobi invariant of the one-cycle obtained from the graph of ψ (which is a one-cycle of $C \times S$) by adding vertical and horizontal one-cycles of $C \times S$ in order to get a homologically trivial one-cycle.

It is well-known that the inclusion

$$H^1(C, \mathbb{R}) \otimes H^2(S, \mathbb{R})_{\mathrm{tr}} \subset H^1(C, \mathbb{C}) \otimes H^2(S, \mathbb{C})_{\mathrm{tr}}$$

induces an isomorphism

$$H^1(C, \mathbb{R}) \otimes H^2(S, \mathbb{R})_{\mathrm{tr}} \cong H^1(C, \mathbb{C}) \otimes H^2(S, \mathbb{C})_{\mathrm{tr}} / F^2(H^1(C) \otimes H^2(S)_{\mathrm{tr}}),$$

and this allows us to identify $J(C\times S)_{\mathrm{tr}}$ to the real torus

$$H^1(C,\mathbb{Z})\otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}.$$

We will view $e_{C,S}$ as an element of this real torus.

Now the zero-cycle Z has an Abel-Jacobi invariant (Albanese image)

$$f_{Z,C}\in J(C)\cong H^1(C,\mathbb{C})/[F^1H^1(C)\oplus H^1(C,\mathbb{Z})].$$

Again, the inclusion $H^1(C,\mathbb{R})\subset H^1(C,\mathbb{C})$ induces an isomorphism $H^1(C,\mathbb{R})\cong H^1(C,\mathbb{C})/F^1H^1(C)$, which provides the identification

$$J(C)\cong H^1(C,\mathbb{Z})\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}.$$

We will view $f_{Z,C}$ as an element of the real torus on the right.

The pairing

$$H^1(C,\mathbb{Z})\otimes_{\mathbb{Z}} H^1(C,\mathbb{Z})\to\mathbb{Z}$$

allows us then to contract $e_{C,S}$ and $f_{Z,C}$ to an element

$$e_{C,S}\cdot f_{Z,C}\in\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}.$$

Defining now $U_2^2\subset\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}$ as the group generated by the elements $e_{C,S}\cdot f_{Z,C}$ defined above, for the triples (C,Z,ψ) such that $\psi_*(Z)=0$ as a zero-cycle of S, it is clear that the projection

$$\overline{e_{C,S}\cdot f_{Z,C}}\in J_2^2(S):=\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{R}/\mathbb{Z}\otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}/U_2^2$$

depends only on the zero-cycle $\psi_*(Z)$. The resulting map

$$\psi_2^2: Z_0(S)_{\mathrm{hom}}\to J_2^2(S)$$

is then easily seen to factor through rational equivalence, so that ψ_2^2 is actually defined on $\mathrm{CH}_0^0(S)$.

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2. The noninjectivity of ψ_2^2

In this section we construct a counterexample to the conjectured injectivity of the map

$$\psi_2^2:\mathrm{CH}_0(S)_{\mathrm{alb}}\to J_2^2(S).$$

The counterexample is based on a refinement of the following argument.

First of all, if $\Gamma\subset C\times S$ is a correspondence homologous to zero, with Abel-Jacobi invariant

$$e_\Gamma\in J(C\times S)_{\mathrm{tr}},$$

we show that

$$\psi_2^2 \circ \Gamma_* : \mathrm{CH}_0(C)_{\mathrm{hom}} \to J_2^2(S)$$

is given by

$$\psi_2^2 \circ \Gamma_*(z) = e_\Gamma \cdot f_z \bmod U_2^2,$$

where $f_z = \mathrm{alb}(z) \in J(C) = H^1(C, \mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$. Now we view e_Γ as an element of $\mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{,} H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z})$ and we note that if its image is contained in a proper real subtorus T of $H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$, there is a nontrivial real subtorus $T^\perp$ of $H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$ such that, if $f_z \in T^\perp$,

$$e_\Gamma \cdot f_z = 0 \ \text{ in } \ H^2(S,\mathbb{Z})_{\mathrm{tr}} \otimes \mathbb{R}/\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z}.$$

Then the injectivity of ψ_2^2 would imply that $T^\perp \subset \operatorname{Ker} \Gamma_*$, and if $J(C)$ is simple, this would imply that $\Gamma_* = 0$, and then e_Γ would be a torsion point in $J(C \times S)_{\mathrm{tr}}$. So it suffices to find C, Γ as above with e_Γ not of torsion (or $\Gamma_* \neq 0$), $J(C)$ simple and $\operatorname{Im} e_\Gamma$ contained in a proper real subtorus T of $H^1(C,\mathbb{Z}) \otimes \mathbb{R}/\mathbb{Z}$ to contradict the injectivity of ψ_2^2.

We start with the simple Lemma 1 below which allows us to extend slightly the definition of ψ_2^2. Let S be a regular surface, C be a smooth curve and $\Gamma \in \mathrm{CH}_1(C \times S)$ be a one-cycle; the homology class of Γ lies in $H_2(C) \oplus H_2(S)_{\mathrm{alg}}$, so that adding to Γ vertical and horizontal cycles we can get a cycle Γ' homologous to zero: Then the induced morphisms

$$\Gamma_* : \mathrm{CH}_0^0(C) \to \mathrm{CH}_0^0(S),\ \Gamma'_* : \mathrm{CH}_0^0(C) \to \mathrm{CH}_0^0(S)$$

coincide, and the Abel-Jacobi image of Γ' in $J(C \times S)_{\mathrm{tr}}$ (see Section 1) does not depend on the choice of Γ'. We will denote it by e_Γ. As in Section 1, we can view e_Γ as an element of the real torus

$$H^1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

By contraction and use of the intersection pairing on $H^1(C,\mathbb{Z})$, e_Γ gives a map

$$[e_\Gamma] : JC \cong H^1(C,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}.$$

We have now:

LEMMA 1. *For $z \in JC$, $\psi_2^2(\Gamma_*(z)) \in J_2^2(S)$ is equal to the projection of $[e_\Gamma](z)$ modulo $U_2^2(S)$, using the definition*

$$J_2^2(S) := \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}/U_2^2$$

of Section 1.

Proof. This is true by definition if Γ is the graph Γ_ϕ of a morphism ϕ from C to S, generically one-to-one on its image. Now let $C_1 \xrightarrow{\phi} S$ be the desingularization of the inclusion of $\mathrm{pr}_2(\mathrm{Supp}\,\Gamma)$ in S. Then Γ lifts to a one-cycle $\Gamma_1 \in C \times C_1$, such that

$$\Gamma_* = \phi_* \circ \Gamma_{1*} : JC \to \mathrm{CH}_0^0(S).$$

We have then

$$\begin{array}{rcl} \psi_2^2(\Gamma_*(z)) & = & \psi_2^2(\phi_*(\Gamma_{1*}(z))) \\ & = & \text{projection of } \ [e_{\Gamma_\phi}](\Gamma_{1*}(z)) \ in \ J_2^2(S). \end{array}$$

Now it suffices to prove that

$$[e_\Gamma] = [e_{\Gamma_\phi}] \circ \Gamma_{1*} : J(C) \to \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}.$$

But Γ_1 induces naturally a correspondence $\tilde{\Gamma}_1$ between $C \times S$ and $C_1 \times S$; hence a morphism

$$\tilde{\Gamma}_1^* : \mathrm{CH}_1(C_1 \times S) \to \mathrm{CH}_1(C \times S),$$

such that $\Gamma \equiv_{\mathrm{rat}} \tilde{\Gamma}_1^*(\Gamma_\phi)$. It follows that $e_\Gamma = \tilde{\Gamma}_1^*(e_{\Gamma_\phi})$ in $J(C \times S)_{\mathrm{tr}}$, where $\tilde{\Gamma}_1^*$ also denotes the induced morphism between the intermediate jacobians $J(C_1 \times S)_{\mathrm{tr}}$ and $J(C \times S)_{\mathrm{tr}}$.

