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Discrete orthogonal polynomial ensembles and the Plancherel measure

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Abstract

We consider discrete orthogonal polynomial ensembles which are discrete analogues of the orthogonal polynomial ensembles in random matrix theory. These ensembles occur in certain problems in combinatorial probability and can be thought of as probability measures on partitions. The Meixner ensemble is related to a two-dimensional directed growth model, and the Charlier ensemble is related to the lengths of weakly increasing subsequences in random words. The Krawtchouk ensemble occurs in connection with zig-zag paths in random domino tilings of the Aztec diamond, and also in a certain simplified directed first-passage percolation model. We use the Charlier ensemble to investigate the asymptotics of weakly increasing subsequences in random words and to prove a conjecture of Tracy and Widom. As a limit of the Meixner ensemble or the Charlier ensemble we obtain the Plancherel measure on partitions, and using this we prove a conjecture of Baik, Deift and Johansson that under the Plancherel measure, the distribution of the lengths of the first k rows in the partition, appropriately scaled, converges to the asymptotic joint distribution for the k largest eigenvalues of a random matrix from the Gaussian Unitary Ensemble. In this problem a certain discrete kernel, which we call the discrete Bessel kernel, plays an important role.

1. Introduction and results

During the last years there has been a lot of activity around the problem of the distribution of the length of a longest increasing subsequence of a random permutation, its generalizations and their connection with random matrices, see for example [Ge], [Ra], [BDJ1], [Jo3], [Ok], [BR2], [Bi], and also [AD] for connections with patience and the history of the problem. Let π be a random permutation from the symmetric group S_N with uniform distribution $\mathbb{P}_{\text{perm},N}$ and let $L(\pi)$ denote the length of a longest increasing subsequence in π. It is proved by Baik, Deift and Johansson in [BDJ1] that

$$\lim_{N\to\infty} \mathbb{P}_{\text{perm},N}[L(\pi) \le 2\sqrt{N} + tN^{1/6}] = F(t), \tag{1.1}$$

where $F(t)$ is the *Tracy-Widom distribution*, (1.5) for the appropriately scaled largest eigenvalue of a random $M \times M$ matrix from the *Gaussian Unitary Ensemble* (GUE) in the limit $M \to \infty$, see [TW1]. The probability density function on $\mathbb{R}^M$ for the M eigenvalues $x_1, \dots, x_M$ of an $M \times M$ GUE matrix is

$$\phi_{\mathrm{GUE},M}(x) = \frac{1}{Z_M} \prod_{1 \le i < j \le M} (x_i - x_j)^2 \prod_{j=1}^{M} e^{-x_j^2}, \tag{1.2}$$

where $Z_M = (2\pi)^{M/2} 2^{-M^2/2} \prod_{j=1}^{M} (j!)^{-1}$. This probability density can be analyzed using the Hermite polynomials, which are orthogonal with respect to the weight $\exp(-x^2)$ occurring in (1.2). Using standard techniques from random matrix theory, see [Me] or [TW2], we can write

$$\mathbb{P}_{\mathrm{GUE},M}\left[\max_{1 \le k \le M} x_k \le \sqrt{2M} + \frac{t}{\sqrt{2}M^{1/6}}\right] = \det(I - \mathcal{K}_M)\big|_{L^2(t,\infty)}, \tag{1.3}$$

where

$$\mathcal{K}_M(\xi, \eta) = \frac{1}{\sqrt{2}M^{1/6}} K_M\left(\sqrt{2M} + \frac{\xi}{\sqrt{2}M^{1/6}}, \sqrt{2M} + \frac{\eta}{\sqrt{2}M^{1/6}}\right).$$

Here K_M is the *Hermite kernel*,

$$K_M(x, y) = \frac{\kappa_{M-1}}{\kappa_M} \frac{h_M(x) h_{M-1}(y) - h_{M-1}(x) h_M(y)}{x - y} e^{-(x^2+y^2)/2}$$

with $h_m(x) = \kappa_m x^m + \dots$, $\int_{\mathbb{R}} h_n(x) h_m(x) \exp(-x^2) dx = \delta_{nm}$, the normalized Hermite polynomials. It follows from standard asymptotic results for Hermite polynomials that

$$\lim_{M \to \infty} \mathcal{K}_M(\xi, \eta) = A(\xi, \eta) \doteq \frac{\mathrm{Ai}\,(\xi)\mathrm{Ai}\,'(\eta) - \mathrm{Ai}\,'(\xi)\mathrm{Ai}\,(\eta)}{\xi - \eta}, \tag{1.4}$$

the *Airy kernel*, and also that the Fredholm determinant in the right-hand side of (1.3) converges to

$$F(t) = \det(I - A)\big|_{L^2(t,\infty)} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(t,\infty)^k} \det[A(\xi_i, \xi_j)]_{i,j=1}^{k} d^k\xi, \tag{1.5}$$

the Tracy-Widom distribution.

The problem of the length of the longest increasing subsequence in a random permutation is closely related to the so called Plancherel measure on partitions, which occurs as a natural probability measure on the set of all equivalence classes of irreducible representations of the symmetric group. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell, 0, 0, \dots)$, $\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_\ell \ge 1$, $\sum_j \lambda_j = N$, be a partition of N, which can be represented in the usual way by a Young diagram with ℓ rows and λ_j boxes in the j^{th} row, see e.g. [Sa], [Fu]. Let f^λ be the number of standard Young tableaux of shape λ. The *Plancherel measure* assigns to λ the probability

$$\mathbb{P}_{\mathrm{Plan},N}[\{\lambda\}] = \frac{(f^\lambda)^2}{N!}. \tag{1.6}$$

The probability measure (1.6) is the push-forward of the uniform distribution on S_N by the Robinson-Schensted-Knuth (RSK)-correspondence, see e.g. [Sa] or [Fu], which maps a permutation π to a pair of standard Young tableaux of the same shape λ, and the length λ_1 of the first row is equal to $L(\pi)$. Thus, the length of the first row behaves in the limit as $N \to \infty$, as the largest eigenvalue of a GUE matrix. It was proved in [BDJ2] that the distribution of the rescaled length of the second row, $\mathbb{P}_{\mathrm{Plan},N}[\lambda_2 \le 2\sqrt{N} + tN^{1/6}]$, converges to the Tracy-Widom distribution for the second largest eigenvalue of a GUE matrix, [TW2], and it was conjectured that the analogous result holds for the k^{th} row. This conjecture will be proved in the present paper. It has recently been independently proved by Borodin, Okounkov and Olshanski, [BOO], see below. The conjecture also follows from the result by Okounkov in [Ok]. His proof uses interesting geometric/combinatorial methods. There are many earlier indications of connections between the Plancherel measure and random matrices for instance in the work of Regev, [Re], and Kerov, [Ke1], [Ke2].