Let $\Gamma_{1\mathbb{Z}}^* : H^1(C_1,\mathbb{Z}) \to H^1(C,\mathbb{Z})$ be the morphism of Hodge structures induced by the cohomology class of Γ_1 in $C \times C_1$; then the morphism $\tilde{\Gamma}_1^*$ is induced by the morphism of Hodge structures

$$\Gamma_{1\mathbb{Z}}^* \otimes \mathrm{Id} : H^1(C_1,\mathbb{Z}) \otimes H^2(S,\mathbb{Z})_{\mathrm{tr}} \to H^1(C,\mathbb{Z}) \otimes H^2(S,\mathbb{Z})_{\mathrm{tr}},$$

and it follows that we have a commutative diagram

$$\begin{array}{lccc} \Gamma_{1\mathbb{R}}^* \otimes \mathrm{Id} & : H^1(C_1,\mathbb{R}) \otimes H^2(S,\mathbb{R})_{\mathrm{tr}} & \to & H^1(C,\mathbb{R}) \otimes H^2(S,\mathbb{R})_{\mathrm{tr}} \\ & \downarrow & & \downarrow \\ \tilde{\Gamma}_{1\mathbb{C}}^* \bmod F^2 & : H^1(C_1,\mathbb{C}) \otimes H^2(S,\mathbb{C})_{\mathrm{tr}}/F^2 & \to & H^1(C,\mathbb{C}) \otimes H^2(S,\mathbb{C})_{\mathrm{tr}}/F^2, \end{array}$$

where the vertical arrows are the identifications already used between real cohomology and complex cohomology mod F^2, and the last horizontal map induces

$$\tilde{\Gamma}_1^* : J(C_1 \times S)_{\mathrm{tr}} \to J(C \times S)_{\mathrm{tr}}$$

by passing to the quotient modulo integral cohomology. This means that, viewed as elements of $J(C_1) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}$ and $J(C) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}$ respectively, e_{Γ_ϕ} and e_Γ satisfy the relation

$$e_\Gamma = \Gamma_1^* \otimes \mathrm{Id}(e_{\Gamma_\phi}).$$

Now it suffices to note that the contraction maps

$$\begin{array}{lll} \langle\,,\rangle_{C_1} : & J(C_1) \otimes_{\mathbb{Z}} J(C_1) & \to \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}, \\ \langle\,,\rangle_{C} : & J(C) \otimes_{\mathbb{Z}} J(C) & \to \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \end{array}$$

satisfy the relation

$$\langle \Gamma_{1*}(z), w\rangle_{C_1} \;=\; \langle z, \Gamma_1^*(w)\rangle_C,\; z \in J(C),\; w \in J(C_1),$$

to get

$$\begin{aligned}[e_{\Gamma_\phi}](\Gamma_{1*}(z)) &= \langle \Gamma_{1*}(z), e_{\Gamma_\phi}\rangle_{C_1} \\ &= \langle z, \Gamma_1^* \otimes \mathrm{Id}(e_{\Gamma_\phi})\rangle_C = \langle z, e_\Gamma\rangle_C = [e_\Gamma](z),\end{aligned}$$

as desired. □

The following Lemma 2 is quite standard (cf. [10]); let $\Gamma \in \mathrm{CH}_1(C \times S)$ be a correspondence, and let

$$\Gamma_* : J(C) \to \mathrm{CH}_0^0(S)$$

be the induced morphism; we have:

Lemma 2. $\operatorname{Ker}\Gamma_*$ *is a countable union of translates of an abelian subvariety of* $J(C)$.

Proof. $\operatorname{Ker}\Gamma_*$ is a subgroup of $J(C)$, and is a countable union of algebraic subsets of $J(C)$. The union of the irreducible algebraic subsets of $J(C)$ passing through 0 and contained in $\operatorname{Ker}\Gamma_*$ is stable under difference which implies that it can be written as an increasing union of irreducible algebraic subsets of $J(C)$. So it must be in fact an algebraic subset of $J(C)$, stable under difference, that is an abelian subvariety of $J(C)$. Hence the result. □

Now assume some real subtorus T of $J(C) = H^1(C,\mathbb{R})/H^1(C,\mathbb{Z})$ is contained in $\operatorname{Ker}\Gamma_*$; then if $A \subset J(C)$ is the maximal abelian subvariety contained in $\operatorname{Ker}\Gamma_*$, so that by Lemma 2, $\operatorname{Ker}\Gamma_* = \bigcup_{m\in\mathbb{Z}} A + t_m$ for some $t_m \in J(C)$, then

$$T = \bigcup_{m\in\mathbb{Z}} T \cap (A + t_m).$$

It follows that some $T \cap (A + t_m)$ must contain an open set of T, and this implies easily that in fact T is contained in A. So we have proved:

Lemma 3. *Let T be a real subtorus of $J(C)$ contained in* $\operatorname{Ker}\Gamma_*$; *then there is an abelian subvariety A of $J(C)$ such that $T \subset A \subset$* $\operatorname{Ker}\Gamma_*$. *In particular, if T is nontrivial and $T \subset B$ where B is a simple abelian subvariety of $J(C)$ (i.e. there is no proper nontrivial abelian subvariety of B), then $B \subset$* $\operatorname{Ker}\Gamma_*$.

We want to apply these observations to show the noninjectivity of the higher Abel-Jacobi map $\psi_2^2 : \mathrm{CH}_0^0(S) \to J_2^2(S)$. Let C be a curve, and $\Gamma \in \mathrm{CH}_1(C \times S)$ be a correspondence. Let $e_\Gamma \in J(C) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}}$ be the corresponding Abel-Jacobi invariant. We can view e_Γ as an element

$$[e_\Gamma]^* \in \mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, J(C)).$$

Assume there is a proper real subtorus T of $J(C)$ containing $\mathrm{Im}\,[e_\Gamma]^*$; i.e. there is a proper sublattice $T_{\mathbb{Z}}$ of $H^1(C,\mathbb{Z})$ such that $\mathrm{Im}\,[e_\Gamma]^* \subset T_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$. Then by definition of $[e_\Gamma]$

$$T^\perp := T_{\mathbb{Z}}^\perp \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \subset \mathrm{Ker}\,[e_\Gamma].$$

Similarly, if $B \overset{i_B}{\hookrightarrow} J(C)$ is an abelian subvariety, and $\check{B}$ is the corresponding quotient of $J(C)$, let $J(C\times S)^B_{\mathrm{tr}}$ be the induced quotient of $J(C\times S)_{\mathrm{tr}}$; that is, writing $\check{B} = \check{B}_{\mathbb{C}}/\check{B}^{1,0} \oplus \check{B}_{\mathbb{Z}}$, then

$$J(C\times S)^B_{\mathrm{tr}} := \check{B}_{\mathbb{C}} \otimes H^2(S,\mathbb{C})_{\mathrm{tr}}/F^2(\check{B}_{\mathbb{C}} \otimes H^2(S,\mathbb{C})_{\mathrm{tr}}) \oplus \check{B}_{\mathbb{Z}} \otimes H^2(S,\mathbb{Z})_{\mathrm{tr}}.$$

Let

$$[e_\Gamma]^*_B \in \mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, \check{B})$$

be the composition of $[e_\Gamma]^*$ with the projection $J(C) \to \check{B}$. Let $e^B_\Gamma \in J(C\times S)^B_{\mathrm{tr}}$ be the projection of e_Γ. Note that $[e_\Gamma]^*_B$ is simply e^B_Γ viewed as an element of $\mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, \check{B})$ using the real representations of the (intermediate) jacobians

$$J(C\times S)^B_{\mathrm{tr}} \cong \mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, \check{B}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}).$$

If $\mathrm{Im}\,[e_\Gamma]^*_B$ is contained in a proper real subtorus T of $\check{B}$, the orthogonal torus $T^\perp \subset B$ is contained in $\mathrm{Ker}\,[e_\Gamma]_{|B}$.

In this situation, assume now that ψ_2^2 is injective and that B is simple: then by Lemma 1, one finds that Γ_* vanishes on $T^\perp \subset B$, and by Lemma 3, one concludes that Γ_* vanishes on B. Now this implies:

Proposition 1. *Under the above assumptions, the projection e^B_Γ of e_Γ in $J(C\times S)^B_{\mathrm{tr}}$ is in fact a torsion point.*

This follows from $\Gamma_{*|B} = 0$ and from the following lemma (cf. [4], [2]) applied to the correspondence $\Gamma \circ \pi_B$ where π_B is a multiple of a projector from $J(C)$ to B:

Lemma 4. *Let $\Gamma \in \mathrm{CH}_1(C\times S)$ be a correspondence such that the corresponding map $\Gamma_* : J(C) \to \mathrm{CH}_0^0(S)$ is zero; then the Abel-Jacobi invariant e_Γ is a torsion point of $J(C\times S)_{\mathrm{tr}}$.*

In order to contradict the injectivity of ψ_2^2 it suffices then to find a smooth curve C, a simple abelian subvariety B of $J(C)$ and a correspondence $\Gamma \in \mathrm{CH}_1(C\times S)$ satisfying the following properties:

– The projection e^B_Γ of the Abel-Jacobi invariant $e_\Gamma \in J(C\times S)_{\mathrm{tr}}$ in $J(C\times S)^B_{\mathrm{tr}}$ is not a torsion point.

– The image of the map

$$[e_\Gamma]^*_B : H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{} \to \check{B}$$

is contained in a proper real subtorus of $\check{B}$.

To get an explicit example, we use a construction due to Paranjape ([9]). Consider a $K3$ surface S which is the desingularization of a general double cover of $\mathbb{P}^2$ branched along the union of six lines. Then $\operatorname{rk} NS(S) = 16$, hence $b_2(S)_{\mathrm{tr}} = 6$. Paranjape constructs a genus 5 curve C, which is a ramified cover of an elliptic curve E, with an automorphism j of order 4, acting on $B := (\operatorname{Ker} Nm : J(C) \to J(E))^0$, a four dimensional abelian variety. The $K3$ surface S is birational to a quotient of $C\times C$ by a finite group. Let $r : C\times C \to S$ be the quotient (rational) map; for generic $c \in C$, r is everywhere defined along $c \times C$ and we get a family of one-cycles of $C \times S$ parametrized by C,

$$c \in C_{\mathrm{gen}} \mapsto \Gamma_c := \text{ graph of } r_{|c\times C} \subset C \times S.$$

This family induces an Abel-Jacobi map

$$\Gamma_* : J(C) \to J(C \times S).$$

Using the projection

$$J(C \times S) \to J(C \times S)_{\mathrm{tr}} \to J(C \times S)^B_{\mathrm{tr}}$$

and restricting the map Γ_* to $B \subset J(C)$, we get a morphism (of complex tori)

$$\Gamma^B_* : B \to J(C \times S)^B_{\mathrm{tr}}.$$

This morphism corresponds to a morphism of Hodge structures

$$\phi_\Gamma : B_\mathbb{Z} \to \check{B}_\mathbb{Z} \otimes H^2(S,\mathbb{Z})_{\mathrm{tr}},$$

where $B = B_\mathbb{C}/F^1B_\mathbb{C} \oplus B_\mathbb{Z}$, $B_\mathbb{Z} = \check{B}_\mathbb{Z}\check{}$. One verifies easily that the corresponding morphism of Hodge structures

$$\psi_\Gamma : H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{} \to \check{B}_\mathbb{Z} \otimes \check{B}_\mathbb{Z}$$

is the composite of the pull-back map

$$r^* : H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{} \to H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z})$$

and of the projection map

$$H^1(C,\mathbb{Z}) \otimes H^1(C,\mathbb{Z}) \to \check{B}_\mathbb{Z} \otimes \check{B}_\mathbb{Z}.$$

Paranjape ([9]) shows that ψ_Γ is injective. It follows that ϕ_Γ is nonzero.

Now let $u \in B_\mathbb{Z}$ be such that $\phi_\Gamma(u) \neq 0$. There are at most countably many points u_i in the real torus $(\mathbb{R}/\mathbb{Z}) \cdot u$ such that $\Gamma^B_*(u_i)$ is of torsion in $J(C \times S)^B_{\mathrm{tr}}$. Let $\alpha \in \mathbb{R}/\mathbb{Z}$ be such that $\Gamma^B_*(\alpha \cdot u)$ is not of torsion; now view $\phi_\Gamma(u)$ as an element $[\phi_\Gamma(u)]$ of $\operatorname{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, \check{B}_\mathbb{Z})$. Since $b_{2,tr}(S) = 6$ and $\operatorname{rk} \check{B}_\mathbb{Z} = 8$, the image of $[\phi_\Gamma(u)]$ is contained in a proper sublattice of $\check{B}_\mathbb{Z}$. It follows that $[\phi_\Gamma(u)] \otimes \alpha \in \operatorname{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, \check{B}_\mathbb{Z} \otimes \mathbb{R}/\mathbb{Z})$ has its image contained in a proper subtorus of $B_\mathbb{Z}\check{} \otimes \mathbb{R}/\mathbb{Z}$.

Since Γ^B_* is induced by the Abel-Jacobi map, there is a one-cycle $\Gamma_{u\cdot\alpha}$ in $C \times S$ such that $e^B_{\Gamma_{u\cdot\alpha}} = \Gamma^B_*(\alpha \cdot u)$. Consider now the corresponding element

$[e_{\Gamma_{u\cdot\alpha}}]^*_B$ of $\mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}, \check{B}_{\mathbb{Z}}\otimes\mathbb{R}/\mathbb{Z})$. Since ϕ_Γ is a morphism of Hodge structures, we have a commutative diagram

$$\begin{array}{cccc} \phi_\Gamma\otimes\mathbb{R}/\mathbb{Z}: & B_{\mathbb{Z}}\otimes\mathbb{R}/\mathbb{Z} & \rightarrow & \mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}, \check{B}_{\mathbb{Z}}\otimes\mathbb{R}/\mathbb{Z}) \\ & \downarrow & & \downarrow \\ \Gamma_*^B: & B & \rightarrow & J(C\times S)^B_{\mathrm{tr}} \end{array}$$

where the vertical maps are the identifications already used above. It follows that $[e_{\Gamma_{u\cdot\alpha}}]^*_B$ is equal to $[\phi_\Gamma(u)]\otimes\alpha$, hence has its image contained in a proper subtorus of $\check{B}_{\mathbb{Z}}\otimes\mathbb{R}/\mathbb{Z}$.

To conclude that this is the desired counterexample, it suffices to note that for general S, B is a simple abelian variety. This follows from the fact that (B,j) determines the Hodge structure on $H^2(S)_{\mathrm{tr}}$ (cf. [9]), which implies that B has at least four moduli. Then a dimension count shows that the moduli space of nonsimple abelian varieties A of dimension 4 admitting an automorphism of order 4, acting on $H^{1,0}(A)$ with two eigenvalues equal to i and two eigenvalues equal to $-i$, as is the case in Paranjape's family, is of dimension strictly less than 4. □

The counterexample given here is quite special, but it seems from the line of the argument that the noninjectivity of ψ_2^2 for a surface with infinite dimensional CH_0 is a general fact; indeed take any such surface S (regular for simplicity) and choose a finite sufficiently ample and generic morphism $\phi: S\rightarrow\mathbb{P}^2$. Let C be a sufficiently general and ample curve in $\mathbb{P}^2$ such that $\tilde{C}=\phi^{-1}(C)$ is smooth, $J(C)$ is simple, and $j_*: B\rightarrow \mathrm{CH}_0(S)$ has an at most countable kernel, where j is the inclusion of $\tilde{C}$ in S and $B:=(\mathrm{Ker}\,Nm: J(\tilde{C})\rightarrow J(C))_0$. Now choose a dimension-1 real subtorus T of $\phi^*(J(C))$ and let $T^\perp\subset J(\tilde{C})$ be its orthogonal. Consider a general small deformation $\tilde{C}_t$ of $\tilde{C}$. The associated element $e_{\tilde{C}_t,S}$ of $J(\tilde{C}_t\times S)_{\mathrm{tr}}$ varies holomorphically with t and the corresponding element $[e_{\tilde{C}_t,S}]^*\in\mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}, H^1(\tilde{C}_t,\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z})\cong \mathrm{Hom}(H^2(S,\mathbb{Z})_{\mathrm{tr}}, H^1(\tilde{C},\mathbb{Z})\otimes\mathbb{R}/\mathbb{Z})$ varies in a real analytic way. By construction, we have $\mathrm{Im}\,[e_{\tilde{C}_0,S}]^*\subset T^\perp$, and the locus where $\mathrm{Im}\,[e_{\tilde{C}_t,S}]^*$ remains contained in $T^\perp$ is defined by $b_2(S)_{\mathrm{tr}}$ real analytic equations. Now, the arguments developed above show that if ψ_2^2 is injective, for t in this locus, there is an abelian subvariety A_t of $J(\tilde{C}_t)$ such that

$$T\subset A_t\subset \mathrm{Ker}\,j_{t*}.$$

The simplicity of $J(C)$ and the fact that $\phi^*(J(C))$ is the maximal abelian subvariety of $J(\tilde{C}_0)$ contained in $\mathrm{Ker}\,j_{0*}$ imply now that on a connected positive dimensional component of this locus containing 0, $A_t\subset J(\tilde{C}_t)$ is a deformation of $\phi^*(J(C))\subset J(\tilde{C}_0)$.