Another measure on partitions, coming from pairs of semi-standard tableaux, arises in [Jo3], where a certain random growth model is investigated. This measure relates to a discrete Coulomb gas on $\mathbb{N}$ of the form

$$\frac{1}{Z_M} \prod_{1\le i<j\le M} (h_i - h_j)^2 \prod_{j=1}^{M} w(h_j), \quad h \in \mathbb{N}^M, \tag{1.7}$$

where Z_M is a normalization constant. The weight $w(x) = \binom{x+K-1}{x} q^x$, is the weight function on $\mathbb{N}$ for the Meixner polynomials, $m_n^{K,q}(x)$, see [NSU]. This measure on $\mathbb{N}^M$ can be analyzed using the *Meixner kernel*

$$K^{K,q}_{\mathrm{Me},M}(x,y) = \frac{-q}{(1-q)d_{M-1}^2} \frac{m_M(x)m_{M-1}(y) - m_{M-1}(x)m_M(y)}{x-y} (w(x)w(y))^{1/2}, \tag{1.8}$$

with $d_n = n!(n+K-1)!(1-q)^{-K} q^{-n}[(K-1)!]^{-1}$, in much the same way as (1.2) is analyzed using the Hermite kernel. The Meixner kernel occurs in connection with probability measures on partitions also in the work of Borodin and Olshanski, [BO1]. The connection between certain measures on partitions and discrete Coulomb gases with their associated orthogonal polynomials is central in the present paper, and give them a very interesting statistical mechanical interpretation very similar to Dyson's Coulomb gas picture of the eigenvalues of random matrices. The difference is that in (1.7) we have a Coulomb gas on the integer lattice instead of on the real line. Other statistical mechanical aspects of measures on partitions have been investigated by Vershik, see [Ve] and references therein. We will refer to (1.7) as a *discrete orthogonal polynomial ensemble*. We will also be concerned with the cases $w(x) = \alpha^x e^{-\alpha}/x!$, $x \in \mathbb{N}$, the *Charlier ensemble*, $w(x) = \binom{N}{x} p^x q^{N-x}$, $x \in \{0, \dots, N\}$, the *Krawtchouk ensemble* and $w(x)$ given by (5.19), the *Hahn ensemble*.

Consider the *Poissonized Plancherel measure*,

$$\mathbb{P}^{\alpha}_{\text{Plan}}[\{\lambda\}] = e^{-\alpha} \sum_{N=0}^{\infty} \mathbb{P}_{\text{Plan},N}[\{\lambda\}] \frac{\alpha^N}{N!}, \tag{1.9}$$

on the set of all partitions, $\mathbb{P}_{\text{Plan},N}[\{\lambda\}] = 0$ if $\sum_j \lambda_j \neq N$. We will prove that this measure is a limit as $q \to 0$ of the Meixner ensemble. The Meixner kernel (1.8) converges in this limit, ($q = \alpha/M^2$, $K = 1$, $M \to \infty$), to the *discrete Bessel kernel*

$$B^{\alpha}(x, y) = \sqrt{\alpha} \frac{J_x(2\sqrt{\alpha}) J_{y+1}(2\sqrt{\alpha}) - J_{x+1}(2\sqrt{\alpha}) J_y(2\sqrt{\alpha})}{x - y}. \tag{1.10}$$

This result can be used to give a new proof of (1.1), and also to verify the k^{th} row conjecture of [BDJ2], as well as to obtain asymptotic results in the "bulk" of the Young diagram. These results have recently been independently obtained by Borodin, Okounkov and Olshanski, [BOO], as a limiting case of the results in [BO1]. See the paper [BO2] for a discussion of the connections between [BOO] and the present paper.

The results for the Poissonized Plancherel measure can also be obtained as a limit of the Charlier ensemble. This ensemble arises in the problem of the distribution of the length of a longest weakly increasing subsequence in a random word which will be studied below. The random word problem has recently been investigated by Tracy and Widom, [TW3], using Toeplitz determinants and Painlevé equations, see also [AD].

Before stating our results precisely we must introduce some notation. Let

$$\mathcal{P} = \{\lambda \in \mathbb{N}^{\mathbb{Z}_+} ; \lambda_1 \geq \lambda_2 \geq \dots \text{ and } \sum_j \lambda_j < \infty\}$$

denote the set of all partitions, and $\mathcal{P}^{(N)} = \{\lambda \in \mathcal{P} ; \sum_j \lambda_j = N\}$, $N \geq 0$, the set of all partitions of N. Set $\ell(\lambda) = \max\{k ; \lambda_k > 0\}$, the *length* of λ. We will consider functions on $\mathcal{P}$ of the following form. Let $f : \mathbb{Z} \to \mathbb{C}$ be a bounded function which satisfies $f(n) = 1$ if $n < 0$. For a given $L \geq 0$ we define $g : \mathcal{P} \to \mathbb{C}$ by

$$g(\lambda) = \prod_{i=1}^{\infty} f(\lambda_i + L - i). \tag{1.11}$$

We say that g is generated by f. Let $\mathcal{G}_L$ denote the set of all functions g obtained in this way and write $c(g) = ||f||_\infty$. Let $\mathcal{P}_M = \{\lambda \in \mathcal{P}\,;\, \ell(\lambda) \le M\}$ and $\mathcal{P}_M^{(N)} = \mathcal{P}_M \cap \mathcal{P}^{(N)}$. We also define, for $M \ge 1$, $N \ge 0$,

$$\Omega_M = \{\lambda \in \mathbb{N}^M\,;\, \lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_M\},$$
$$\Omega_M^{(N)} = \{\lambda \in \Omega_M\,;\, \sum_{j=1}^M \lambda_j = N\}.$$