A contradiction would follow by proving the following facts:

– The small deformations $\tilde{C}_t$ of $\tilde{C} = \tilde{C}_0$ such that $J(\tilde{C}_t)$ contains a deformation of $\phi^*(J(C))$ are the curves of the form $\phi_t^{-1}(C_t)$ where ϕ_t is a deformation of ϕ and C_t is a deformation of C. In particular they form a sublocus of the family of deformations of $\tilde{C}_t$ of arbitrarily large codimension.

– The locus where $\operatorname{Im}[e_{\tilde{C}_t,S}]^*$ remains contained in $T^\perp$ is actually of real codimension less or equal to $b_2(S)_{\mathrm{tr}}$. (This is not clear since the equations are only real analytic, and not holomorphic, but this could be proved by an infinitesimal study: it would suffice to show that the equations have independent differentials at 0.)

3. A formula for the pull-back of holomorphic two-forms

Let S be a regular surface. Let W be a complex ball parametrizing the following data:

$\mathcal{C}$ is a smooth complex variety, $\pi : \mathcal{C} \to W$ is a proper submersive holomorphic map of relative dimension 1.

$\mathcal{S}$ is a smooth complex variety, $\rho : \mathcal{S} \to W$ is a proper submersive holomorphic map of relative dimension 2.

There exists a holomorphic map $\tau : \mathcal{S} \to W \times S$, making the following diagram commutative

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\tau} & W \times S \\ \rho \downarrow & & \mathrm{pr}_1 \downarrow \\ W & = & W \end{array} .$$

Furthermore, $\tau_{|S_w} : S_w \to S$ is a birational map for each $w \in W$.

Let $\phi : \mathcal{C} \to \mathcal{S}$ be a holomorphic immersion making the following diagram commutative

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{S} \\ \pi \downarrow & & \rho \downarrow \\ W & = & W \end{array} .$$

Finally, let $\sigma_1, \ldots, \sigma_N$ be holomorphic sections of π, and let $m_1, \ldots, m_N$ be integers such that the zero-cycle $Z_w = \sum_i m_i \sigma_i(w)$ is of degree 0 on each component of the curve C_w, for each $w \in W$.

For each i, we get a holomorphic map

$$\alpha_i = \mathrm{pr}_2 \circ \tau \circ \phi \circ \sigma_i : W \to S,$$

and for each complex valued two-form ω on S, we get a two-form

$$\tilde{\omega} = \sum_i m_i \alpha_i^*(\omega)$$

on W. This two-form $\tilde{\omega}$ is Mumford's pull-back of the two-form ω on S (see [7]), for the family of zero-cycles $(\mathrm{pr}_2 \circ \tau \circ \phi(Z_w))_{w\in W}$ of S parametrized by W.

On the other hand, for each $w \in W$, we have the Abel-Jacobi invariant $e_w := e_{C_w,S} \in J(C_w \times S)_{\mathrm{tr}}$ or its real version

$$e_w := e_{C_w,S} \in H^1(C_w, \mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S, \mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Canonically identifying $H^1(C_w, \mathbb{Z})$ and $H^1(C_0, \mathbb{Z})$, we can view $(e_w)_{w\in W}$ as a map

$$e : W \to H^1(C_0, \mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S, \mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Clearly e is differentiable (and in fact real analytic since the Abel-Jacobi invariants vary holomorphically with the parameters).

Next, for $w \in W$, the 0-cycle Z_w is homologous to 0 on C_w, hence has a corresponding Abel-Jacobi invariant $f_w \in J(C_w)$, or its real version $f_w \in H^1(C_w, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}$. Identifying canonically $H^1(C_w, \mathbb{Z})$ and $H^1(C_0, \mathbb{Z})$, we can view $(f_w)_{w\in W}$ as a map

$$f : W \to H^1(C_0, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

Again it is easy to see that f is real analytic.

Now we differentiate e and f to get one-forms

$$de \in \Omega_W^{\mathbb{R}} \otimes_{\mathbb{Z}} H^1(C_0, \mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S, \mathbb{Z})_{\mathrm{tr}},\ df \in \Omega_W^{\mathbb{R}} \otimes_{\mathbb{Z}} H^1(C_0, \mathbb{Z}).$$

Finally we can contract $de \wedge df$ using the intersection pairing on $H^1(C_0, \mathbb{Z})$, to get a two-form

$$de \wedge df \in \bigwedge^2 \Omega_W^{\mathbb{R}} \otimes_{\mathbb{Z}} H^2(S, \mathbb{Z})_{\mathrm{tr}}.$$

We can view $de \wedge df$ as an element $[de \wedge df]$ of $\mathrm{Hom}_{\mathbb{Z}}(H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{}, \bigwedge^2 \Omega_W^{\mathbb{R}})$, which we can extend by $\mathbb{C}$-linearity to an element $[de \wedge df]$ of $\mathrm{Hom}_{\mathbb{C}}(H^2(S, \mathbb{C})_{\mathrm{tr}}\check{}, \bigwedge^2 \Omega_W^{\mathbb{C}})$.

Now let ω be a $(2, 0)$-form on S, with class $[\omega] \in H^2(S, \mathbb{C})_{\mathrm{tr}}\check{}$. Our main result in this section is the following:

Proposition 2. *For a holomorphic two-form ω on S, there is the pointwise equality of two-forms on W*

$$\tilde{\omega} = [de \wedge df]([\omega]). \tag{3.3}$$

The proof of formula (3.3) given below is a simplification of the original proof, following a suggestion of P. Griffiths. It goes essentially as follows: Note first that

$$de \wedge df([\omega]) = de([\omega]) \wedge df, \tag{3.4}$$

where $\mathrm{de} \in \mathrm{Hom}\,(H^2(S, \mathbb{C})_{\mathrm{tr}}\check{}, H^1(C_0, \mathbb{C}) \otimes \Omega_W^{\mathbb{C}})$ is the $\mathbb{C}$-linear extension of $\mathrm{de} \in \mathrm{Hom}\,(H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_0, \mathbb{Z}) \otimes \Omega_W^{\mathbb{R}})$.

Then if $\tilde{f} \in \mathcal{C}^\infty(W) \otimes H^1(C_0, \mathbb{R})$ is a lifting of f, we have

$$de([\omega]) \wedge df = -d(\langle de([\omega])\tilde{f}\rangle). \tag{3.5}$$

Now let ω' be the two-form on $\mathcal{C}$ induced by ω via $\mathrm{pr}_2 \circ \tau \circ \phi$. Then ω' induces a section of $\Omega_{\mathcal{C}/W} \otimes \pi^*\Omega_W$ on $\mathcal{C}$, that is a section β_ω of $\mathcal{H}^{1,0} \otimes \Omega_W$ on W. The first step is to show (see Lemma 5) that

$$de([\omega]) = \beta_\omega, \tag{3.6}$$

via the natural inclusion

$$\mathcal{H}^{1,0} \otimes \Omega_W \subset H^1_{\mathbb{C}} \otimes \Omega_W^{\mathbb{C}} \cong H^1(C_0, \mathbb{C}) \otimes \Omega_W^{\mathbb{C}}.$$

Next we use the definition of the Abel-Jacobi map which says that there exists a differentiably varying path γ_w on C_w such that $\partial\gamma_w = Z_w$, and for any $\eta \in H^{1,0}(C_w)$

$$\langle \eta, \tilde{f}_w \rangle \;=\; \int_{\gamma_w} \eta. \tag{3.7}$$

Combining (3.4), (3.6), and (3.7), we see that we have to show

$$\tilde{\omega} = -d(\int_\gamma \beta_\omega), \tag{3.8}$$

where $\int_\gamma \beta_\omega$ is the one-form ψ on W defined by $\psi(u) = \int_{\gamma_w} \beta_\omega(u)$ for $u \in T_{W,w}$. But (3.8) is essentially the homotopy formula since ω' is closed.

We now check the details of this outline of the proof and consider first the form de; we can view it as a map

$$[de] : H^2(S, \mathbb{Z})_{\mathrm{tr}}\check{} \to \Omega_W^{\mathbb{R}} \otimes_{\mathbb{Z}} H^1(C_0, \mathbb{Z}),$$

which can be extended by $\mathbb{C}$-linearity to a map

$$[de] : H^2(S, \mathbb{C})_{\mathrm{tr}}\check{} \to \Omega_W^{\mathbb{C}} \otimes_{\mathbb{C}} H^1(C_0, \mathbb{C}).$$

On the other hand, we have on $\mathcal{C}$ the exact sequence

$$0 \to \pi^*\Omega_W^2 \to \Omega_{\mathcal{C}}^2 \to \pi^*\Omega_W \otimes \Omega_{\mathcal{C}/W} \to 0.$$

The form $\omega' = \phi^*\tau^*\omega$ then has an image

$$\beta_\omega \in \Omega_W \otimes \mathcal{H}^{1,0},\ \Omega_W = \Omega_W^{1,0}$$

where $\mathcal{H}^{1,0} = \pi_*\Omega_{\mathcal{C}/W}$ is the Hodge bundle with fiber $H^{1,0}(C_w) \subset H^1(C_w, \mathbb{C})$.

LEMMA 5. *For any $w \in W$, the following equality*

$$[de]([\omega])_w = (\beta_\omega)_w$$

holds via the inclusion

$$\Omega_{W,w} \otimes H^{1,0}(C_w) \subset \Omega^{\mathbb{C}}_{W,w} \otimes H^1(C_w,\mathbb{C}) \cong \Omega^{\mathbb{C}}_{W,w} \otimes H^1(C_0,\mathbb{C}).$$

Proof. Recall that $e_w \in \mathrm{Hom}\,(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z})$ is obtained from the mixed Hodge structure on $H^2(S_w, C_w, \mathbb{Z})_{\mathrm{tr}}$, which fits into the exact sequence

$$0 \to H^1(C_w,\mathbb{Z}) \to H^2(S_w,C_w,\mathbb{Z})_{\mathrm{tr}} \to H^2(S,\mathbb{Z})_{\mathrm{tr}} \to 0, \tag{3.9}$$

as follows: the extension class of this extension is the class of the difference $\sigma_H - \sigma_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{C}))$ in the quotient

$$\mathrm{Hom}_{\mathbb{C}}(H^2(S)_{\mathrm{tr}}\check{}, H^1(C_w))/[F^0\mathrm{Hom}_{\mathbb{C}}(H^2(S)_{\mathrm{tr}}\check{}, H^1(C_w)) \oplus \mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{Z}))]$$

where σ_H is a Hodge splitting of the sequence 3.9, and $\sigma_{\mathbb{Z}}$ is an integral splitting of the sequence 3.9. The identification

$$\begin{aligned} &\mathrm{Hom}_{\mathbb{C}}(H^2(S,\mathbb{C})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{C}))/F^0\mathrm{Hom}\,(H^2(S,\mathbb{C})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{C}))\\ &\cong \mathrm{Hom}_{\mathbb{R}}(H^2(S,\mathbb{R})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{R})) \cong \mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{R}))\end{aligned}$$

means simply that there is a unique splitting $\sigma_{H,\mathbb{R}}$ of the sequence 3.9 which is both Hodge and real. Then e_w is the class of

$$\sigma_{H,\mathbb{R}} - \sigma_{\mathbb{Z}} \in \mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{R}))$$

in the quotient $\mathrm{Hom}_{\mathbb{Z}}(H^2(S,\mathbb{Z})_{\mathrm{tr}}\check{}, H^1(C_w,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z})$.

Now we have the following:

LEMMA 6. *For ω a holomorphic two-form on S, $\sigma_{H,\mathbb{R}}([\omega])(w)$ is the class of $\tau_w^*(\omega)$ in $H^2(S_w,C_w,\mathbb{C})_{\mathrm{tr}}$, (which is well-defined since $\tau_w^*\omega$ vanishes on C_w).*

Proof. This follows from the fact that

$$F^2H^2(S_w,C_w)_{\mathrm{tr}} \cong F^2H^2(S_w)_{\mathrm{tr}} \cong F^2H^2(S)_{\mathrm{tr}},$$

so that there is a unique Hodge splitting of the sequence 3.9 over $F^2H^2(S)_{\mathrm{tr}}$. On the other hand the map $[\omega] \mapsto$ class of $\tau_w^*(\omega)$ in $H^2(S_w,C_w,\mathbb{C})_{\mathrm{tr}}$ gives such a splitting as does $\sigma_{H,\mathbb{R}|H^{2,0}(S)}$. ◻

Let $\mathcal{H}^1_{\mathbb{C}}$ be the flat vector bundle on W with fiber $H^1(C_w,\mathbb{C})$, and ∇^C be its Gauss-Manin connection. Similarly let $\mathcal{H}^2_{\mathbb{C},\mathcal{S}/\mathcal{C}}$ be the flat vector bundle on W with fiber $H^2(S_w,C_w,\mathbb{C})_{\mathrm{tr}}$, and $\nabla^{\mathcal{S}/\mathcal{C}}$ be its Gauss-Manin connection. By definition, and by Lemma 6 we have the equality:

$$[de]([\omega]) = \nabla^{\mathcal{S}/\mathcal{C}}([\tau^*\omega]), \tag{3.10}$$

where $[\tau^*\omega]$ denotes the section of $\mathcal{H}^2_{\mathbb{C},\mathcal{S}/\mathcal{C}}$ whose value at w is the class of $\tau^*_w\omega$ in $H^2(S_w, C_w, \mathbb{C})$. (Notice that $\nabla^{\mathcal{S}/\mathcal{C}}([\tau^*\omega])$ belongs to $\Omega^{\mathbb{C}}_W \otimes \mathcal{H}^1_{\mathbb{C}}$, since the projection of $[\tau^*\omega]$ in the quotient bundle $\mathcal{H}^2_{\mathbb{C},\mathcal{S}}$ with fiber $H^2(S_w, \mathbb{C})_{\mathrm{tr}}$ is obviously flat.) The proof of Lemma 5 follows now from the equality 3.10, and from the following general statement:

LEMMA 7. *Consider a commutative diagram of differentiable smooth fibrations*

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathcal{S} \\ \pi \downarrow & & \downarrow \rho \\ W & = & W \end{array} \quad ,$$

and let Ω be a closed r-form on $\mathcal{S}$, such that $\Omega_{|C_w} = 0$, for any $w \in W$. Then for the corresponding section $[\Omega]$ of the bundle $\mathcal{H}^r_{\mathcal{S}/\mathcal{C}}$, $\nabla^{\mathcal{S}/\mathcal{C}}([\Omega])$ (which belongs to $\Omega_W \otimes \mathcal{H}^{r-1}_{\mathcal{C}}/\mathcal{H}^{r-1}_{\mathcal{S}}$) can be described as follows: the restriction of Ω to $\mathcal{C}$ projects naturally to a section of $\Omega^{r-1}_{\mathcal{C}/W} \otimes \pi^*(\Omega_W)$, which is in fact vertically closed, hence gives a section β_Ω of $\Omega_W \otimes \mathcal{H}^{r-1}_{\mathcal{C}}$; its image in $\Omega_W \otimes \mathcal{H}^{r-1}_{\mathcal{C}}/\mathcal{H}^{r-1}_{\mathcal{S}}$ is equal to $\nabla^{\mathcal{S}/\mathcal{C}}([\Omega])$.