Note that there is a natural bijection between $\mathcal{P}_M$ and Ω_M (and $\mathcal{P}_M^{(N)}$ and $\Omega_M^{(N)}$). If $M \ge L$, $g \in \mathcal{G}_L$ and $\lambda \in \mathcal{P}_M$, then

$$g(\lambda) = \prod_{i=1}^M f(\lambda_i + L - i), \tag{1.12}$$

since $f(n) = 1$ if $n < 0$, and we take (1.12) as our definition of g on Ω_M.

For $m \ge 1$ and $\lambda \in \mathcal{P}$ we define

$$V_m(\lambda) = \prod_{1 \le i < j \le m} (\lambda_i - \lambda_j + j - i),$$

and

$$W_m(\lambda) = \prod_{i=1}^m \frac{1}{(\lambda_i + m - i)!}.$$

According to a formula of Frobenius, see e.g. [Sa] or [Fu], the quantity f^λ above can be expressed as

$$f^\lambda = N! V_{\ell(\lambda)}(\lambda) W_{\ell(\lambda)}(\lambda). \tag{1.13}$$

Let $q \in (0,1)$ and $N \ge M$. We define the *Meixner ensemble* on Ω_M by

(1.14)

$$\mathbb{P}^q_{\mathrm{Me},M,N}[\{\lambda\}] = (1-q)^{MN} \prod_{j=0}^{M-1} \frac{(N-M)!}{j!(N-M+j)!} V_M(\lambda)^2 \prod_{i=1}^M \binom{\lambda_i + N - i}{\lambda_i + M - i} q^{\lambda_i}.$$

Note that if we make the change of variables $h_i = \lambda_i + M - i$ this gives us the discrete Coulomb gas (1.7) with the Meixner weight $w(x) = \binom{x+K-1}{x} q^x$, where $K = N - M + 1$. For more about the Meixner ensemble and its probabilistic interpretations see [Jo3]. We can now state our first theorem.

THEOREM 1.1. *For any $g \in \mathcal{G}_L$, $L \ge 0$, and $\alpha > 0$ we have that*

$$\mathbb{E}^\alpha_{\mathrm{Plan}}[g] = \lim_{N\to\infty} \mathbb{E}^{\alpha/N^2}_{\mathrm{Me},N,N}[g]. \tag{1.15}$$

Thus the Poissonized Plancherel measure can be obtained as a limit of the Meixner ensemble. The theorem will be proved in Section 2.

Next, we define the *Charlier ensemble* on Ω_M, which can be obtained as a limit of the Meixner ensemble, see (3.1). Given $\alpha > 0$ we define

$$\mathbb{P}^{\alpha}_{\mathrm{Ch},M}[\{\lambda\}] = \left(\prod_{j=1}^{M-1} \frac{1}{j!}\right) V_M(\lambda)^2 W_M(\lambda) \prod_{i=1}^{M} \left[\left(\frac{\alpha}{M}\right)^{\lambda_i} e^{-\alpha/M}\right] \tag{1.16}$$

on Ω_M. Again, the change of variables $h_i = \lambda_i + M - i$ gives a discrete Coulomb gas, (1.7). The Poissonized Plancherel measure can also be obtained as a limit of the Charlier ensemble.

THEOREM 1.2. *For any $g \in \mathcal{G}_L$, $L \geq 0$, and $\alpha > 0$,*

$$\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] = \lim_{M\to\infty} \mathbb{E}^{\alpha}_{\mathrm{Ch},M}[g]. \tag{1.17}$$

The Charlier ensemble has a probabilistic interpretation in terms of random words, see Proposition 1.5. Since the Meixner and Charlier ensembles both correspond to discrete orthogonal polynomial ensembles they can be analyzed in a way similar to that in which the Hermite ensemble (GUE) is analyzed. This makes it possible to prove the following theorem, compare with [BOO].

THEOREM 1.3. *Let $g \in \mathcal{G}_L$, $L \geq 0$, be generated by f, see* (1.11), *and write $\phi = f - 1$. Then,*

$$\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{h\in\mathbb{N}^k} \prod_{j=1}^{k} \phi(h_j) \det[B^{\alpha}(h_i - L, h_j - L)]_{i,j=1}^{k}, \tag{1.18}$$

where B^{α} is the discrete Bessel kernel, (1.10). *Note that the right-hand side is the Fredholm determinant of the operator on $\ell^2(\mathbb{N})$ with kernel $B^{\alpha}(x - L, y - L)\phi(y)$.*

The theorem will be proved in Section 3.

As an example we can take $\phi(t) = -\chi_{(n,\infty)}(s)$ and $L = 0$. This gives

$$\mathbb{P}^{\alpha}_{\mathrm{Plan}}[\lambda_1 \leq n] = \det(I - B^{\alpha})\big|_{\ell^2(\{n,n+1,\dots\})}.$$

By Gessel's formula the left-hand side is also a certain Toeplitz determinant, see e.g. [BDJ1], and hence we get an interesting identity between a Toeplitz determinant and a certain Fredholm determinant on a discrete space. This formula has recently been generalized by Borodin and Okounkov, [BoOk].