Proof. Since the result is local, we may assume that our diagram of fibrations is trivial, that is, identifies to the inclusion $C \times W \subset S \times W$ for some $C \subset S$. For $w \in W$, $u \in T_{W,w}$, $\nabla^{\mathcal{S}/\mathcal{C}}_u([\Omega])$ is the class of the form $(d(\mathrm{int}_{\tilde{u}}\Omega) + \mathrm{int}_{\tilde{u}}(d\Omega))_{|S\times w}$, which is closed and restricts to 0 on C_w, in $H^r(S_w, C_w)$, where $\tilde{u}$ is the section of $T_{S\times W}$, defined along $S \times w$ and lifting u. Since Ω is closed, we get

$$\nabla^{\mathcal{S}/\mathcal{C}}_u([\Omega]) = \text{ class of } d(\mathrm{int}_{\tilde{u}}\Omega)_{|S\times w} \text{ in } H^r(S, C).$$

Of course $d(\mathrm{int}_{\tilde{u}}\Omega)_{|C\times w} = 0$, and the class of $\mathrm{int}_{\tilde{u}}\Omega_{|C\times w}$ in $H^{r-1}(C)$ is by definition equal to $\beta_\Omega(u)$. To conclude, it suffices to note that for an exact r-form $\beta = d\gamma$ on S vanishing on C, its class in $H^{r-1}(C)/H^{r-1}(S) \subset H^r(S, C)$ is the projection of the class of $\gamma_{|C} \in H^{r-1}(C)$. So, Lemma 7, hence Lemma 5 are proved. □

Now let $\tilde{f}$ be a $\mathcal{C}^\infty$ lifting of f to a function with value in $H^1(C_0, \mathbb{R})$. It is clear that we have

$$de \wedge df([\omega]) = -d(\langle de([\omega]), \tilde{f}\rangle), \tag{3.11}$$

where $\langle\, ,\rangle$ is the intersection form on $H^1(C_0, \mathbb{C})$. Now we use the definition of the Abel-Jacobi map or Albanese map to compute this bracket; the point $\tilde{f}_w \in H^1(C_w, \mathbb{R})$ projects to

$$f^{0,1}_w \in H^{0,1}(C_w) \cong (H^{1,0}(C_w))^*$$

and we have the equality, for $\eta \in H^{1,0}(C_w)$)

$$\langle \eta, \tilde{f}_w\rangle \;=\; \langle \eta, \tilde{f}_w^{0,1}\rangle \;=\; \int_{\gamma_w}\eta, \tag{3.12}$$

for an adequate choice of a path γ_w in C_w such that $\partial\gamma_w = Z_w$.

Next, by Lemma 5, we can use this formula to compute $\langle de([\omega]), \tilde{f}\rangle$ and this gives

$$\langle [de]([\omega]), \tilde{f}\rangle \;=\; \langle \beta_\omega, \tilde{f}^{0,1}\rangle \;=\; \int_\gamma \beta_\omega, \tag{3.13}$$

where the right-hand side is the one-form ψ on W defined by

$$\psi(u) = \int_{\gamma_w}\beta_\omega(u),$$

for $u \in T_{W,w}$.

By (3.11) and (3.13), to conclude the proof of Proposition 2 we have now only to prove the following:

Lemma 8. *Let ω' be a closed holomorphic two-form on $\mathcal{C}$ with induced section β_ω of $\mathcal{H}^{1,0}\otimes\Omega_W$. Let $\gamma_w \subset C_w$ be a differentiably varying family of paths such that $\partial\gamma_w = Z_w$; then*

$$\sum_i m_i\sigma_i^*\omega' = -d\left(\int_\gamma \beta_\omega\right). \tag{3.14}$$

Proof. It is clear that it suffices to prove equality (3.14) when we have only two sections σ_1, σ_2, and $m_1 = 1$, $m_2 = -1$. We may furthermore assume $\sigma_1(w) \neq \sigma_2(w)$ for all $w \in W$, since it suffices by continuity to prove the equality at the generic point of W, where this is true. (Otherwise the two sections coincide, and both sides of the equality are equal to zero.) Next, since the result is local, we can assume there is a $\mathcal{C}^\infty$ trivialization of the family $\pi : \mathcal{C} \to W$ in such a way that the two sections become constant and that there is an induced trivialization of the family of paths γ_w:

$$\begin{array}{ccc} \mathcal{C} & \cong & W\times C \\ \downarrow\pi & & \downarrow \mathrm{pr}_1 \\ W & = & W \end{array},$$

with $\sigma_i(w) = (w, c_i)$ and $\gamma_w(t) = (w, \gamma(t))$, $t \in [0,1]$, with $\gamma(0) = c_2$, $\gamma(1) = c_1$. Denote by $\Gamma : W\times[0,1] \to \mathcal{C}$ the map

$$\mathrm{Id}\times\gamma : W\times[0,1] \to W\times C \cong \mathcal{C},$$

and let $\omega'' = \Gamma^*(\omega')$. The form ω'' can be written

$$\omega'' = \eta + \delta\wedge dt, \tag{3.15}$$

where $\eta \in \mathrm{pr}_1^*\Omega^2_{W,\mathbb{C}}$, $\delta \in \mathrm{pr}_1^*\Omega_{W,\mathbb{C}}$. Then we have

$$\sigma_1^*(\omega') - \sigma_2(\omega') = \eta_{|W\times 1} - \eta_{|W\times 0}. \tag{3.16}$$

Furthermore, since ω'' is closed, the homotopy formula says

$$\eta_{|W\times 1} - \eta_{|W\times 0} = -d(\int_0^1 \delta_t dt). \tag{3.17}$$

Finally let $u \in T_W$ and let $\tilde{u}$ be its natural lifting in $T_{\mathcal{C}}$ given by the trivialization: then by definition of β_ω we have $\beta_\omega(u) = \mathrm{int}_{\tilde{u}}(\omega')_{|T_{\mathcal{C}/W}}$. Pulling this back to $W \times [0,1]$ via Γ, and using (3.15) we get

$$\delta_{w,t}(\tilde{u})dt = \gamma^*((\beta_\omega)_w(u)).$$

Fixing w, u and integrating over t, we get

$$\int_{\gamma_w} (\beta_\omega)_w(u) = \int_0^1 \delta_{w,t}(u)dt;$$

that is,

$$\int_\gamma \beta_\omega = \int_0^1 \delta_t dt.$$

Then formula 3.14 follows from the equality above and from (3.16), (3.17). Thus Lemma 8 and Proposition 2 are proved. $\square$

4. The nontriviality of ψ_2^2

We will prove in this section the following theorem:

Theorem 3. *Let S be a surface with $h^{2,0} \neq 0$; then $\psi_2^2(S)$ is nontrivial modulo torsion.* (*In fact the proof will show that* $\mathrm{Im}\,\psi_2^2(S)$ *mod. torsion is infinite dimensional.*)

The proof will be based on Propositions 3 and 4, which allow us to apply Proposition 2.

We work with the notation introduced at the beginning of Section 3, that is with the diagram

$$\begin{array}{ccccc} \mathcal{C} & \xrightarrow{\phi} & \mathcal{S} & \xrightarrow{\tau} & W \times S \\ \pi \downarrow & & \rho \downarrow & & \mathrm{pr}_1 \downarrow \\ W & = & W & = & W \end{array},$$

together with sections σ_i of π, and integers m_i, defining a family of zero-cycles Z_w homologous to zero on C_w. They allow us to define functions

$$w \mapsto e_w \in H^1(C_0,\mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S,\mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z},$$

$$w \mapsto f_w \in H^1(C_0,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z},$$

and by definition $\psi_2^2((\mathrm{pr}_2 \circ \tau \circ \phi)_* Z_w)$ is the projection modulo $U_2^2(S)$ of the product

$$e_w \cdot f_w \in H^2(S,\mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}.$$

This product has the following explicit form: let $\{\alpha_i, \beta_i\}$, $1 \leq i \leq g$, be a symplectic basis of $H^1(C_0, \mathbb{Z})$ and let $\{\gamma_j\}$ be a basis of $H^2(S, \mathbb{Z})_{\mathrm{tr}}$; then we can write

$$\begin{aligned} e_w &= \sum_{i,j} \rho_{i,j}(w) \otimes \alpha_i \otimes \gamma_j + \sum_{i,j} \chi_{i,j}(w) \otimes \beta_i \otimes \gamma_j, \\ f_w &= \sum_i \phi_i(w) \otimes \alpha_i + \sum_i \psi_i(w) \otimes \beta_i, \end{aligned}$$

and we have

$$e_w \cdot f_w = \sum_j \Big(\sum_i \rho_{i,j}(w) \otimes_{\mathbb{Z}} \psi_i(w) - \sum_i \chi_{i,j}(w) \otimes_{\mathbb{Z}} \phi_i(w)\Big) \otimes \gamma_j. \tag{4.18}$$

We prove now:

PROPOSITION 3. *Let $V \subset W$ be a smooth real analytic subset, such that for any $w \in V$, the product $e_w \cdot f_w$ vanishes in $H^2(S, \mathbb{Q})_{\mathrm{tr}} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$. Then the $H^2(S, \mathbb{R})_{\mathrm{tr}}$-valued two-form* $\mathrm{d}e \wedge df$ *(see Section 3) vanishes on V.*

Proof. It suffices to prove that for any index j the hypothesis $(e_w \cdot f_w)_j = 0$ in $\mathbb{R}/\mathbb{Q}$, for any $w \in V$, implies that $(de \wedge df)_j = 0$ on V. We may assume that V is connected. We have then:

LEMMA 9. *There exist I_1, $I_2 \subset \{1, \ldots, g\}$, a dense subset $V' \subset V$ and coefficients*

$$\begin{aligned} \mu_{ik} &\in \mathbb{Q},\ i \in I_1,\ k \in \{1, \ldots, g\} - I_1, \\ \mu'_{lk} &\in \mathbb{Q},\ l \in I_2,\ k \in \{1, \ldots, g\} - I_1, \\ \nu_{im} &\in \mathbb{Q},\ i \in I_1,\ m \in \{1, \ldots, g\} - I_2, \\ \nu'_{lm} &\in \mathbb{Q},\ l \in I_2,\ m \in \{1, \ldots, g\} - I_2, \end{aligned}$$

such that for $v \in V'$, the elements $\phi_i(v)_{i \in I_1}$, $\psi_l(v)_{l \in I_2}$ form a $\mathbb{Q}$-basis of the $\mathbb{Q}$-vector subspace of $\mathbb{R}/\mathbb{Q}$ generated by the $\phi_i(v)$, $\psi_i(v)$, $1 \leq i \leq g$. Also, the following relations hold everywhere on V

$$\begin{aligned} \phi_k(w) &= \sum_{i \in I_1} \mu_{ik} \phi_i(w) + \sum_{l \in I_2} \mu'_{lk} \psi_l(w) \ \ \textit{in}\ \ \mathbb{R}/\mathbb{Q},\ k \in \{1, \ldots, g\} - I_1 \\ \psi_m(w) &= \sum_{i \in I_1} \nu_{im} \phi_i(w) + \sum_{l \in I_2} \nu'_{lm} \psi_l(w) \ \ \textit{in}\ \ \mathbb{R}/\mathbb{Q},\ m \in \{1, \ldots, g\} - I_2. \end{aligned} \tag{4.19}$$

Proof. Any relation $\sum_i \gamma_i \phi_i(w) + \sum_i \gamma'_i \psi_i(w) = 0$ in $\mathbb{R}/\mathbb{Q}$ holds everywhere on V or only on a countable union of proper real analytic subsets of V. If we consider all possible such relations, it follows from Baire's theorem that there is a dense subset $V' \subset V$ such that for any $w_0 \in V'$, any relation $\sum_i \gamma_i \phi_i(w_0) + \sum_i \gamma'_i \psi_i(w_0) = 0$ in $\mathbb{R}/\mathbb{Q}$ implies that $\sum_i \gamma_i \phi_i(w) + \sum_i \gamma'_i \psi_i(w) = 0$

in $\mathbb{R}/\mathbb{Q}$, for any $w \in V$. Choosing for such w_0 a basis $\phi_i(w_0)$, $\psi_l(w_0)$, $i \in I_1$, $l \in I_2$ of the $\mathbb{Q}$-vector subspace of $\mathbb{R}/\mathbb{Q}$ generated by the $\phi_i(w_0)$, $\psi_i(w_0)$, $1 \leq i \leq g$, gives the result. $\square$

Now in formula (4.18) we replace $\phi_k(w)$ and $\psi_m(w)$ by their expressions in (4.19), which gives

$$\begin{aligned}(e_w \cdot f_w)_j \quad = \quad & \sum_{l \in I_2} \rho_{lj}(w) \otimes \psi_l(w) + \sum_{m \notin I_2} \rho_{mj}(w) \\ & \otimes (\sum_{i \in I_1} \nu_{im}\phi_i(w) + \sum_{l \in I_2} \nu'_{lm}\psi_l(w)) \\ & - \sum_{i \in I_1} \chi_{ij}(w) \otimes \phi_i(w) - \sum_{k \notin I_1} \chi_{kj}(w) \\ & \otimes \sum_{i \in I_1} \mu_{ik}\phi_i(w) + \sum_{l \in I_2} \mu'_{lk}\psi_l(w)),\end{aligned}$$

where the equality holds in $\mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$.

Now we use the fact that $\phi_i(w)$ and $\psi_l(w)$ are independent over $\mathbb{Q}$ for w in the dense subset V'. Then for $w \in V'$ the condition $(e_w \cdot f_w)_j = 0$ in $\mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$ implies

$$(4.20) \qquad \begin{aligned} & \rho_{lj}(w) + \sum_{m \notin I_2} \nu'_{lm}\rho_{mj}(w) - \sum_{k \notin I_1} \mu'_{lk}\chi_{kj}(w) = 0 \quad \text{in } \mathbb{R}/\mathbb{Q}, \forall l \in I_2 \\ & -\chi_{ij}(w) + \sum_{m \notin I_2} \nu_{im}\rho_{mj}(w) - \sum_{k \notin I_1} \mu_{ik}\chi_{kj}(w) = 0 \quad \text{in } \mathbb{R}/\mathbb{Q}, \forall i \in I_1.\end{aligned}$$

But recall that V' is the complementary set in V of a countable union of proper real analytic subsets. So the equalities (4.20), being satisfied on V', must hold everywhere on V.

Now we can differentiate (4.19) and (4.20): indeed these equalities mean that for liftings of the functions $\phi_i, \psi_i, \chi_{ij}, \rho_{kj}$ to functions with values in $\mathbb{R}$, the corresponding equalities hold modulo some (necessarily constant) rational numbers. This gives

$$(4.21) \qquad \begin{aligned} d\phi_k \quad &= \quad \sum_{i \in I_1} \mu_{ik} d\phi_i + \sum_{l \in I_2} \mu'_{lk} d\psi_l, \ k \in \{1, \ldots, g\} - I_1 \\ d\psi_m \quad &= \quad \sum_{i \in I_1} \nu_{im} d\phi_i + \sum_{l \in I_2} \nu'_{lm} d\psi_l, \ m \in \{1, \ldots, g\} - I_2.\end{aligned}$$

$$(4.22) \qquad \begin{aligned} d\rho_{lj} + \sum_{m \notin I_2} \nu'_{lm} d\rho_{mj} - \sum_{k \notin I_1} \mu'_{lk} d\chi_{kj} \quad &= \quad 0, \quad \text{for all } l \in I_2 \\ -d\chi_{ij} + \sum_{m \notin I_2} \nu_{im} d\rho_{mj} - \sum_{k \notin I_1} \mu_{ik} d\chi_{kj} \quad &= \quad 0, \quad \text{for all } i \in I_1.\end{aligned}$$

Next we have

$$(de \wedge df)_j = (de)_j \wedge df = \sum_i d\rho_{ij} \wedge d\psi_i - \sum_i d\chi_{ij} \wedge d\phi_i,$$

which shows that this is equal to zero, by (4.21) and (4.22). Now Proposition 3 is proved. □

Combining Proposition 2 and Proposition 3, we conclude:

COROLLARY 1. *Under the assumptions of Proposition* 3, *the pull-back*

$$\tilde{\omega} = \sum_i m_i (\mathrm{pr}_2 \circ \tau \circ \phi \circ \sigma_i)^*(\omega)$$

of any holomorphic two-form ω *on* S *vanishes on* V.