By letting α go to infinity we can use (1.18) combined with de-Poissonization techniques to prove asymptotic properties of the Plancherel measure. In particular the next theorem generalizes the results of [BDJ1] and [BDJ2]. Note, however, that we do not prove convergence of moments of the appropriately rescaled random variables. In Section 3 we will prove

THEOREM 1.4. *Let $x^{(j)}$ denote the j^{th} largest eigenvalue among the eigenvalues $x_1, \dots, x_M$ of a random $M \times M$ matrix from* GUE *with measure* (1.2). *There is a distribution function $F(t_1, \dots, t_k)$ on $\mathbb{R}^k$, see* (3.48), *such that*

$$\lim_{M\to\infty} \mathbb{P}_{\mathrm{GUE},M}\left[x^{(j)} \leq \sqrt{2M} + \frac{t_j}{\sqrt{2}M^{1/6}}, j = 1, \dots, k\right] = F(t_1, \dots, t_k), \tag{1.19}$$

for $(t_1, \dots, t_k) \in \mathbb{R}^k$, and

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{Plan},N}[\lambda_j \leq 2\sqrt{N} + t_j N^{1/6}, j = 1, \dots, k] = F(t_1, \dots, t_k), \tag{1.20}$$

for $(t_1, \dots, t_k) \in \mathbb{R}^k$.

We turn now to the random word problem. By a *word* of length N on M letters, $M, N \geq 1$, we mean a map $w : \{1, \dots, N\} \to \{1, \dots, M\}$. Let $W_{M,N}$ denote the set of all such words, and let $\mathbb{P}_{\mathrm{W},M,N}[\cdot]$ be the uniform probability distribution on $W_{M,N}$ where all M^N words have the same probability. A weakly increasing subsequence of w is a subsequence $w(i_1), \dots, w(i_m)$ such that $i_1 < \cdots < i_m$ and $w(i_1) \leq \cdots \leq w(i_m)$. Let $L(w)$ be the length of a longest weakly increasing subsequence in w. The RSK-correspondence defines a bijection from $W_{M,N}$ to the set of all pairs of Young tableaux (P, Q) of the same shape $\lambda \in \mathcal{P}^{(N)}$, where P is semistandard with elements in $\{1, \dots, M\}$ and Q is standard with elements in $\{1, \dots, N\}$. Under this correspondence $L(w) = \lambda_1$, the length of the first row. Note that we must have $\ell(\lambda) \leq M$, so $\ell \in \mathcal{P}_M^{(N)}$ which we can identify with $\Omega_M^{(N)}$. In this way we get a map $S : W_{M,N} \to \Omega_M^{(N)}$.

PROPOSITION 1.5. *The push-forward of the uniform distribution on $W_{M,N}$ by the map $S : W_{M,N} \to \Omega_M^{(N)}$ is*

$$\mathbb{P}_{\mathrm{W},M,N}[S^{-1}(\lambda)] = \mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}] \doteq \frac{N!}{M^N}(\prod_{j=1}^{M-1} \frac{1}{j!}) V_M(\lambda)^2 W_M(\lambda) \tag{1.21}$$

on $\Omega_M^{(N)}$. The Poissonization of this measure is the Charlier ensemble (1.16). *Consequently,*

$$\mathbb{P}_{\mathrm{W},M,N}[L(w) \leq t] = \mathbb{P}_{\mathrm{Ch,M,N}}[\lambda_1 \leq t], \tag{1.22}$$

and for the Poissonized word problem,

$$\mathbb{P}^{\alpha}_{\mathrm{W},M}[L(w) \leq t] \doteq \sum_{N=0}^{\infty} e^{-\alpha} \frac{\alpha^N}{N!} \mathbb{P}_{\mathrm{W},M,N}[L(w) \leq t] = \mathbb{P}^{\alpha}_{\mathrm{Ch},M}[\lambda_1 \leq t]. \tag{1.23}$$

Proof. See Section 4. □

The probability (1.23) can also be expressed as a Toeplitz determinant using Gessel's formula, [Ge], see also [TW3] and [BR1]. The formula (1.21) can be used to prove a conjecture by Tracy and Widom, [TW3]. This conjecture says that the Poissonized measure on Ω_M induced by the uniform distribution on words converges, after appropriate rescaling, to the $M \times M$ GUE measure (1.2). In Section 4 we will prove

THEOREM 1.6. *Let g be a continuous function on $\mathbb{R}^M$. Then*

$$\lim_{N\to\infty} \mathbb{E}_{\mathrm{Ch},M,N}\left[g\left(\frac{\lambda_1 - N/M}{\sqrt{2N/M}}, \dots, \frac{\lambda_M - N/M}{\sqrt{2N/M}}\right)\right] = M!\sqrt{\pi M}\int_{\mathbb{A}_M} g(x)\phi_{\mathrm{GUE},M}(x)dx_1\dots dx_{M-1}, \tag{1.24}$$

where $\mathbb{A}_M = \{x \in \mathbb{R}^M\,;\, x_1 > \dots > x_M$ and $x_1 + \dots + x_M = 0\}$. Furthermore

$$\lim_{\alpha\to\infty} \mathbb{E}^{\alpha}_{\mathrm{Ch},M}\left[g\left(\frac{\lambda_1 - \alpha/M}{\sqrt{2\alpha/M}}, \dots, \frac{\lambda_M - \alpha/M}{\sqrt{2\alpha/M}}\right)\right] = M!\int_{\{x\in\mathbb{R}^M\,;\,x_1>\dots>x_M\}} g(x)\phi_{\mathrm{GUE},M}(x)d^Mx. \tag{1.25}$$

The case when g only depends on λ_1 has been proved in [TW3] using very different methods.

The formula (1.23) can be used to analyze the asymptotics of the random variable $L(w)$ on $W_{M,N}$ as both M and N go to infinity.

THEOREM 1.7. *Let $F(t)$ be the Tracy-Widom distribution function* (1.5). *Then, for all $t \in \mathbb{R}$,*

$$\lim_{\alpha\to\infty} \mathbb{P}^{\alpha}_{\mathrm{W},M}\left[L(w) \le \frac{\alpha}{M} + 2\sqrt{\alpha} + \left(1 + \frac{\sqrt{\alpha}}{M}\right)^{2/3}\alpha^{1/6}t\right] = F(t). \tag{1.26}$$

Assume that $M = M(N) \to \infty$ as $N \to \infty$ in such a way that $(\log N)^{1/6}/M(N) \to 0$. *Then, for all $t \in \mathbb{R}$,*

$$\lim_{N\to\infty} \mathbb{P}_{\mathrm{W},M,N}\left[L(w) \le \frac{N}{M} + 2\sqrt{N} + \left(1 + \frac{\sqrt{N}}{M}\right)^{2/3}N^{1/6}t\right] = F(t). \tag{1.27}$$

Proof. See Section 4. □

Note that when $M \gg \alpha$, the leading order of the mean goes like $2\sqrt{\alpha}$ and the standard deviation like $\alpha^{1/6}$ just as for random permutations. When $M \ll \alpha$, we expect from (1.3) and (1.25) that

$$\begin{aligned} L(w) = \lambda_1 &\approx \alpha/M + \sqrt{2\alpha/M}(\sqrt{2M} + t/\sqrt{2}M^{1/6}) \\ &= \alpha/M + 2\sqrt{\alpha} + t\sqrt{\alpha}/M^{2/3}, \end{aligned}$$

which fits perfectly with (1.26).

In Section 5 we will consider two problems in combinatorial probability that relate to the Krawtchouk ensemble, namely Seppäläinen's simplified model of directed first-passage percolation and zig-zag paths in random domino tilings of the Aztec diamond introduced by Elkies, Kuperberg, Larsen and Propp. Since both problems require some definitions we will not state the results here. A third problem, random tilings of a hexagon by rhombi, which is related to the Hahn ensemble will also be discussed briefly.

2. The Plancherel measure as a limit of the Meixner ensemble

The setting is the same as in [Jo3]. Let $\mathcal{M}_N$ denote the set of all $N \times N$ matrices with elements in $\mathbb{N}$. We define a probability measure, $\mathbb{P}_N^q[\cdot]$ on $\mathcal{M}_N$ by letting each element a_{ij} in $A \in \mathcal{M}_N$ be geometrically distributed with parameter $q \in (0,1)$, and requiring all elements to be independent. Then

$$\mathbb{P}_N^q[A] = (1-q)^{N^2} q^{\Sigma(A)}, \tag{2.1}$$

$A \in \mathcal{M}_N$, where $\Sigma(A) = \sum_{i,j=1}^N a_{ij}$. Let $\mathcal{M}_N(k)$ denote the set of all A in $\mathcal{M}_N$ for which $\Sigma(A) = k$. Note that by (2.1) all matrices in $\mathcal{M}_N(k)$ have the same probability. Furthermore we let $\tilde{\mathcal{M}}_N(k)$ be the set of all matrices A in $\mathcal{M}_N(k)$ for which $\sum_i a_{ij} \le 1$ for each j and $\sum_j a_{ij} \le 1$ for each i; $\tilde{\mathcal{M}}_N = \cup_k \tilde{\mathcal{M}}_N(k)$. By taking the appropriate submatrix of $A \in \tilde{\mathcal{M}}_N(k)$ we get a permutation matrix and hence a unique permutation. This defines a map $R : \tilde{\mathcal{M}}_N(k) \to S_k$, where S_k is the k^{th} symmetric group. Note that if q is very small a typical element in $\mathcal{M}_N$ belongs to $\tilde{\mathcal{M}}_N(k)$ for some k. This is the crucial observation for what follows. The RSK-correspondence defines a map $K : \mathcal{M}_N(k) \to \mathcal{P}^{(k)}$, and also a map $S : S_k \to \mathcal{P}^{(k)}$. The number of elements in S_k that are mapped to the same λ equals $(f^\lambda)^2$. It is not difficult to see that if $A \in \tilde{\mathcal{M}}_N(k)$ then $K(A) = S(R(A))$. Let $g \in \mathcal{G}_L$. It is proved in [Jo3] that

$$\mathbb{E}_N^q[g(K(A))] = \mathbb{E}_{\text{Me},N,N}^q[g(\lambda)]. \tag{2.2}$$

With these preparations we are ready for the

Proof of Theorem 1.1. By (2.2) we see that in order to prove (1.15) it suffices to show that

$$\lim_{N\to\infty} \mathbb{E}_N^{\alpha/N^2}[g(K(A))] = \mathbb{E}_{\text{Plan}}^{\alpha}[g]. \tag{2.3}$$

Note that $\mathbb{P}_N^q[\{A\}\big|\Sigma(A) = k] = 1/\#\mathcal{M}_N(k)$, where $\#\mathcal{M}_N(k) = \binom{N^2-1+m}{m}$, and $\mathbb{P}_N^q[\Sigma(A) = k] = \#\mathcal{M}_N(k)(1-q)^{N^2}q^k$, by (2.1). Thus

(2.4)

$$\begin{aligned}
\mathbb{E}_N^q[g(K(A))\chi_{\tilde{\mathcal{M}}_N}(A)] &= \sum_{k=0}^{\infty} \mathbb{E}_N^q[g(K(A))\chi_{\tilde{\mathcal{M}}_N}(A)\big|\Sigma(A) = k]\mathbb{P}_N^q[\Sigma(A) = k] \\
&= \sum_{k=0}^{\infty} \sum_{A\in\tilde{M}_N(k)} g(K(A))(1-q)^{N^2}q^k \\
&= (1-q)^{N^2}\sum_{k=0}^{\infty} q^k \sum_{\lambda\in\mathcal{P}^{(k)}} g(\lambda)\#\{A\in\tilde{M}_N(k)\,;\, K(A)=\lambda\}.
\end{aligned}$$