Next we have the following:

PROPOSITION 4. *Assume the map* $\psi_2^2(S)$ *vanishes modulo torsion in* $J_2^2(S)$; *then there exist data*

$$\begin{array}{ccccc} \mathcal{C} & \stackrel{\phi}{\rightarrow} & \mathcal{S} & \stackrel{\tau}{\rightarrow} & W \times S \\ \pi \downarrow & & \rho \downarrow & & \mathrm{pr}_1 \downarrow \\ W & = & W & = & W \end{array},$$

together with sections σ_i *of* π, *and integers* m_i, *defining a family of zero-cycles* Z_w *homologous to zero on* C_w *satisfying the following properties*:

- *There exists a map* $\psi = (\psi_1, \psi_2) : W \rightarrow S \times S$ *such that*

  $$(\mathrm{pr}_2 \circ \tau \circ \phi)_* Z_w = \psi_1(w) - \psi_2(w)$$

  as a zero-cycle of S, *for any* $w \in W$.

- *There is a smooth locally closed real analytic subset* $V \subset W$ *such that for any* $w \in V$, $e_w \cdot f_w$ *vanishes in* $H^2(S, \mathbb{Q})_{\mathrm{tr}} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$, *and* $\psi_{|V}$ *is a submersion.*

Clearly this proposition implies Theorem 3; indeed, if $\psi_2^2(S)$ vanishes modulo torsion in $J_2^2(S)$, Proposition 4 and Corollary 1 give a submersive map $\psi : V \rightarrow S \times S$ such that for any holomorphic two-form ω on S,

$$\psi_1^*(\omega) - \psi_2^*(\omega) = \sum_i m_i (\mathrm{pr}_2 \circ \tau \circ \phi \circ \sigma_i)^*(\omega)$$

vanishes on V. It follows that $\mathrm{pr}_1^*(\omega) - \mathrm{pr}_2^*(\omega)$ vanishes on an open set of $S \times S$, hence that $\omega = 0$. So we have proved that $\psi_2^2(S) = 0$ modulo torsion implies $H^{2,0}(S) = \{0\}$, that is, Theorem 3. □

Proof of Proposition 4. By definition of ψ_2^2, the assumption implies that for any $(x_1, x_2) \in S \times S$, there exist a smooth curve C, a zero-cycle Z homologous to zero on C, and an immersion $\phi : C \hookrightarrow \tilde{S}$ of C in a surface $\tilde{S} \xrightarrow{\tau} S$ birational to S, such that $\tau \circ \phi_*(Z) = x_1 - x_2$ and $e_{C,S} \cdot f_{Z,C} = 0$ in $H^2(S, \mathbb{Q})_{\mathrm{tr}} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$. Now note that there are countably many quasi-projective varieties (that we may assume smooth by desingularization) W_m, together with data

$$\begin{array}{ccccc} \mathcal{C}_m & \stackrel{\phi}{\to} & \mathcal{S}_m & \stackrel{\tau}{\to} & W_m \times S \\ \pi_m \downarrow & & \rho_m \downarrow & & \mathrm{pr}_1 \downarrow \\ W_m & = & W_m & = & W_m \end{array},$$

with sections σ_i^m of π_m, and integers m_i^m, defining a family of zero-cycles Z_w^m homologous to zero on C_w^m, and satisfying the following properties:

- There exists a map $\psi_m = (\psi_1^m, \psi_2^m) : W_m \to S \times S$ such that $(\tau_m \circ \phi_m)_* Z_w^m = \psi_1^m(w) - \psi_2^m(w)$ as a zero-cycle of S, for any $w \in W_m$.
- Any set of data $((x_1, x_2), C, Z, \phi, \tau)$ as above, such that $\tau \circ \phi_*(Z) = x_1 - x_2$ identifies with the data parametrized by some point $w \in W_m$, with $(x_1, x_2) = \psi_m(w)$.

On each W_m, we have the locally defined maps

$$\begin{aligned} e_m : W_m &\to H^1(C_0^m, \mathbb{Z}) \otimes_{\mathbb{Z}} H^2(S, \mathbb{Z})_{\mathrm{tr}} \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}, \\ f_m : W_m &\to H^1(C_0^m, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z}, \end{aligned}$$

(which are globally defined as sections of a flat bundle), and their product

$$e_m \cdot f_m : W_m \to H^2(S, \mathbb{Q})_{\mathrm{tr}} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{R}/\mathbb{Q}.$$

So the assumption of Proposition 4 is that

$$S \times S = \bigcup_m \psi_m((e_m \cdot f_m)^{-1}(0)).$$

We have now:

LEMMA 10. *Locally* $(e_m \cdot f_m)^{-1}(0))$ *is a countable union of real analytic subsets of* W_m.

Assuming Lemma 10 we have countably many locally closed real analytic subsets $W_m^n \subset W_m$ on which $e_m \cdot f_m$ vanishes, and such that

$$S \times S = \bigcup_{m,n} \psi_m(W_m^n).$$

Stratifying each W_m^n into smooth real analytic subsets, we may assume the W_m^n are smooth. The theorems of Baire and Sard imply now that for some (m, n), $\psi_{m|W_m^n}$ must be submersive at some point of W_m^n, hence on an open subset V of it. So Proposition 4 is proved, with $W = W_m$, $\psi = \psi_m$. $\square$

Proof of Lemma 10. The proof was almost completed in the course of the proof of Proposition 3. With the notation introduced there (and forgetting the subscript m), it follows from the computations made there that, for the $j^{\rm th}$-component $(e\cdot f)_j$ of $e\cdot f$, $(e\cdot f)_j^{-1}(0)\subset W$ can be written locally as the countable union of the sets $W_{I_1,I_2,\mu_{ik},\mu'\_lk,\nu_{im},\nu'_{lm}}\subset W$ where the equations (4.19) and (4.20) are satisfied. But choosing (locally) liftings of the ϕ_i, ψ_i, ρ_{ij}, χ_{ij} to real analytic functions with values in $\mathbb{R}$, one sees immediately that each $W_{I_1,I_2,\mu_{ik},\mu'_{lk},\nu_{im},\nu'_{lm}}$ is a countable union of real analytic subsets of W. Now the lemma is proved, since $(e\cdot f)^{-1}(0)=\bigcap_j(e\cdot f)_j^{-1}(0)$. □

Remark 1. More generally, we have proved that in $S^{[k]}\times S^{[k]}$, the set Z of points (z_1,z_2) such that $\psi_2^2(z_1-z_2)=0$ mod torsion is covered by a countable union of images of real analytic sets V,$\psi:V\to S^{[k]}\times S^{[k]}$, such that for any holomorphic two-form ω on S with induced form ω_k on $S^{[k]}$, $\psi_1^*\omega_k-\psi_2^*\omega_k$ vanishes on V. Hence, the Mumford argument (see [7]) applies to show that $\operatorname{Im}\psi_2^2$ mod torsion is infinite dimensional. Indeed, if $z\in S^{[k]}$ is a general point, the two-form $\psi_1^*\omega_k-\psi_2^*\omega_k$ is nondegenerate at (z,z), so that its real part is also nondegenerate, and the fact that it vanishes on Z implies that the real dimension of any component of Z passing through (z,z) is at most equal to $\frac{\dim_{\mathbb{R}} S^{[k]}\times S^{[k]}}{2}=\dim_{\mathbb{R}} S^{[k]}$. But any component of Z passing through (z,z) has to dominate an open set of $S^{[k]}$ by the first projection, if z is chosen outside a countable union of real analytic sets. It follows that the map $z'\mapsto\psi_2^2(z'-z)$ has almost all of its fibers countable in some neighbourhood of z. Hence the dimension of its image, defined as $\dim_{\mathbb{R}} S^{[k]}-\dim_{\mathbb{R}}$ (*general fiber*), is equal to $\dim_{\mathbb{R}} S^{[k]}$, which tends to ∞ with k. Here *general* is with respect to the real analytic Zariski topology and the dimension of a fiber is well-defined since the fiber is covered by a countable union of real analytic sets.

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