The number of matrices in $\tilde{M}_N(k)$ which are mapped to the same permutation by R is $\binom{N}{k}^2$, since there are $\binom{N}{k}$ ways of choosing the rows and $\binom{N}{k}$ ways of choosing the columns that select the submatrix. Since $K = S\circ R$ we obtain

$$\#\{A\in\tilde{M}_N(k)\,;\, K(A)=\lambda\} = \binom{N}{k}^2 (f^\lambda)^2.$$

Together with (2.4) this yields

$$\begin{aligned}
\mathbb{E}_N^q[g(K(A))\chi_{\tilde{\mathcal{M}}_N}(A)] &= (1-q)^{N^2}\sum_{k=0}^{\infty}\frac{q^k}{k!}\frac{N!^2}{(N-k)!^2}\sum_{\lambda\in\mathcal{P}^{(k)}} g(\lambda)\frac{(f^\lambda)^2}{k!} \\
&= (1-q)^{N^2}\sum_{k=0}^{\infty}\frac{q^k}{k!}\frac{N!^2}{(N-k)!^2}\mathbb{E}_{\text{Plan},k}[g] \\
&= (1-\alpha/N^2)^{N^2}\sum_{k=0}^{\infty}\frac{\alpha^k}{k!}\left(\frac{N!}{N^k(N-k)!}\right)^2\mathbb{E}_{\text{Plan},k}[g],
\end{aligned}$$

if we pick $q = \alpha/N^2$. Since $N!(N^k(N-k)!)^{-1} \le 1$ and converges to 1 as $N\to\infty$ for each fixed k and furthermore $\mathbb{E}_{\text{Plan},k}[g] \le c(g)^{\max(L,k)}$, it follows from the dominated convergence theorem that

$$\lim_{N\to\infty} \mathbb{E}_N^{\alpha/N^2}[g(K(A))\chi_{\tilde{\mathcal{M}}_N}(A)] = \mathbb{E}_{\text{Plan}}^{\alpha}[g]. \tag{2.5}$$

To deduce (2.3) from (2.5) we have to show that if $\tilde{\mathcal{M}}_N^* = \mathcal{M}_N \setminus \tilde{\mathcal{M}}_N$, then

$$\mathbb{E}_N^{\alpha/N^2}[g(K(A))\chi_{\tilde{\mathcal{M}}_N^*}(A)] = 0. \tag{2.6}$$

By the Cauchy-Schwarz' inequality, the left-hand side of (2.6) is

$$\leq \mathbb{E}_N^{\alpha/N^2}[g(K(A))^2]^{1/2}\mathbb{P}_N^{\alpha/N^2}[\tilde{\mathcal{M}}_N^*]^{1/2}. \tag{2.7}$$

If $\lambda = K(A)$, then $\ell(\lambda) \leq \Sigma(A)$ and from the definition (1.11) of g it follows that

$$|g(K(A))| \leq c(g)^{\max(L,\ell(\lambda))} \leq c(g)^{L+\Sigma(A)}.$$

Thus,

$$\mathbb{E}_N^{\alpha/N^2}[g(K(A))^2] \leq c(g)^{2L}\sum_{k=0}^{\infty} c(g)^{2k}\mathbb{P}_N^{\alpha/N^2}[\Sigma(A) = k].$$

Since,

$$\mathbb{P}_N^{\alpha/N^2}[\Sigma(A) = k] = \binom{N^2-1+k}{k}\left(1-\frac{\alpha}{N^2}\right)^{N^2}\left(\frac{\alpha}{N^2}\right)^k \to e^{-\alpha}\frac{\alpha^k}{k!}$$

as $N \to \infty$, it is not hard to show that

$$\mathbb{E}_N^{\alpha/N^2}[g(K(A))^2] \leq C(\alpha, g), \tag{2.8}$$

for all $N \geq 1$, where $C(\alpha, g)$ depends only on α and $c(g)$.

Next, we note that

$$\tilde{\mathcal{M}}_N^* \subseteq \bigcup_{i=1}^N \left\{\sum_j a_{ij} \geq 2\right\} \cup \bigcup_{j=1}^N \left\{\sum_i a_{ij} \geq 2\right\}$$

and hence

$$\mathbb{P}_N^{\alpha/N^2}[\tilde{\mathcal{M}}_N^*] \leq 2N\mathbb{P}_N^{\alpha/N^2}\left[\sum_j a_{ij} \geq 2\right].$$

Since, $\mathbb{P}_N^q[\sum_j a_{ij} \geq 2] = 1 - (1-q)^N - N(1-q)^{N-1}$, we obtain

$$\mathbb{P}_N^{\alpha/N^2}[\tilde{\mathcal{M}}_N^*] \leq \frac{C\alpha^2}{N}.$$

Together with (2.7) and (2.8) this implies (2.6) and we are done. □

It is also possible to give a more direct proof based on the explicit formulas similarly to what will be done with the Charlier ensemble in the next section. Above we have emphasized the probabilistic and geometric picture.

3. The Plancherel measure as a limit of the Charlier ensemble

3.1. *The limit of the Charlier ensemble.* The Charlier ensemble is defined by (1.16). It can be obtained as a limit of the Meixner ensemble (1.14) by taking $q = \alpha/MN$ and letting $N \to \infty$ with M fixed. In this limit

$$(3.1) \qquad (1-q)^{MN} \prod_{j=0}^{M-1} \frac{(N-M)!}{j!(N-M+j)!} \prod_{i=1}^{M} \binom{\lambda_i + N - i}{\lambda_i + M - i} q^{\lambda_i}$$
$$\to \left(\prod_{j=1}^{M-1} \frac{1}{j!} \right) W_M(\lambda) \prod_{i=1}^{M} \left[\left(\frac{\alpha}{M} \right)^{\lambda_i} e^{-\alpha/M} \right],$$

so we obtain (1.16). In light of Theorem 1.1 we see that it is reasonable to expect that the Poissonized Plancherel measure should be the limit of the Charlier ensemble as $M \to \infty$. The interpretation of the Charlier ensemble in connection with random words, Proposition 1.5, also supports this since a random word in the limit $M \to \infty$ is like a permutation (no letter is used twice), see also [TW3]. We will give an analytical proof of Theorem 1.2 that does not use the RSK-correspondence. We start with the following simple but important lemma.

LEMMA 3.1. *If* $M \geq \ell(\lambda)$, *then*

$$(3.2) \qquad V_M(\lambda) W_M(\lambda) = V_{\ell(\lambda)}(\lambda) W_{\ell(\lambda)}(\lambda).$$

Proof. We may assume that $M > \ell(\lambda)$. Note that, by definition, $\lambda_i = 0$ if $i > \ell(\lambda)$. Hence,

$$V_M(\lambda) = V_{\ell(\lambda)}(\lambda) \prod_{i=1}^{\ell(\lambda)} \prod_{j=\ell(\lambda)+1}^{M} (\lambda_i + j - i) \prod_{\ell(\lambda) < i < j \leq M} (j - i)$$
$$= V_{\ell(\lambda)}(\lambda) \prod_{i=1}^{\ell(\lambda)} \frac{(\lambda_i + M - i)!}{(\lambda_i + \ell(\lambda) - i)!} \prod_{\ell(\lambda) < i < j \leq M} (j - i).$$

Thus in order to prove (3.2) we must show that

$$\prod_{\ell(\lambda) < i < j \leq M} (j - i) = \prod_{i=\ell(\lambda)+1}^{M} (\lambda_i + M - i)!,$$

but this is immediate since $\lambda_i = 0$ if $i > \ell(\lambda)$. □

Proof of Theorem 1.2. It follows from the definition (1.12) of $g(\lambda)$ that

$$(3.3) \qquad |g(\lambda)| \leq c(g)^{\max(\ell(\lambda), L)} \leq c(g)^{\ell(\lambda)+L}.$$

Let $\mathbb{P}_{\mathrm{Ch,M,N}}$ be defined by (1.21). Then,

(3.4)
$$\begin{aligned}\mathbb{E}^{\alpha}_{\mathrm{Ch},M}[g] &= \sum_{\lambda\in\Omega_M} g(\lambda)\left(\prod_{j=1}^{M-1}\frac{1}{j!}\right)V_M(\lambda)^2 W_M(\lambda)\prod_{j=1}^{M}\left[\left(\frac{\alpha}{M}\right)^{\lambda_j} e^{-\alpha/M}\right]\\ &= \sum_{N=0}^{\infty} e^{-\alpha}\frac{\alpha^N}{N!}\sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}].\end{aligned}$$

Thus, by (3.3) and the fact that $\ell(\lambda)\le N$ if $\lambda\in\Omega_M^{(N)}$,

$$\left|\sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}]\right| \le c(g)^{L+N}, \tag{3.5}$$

since $\mathbb{P}_{\mathrm{Ch,M,N}}$ is a probability measure on $\Omega_M^{(N)}$. Given $\varepsilon>0$ we can choose K so large that

$$\left|\sum_{N=K+1}^{\infty} e^{-\alpha}\frac{\alpha^N}{N!}c(g)^{L+N}\right| \le \varepsilon. \tag{3.6}$$

Consequently,

$$\left|\mathbb{E}^{\alpha}_{\mathrm{Ch},M}[g] - \sum_{N=0}^{K} e^{-\alpha}\frac{\alpha^N}{N!}\sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}]\right| \le \varepsilon. \tag{3.7}$$

If $M\ge K\ge N\ge \ell(\lambda)$, $\lambda\in\Omega_M^{(N)}$, we can identify $\Omega_M^{(N)}$ with $\mathcal{P}^{(N)}$ and use (3.2) to write

(3.8)
$$\begin{aligned}&\sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}]\\ &\qquad= \sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)N!V_{\ell(\lambda)}(\lambda)^2 W_{\ell(\lambda)}(\lambda)^2\prod_{j=1}^{M}\frac{(\lambda_j+M-j)!}{M^{\lambda_j}}\prod_{j=1}^{M-1}\frac{1}{j!}\\ &\qquad= \sum_{\lambda\in\mathcal{P}^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Plan},N}[\{\lambda\}]\prod_{j=1}^{\ell(\lambda)}\frac{(\lambda_j+M-j)!}{M^{\lambda_j}(M-j)!},\end{aligned}$$

where the last equality is a straightforward computation using the fact that $\lambda_j=0$ if $j>\ell(\lambda)$. Now,

$$\prod_{j=1}^{\ell(\lambda)}\frac{(\lambda_j+M-j)!}{M^{\lambda_j}(M-j)!} = \prod_{j=1}^{\ell(\lambda)}\left(1-\frac{j-1}{M}\right)\dots\left(1-\frac{j-\ell(\lambda)}{M}\right)$$

which goes to 1 as $M \to \infty$ for a fixed λ. Since the sum in (3.7) is, for a fixed K, a sum over finitely many λ, we obtain

$$\lim_{M\to\infty} \sum_{N=0}^{K} e^{-\alpha} \frac{\alpha^N}{N!} \sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}] = \sum_{N=0}^{K} e^{-\alpha} \frac{\alpha^N}{N!} \sum_{\lambda\in\mathcal{P}^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Plan},N}[\{\lambda\}]. \tag{3.9}$$

Using (3.6) and the fact that $\mathbb{P}_{\mathrm{Plan},N}$ is a probability measure on $\mathcal{P}^{(N)}$, we obtain

$$\left|\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] - \sum_{N=0}^{K} e^{-\alpha} \frac{\alpha^N}{N!} \sum_{\lambda\in\Omega_M^{(N)}} g(\lambda)\mathbb{P}_{\mathrm{Plan},N}[\{\lambda\}]\right| \le \varepsilon. \tag{3.10}$$

The theorem now follows from (3.7), (3.9) and (3.10). □

3.2. *Coulomb gas interpretation of the Plancherel measure.* As $M \to \infty$ the number of particles in the Coulomb gas representation of the Charlier ensemble goes to infinity, so a Coulomb gas interpretation of the Plancherel measure is not immediate. We will now show that we can actually approximate $\mathbb{P}^{\alpha}_{\mathrm{Plan}}$ by a Coulomb gas with K particles, which gives a good approximation if K is chosen large enough (depending on α).

Consider the Poissonization of the restriction of the Plancherel measure to $\mathcal{P}_M^{(N)}$,

$$F_M^{\alpha}[g] = e^{-\alpha} \sum_{N=0}^{\infty} \frac{\alpha^N}{N!} \sum_{\lambda\in P_M^{(N)}} g(\lambda) \frac{(f^{\lambda})^2}{N!}$$

for $g \in \mathcal{G}_L$. If $M \ge L$ it follows from (1.12), (1.13) and Lemma 3.1 that

$$F_M^{\alpha}[g] = e^{-\alpha} \sum_{\lambda\in\Omega_M} g(\lambda) V_M(\lambda)^2 W_M(\lambda)^2 \prod_{i=1}^{M} \alpha^{\lambda_i}.$$

When M is large, we expect that $F_M^{\alpha}[g]$ and $\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g]$ should be close.

LEMMA 3.2. *Assume that $g \in \mathcal{G}_L$ and let $d > 0$ be given. There is a numerical constant C such that if $M \ge \max(L, \alpha\exp(d+1))$, then*

$$\left|\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] - F_M^{\alpha}[g]\right| \le C(c(g)e^{-d})^M. \tag{3.11}$$

Proof. Set

$$R_{N,M}[g] = \mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] - F_M^{\alpha}[g] = \sum_{N=0}^{\infty} e^{-\alpha} \frac{\alpha^N}{N!} \sum_{\lambda\in\mathcal{P}^{(N)}\setminus\mathcal{P}_M^{(N)}} g(\lambda) \frac{(f^{\lambda})^2}{N!}.$$

If $N \le M$, then $R_{N,M}[g] = 0$ since then $\ell(\lambda) \le \sum_i \lambda_i = N \le M$, so $\mathcal{P}^{(N)} = \mathcal{P}_M^{(N)}$. If $N > M \ge L$, then $|g(\lambda)| \le c(g)^N$ since $\lambda_i = 0$ if $i > N$. Thus,

$$\begin{aligned}\left|\mathbb{E}^{\alpha}_{\mathrm{Plan}}[g] - F^{\alpha}_M[g]\right| &\le \sum_{N=M+1}^{\infty} e^{-\alpha}\frac{\alpha^N}{N!}\left|R_{N,M}[g]\right| \\ &\le \sum_{N=M+1}^{\infty} e^{-\alpha}\frac{\alpha^N}{N!}c(g)^N.\end{aligned}$$

This last sum is estimated as follows. By Stirling's formula there is a numerical constant C such that $\exp(-\alpha)\alpha^N/N! \le C\exp(-\alpha f(N/\alpha))$, where $f(x) = x\log x + 1 - x$. If $N/\alpha \ge \exp(d+1)$, then $f(N/\alpha) \ge dN/\alpha$, and so $\exp(-\alpha)\alpha^N/N! \le \exp(-dN)$. The lemma is proved. □

Recall from the introduction that $\mathcal{P}_M$ can be naturally identified with Ω_M. For $K < M$ we define

$$\Omega_{M,K} = \{\lambda \in \Omega_M\,;\, \lambda_{K+1} = \cdots = \lambda_M = 0\},$$

and $\Omega^*_{M,K} = \Omega_M \setminus \Omega_{M,K}$. If $1 \le j \le M-K$ we set

$$\Omega^*_{M,K}(j) = \{\lambda \in \Omega^*_{M,K}\,;\, \lambda_{M+1-j} > 0 \text{ but } \lambda_i = 0, M+1-j < i \le M\},$$

so that $\Omega^*_{M,K} = \cup_{j=1}^{M-K}\Omega^*_{M,K}(j)$. The next lemma asserts that $\ell(\lambda)$ is not too large for typical λ that we will consider.

LEMMA 3.3. *Let $g \in \mathcal{G}_L$ be generated by f. Assume that f satisfies*

$$0 \le f(x) \le C_0 f(x-1) \tag{3.12}$$

for all $x \in \mathbb{Z}$ and some constant C_0. Then

$$e^{-\alpha}\sum_{\lambda\in\Omega^*_{M,K}(j)} g(\lambda)V_M(\lambda)^2 W_M(\lambda)^2\prod_{i=1}^{M}\alpha^{\lambda_i} \le \frac{(C_0\alpha)^{M-j+1}}{(M-j+1)!^2}F^{\alpha}_M[g]. \tag{3.13}$$

Proof. It will be most convenient to use the discrete Coulomb gas representation. Set $x_j = \lambda_{M+1-j} + j - 1$, $j = 1,\dots,M$ and let $\Delta_M(x) = \prod_{1\le i<j\le M}(x_j - x_i)$ be the Vandermonde determinant. Also, set $A = \{x \in \mathbb{N}^M\,;\, 0 \le x_1 < \cdots < x_M\}$ and $A_j = \{x \in A\,;\, x_i < i \text{ for } i < j \text{ and } x_j \ge j\}$, $j = 1,\dots,M$. Note that $\lambda \in \Omega^*_{M,K}(j)$ translates into $x \in A_j$. If $x \in A_j$, then $x_i = i-1$ for $i = 1,\dots,j-1$ and we have the first *hole* in the particle configuration x at $j-1$. Now,

$$\sum_{\lambda\in\Omega^*_{M,K}(j)} g(\lambda)V_M(\lambda)^2 W_M(\lambda)^2\prod_{i=1}^{M}\alpha^{\lambda_i} = \sum_{x\in A_j}\Delta_M(x)^2\prod_{i=1}^{M}\frac{\alpha^{x_i}}{x_i!^2}f(x_i+K-M). \tag{3.14}$$

We want to show that, with high probability, the first hole must be fairly close to M. Define $T_j : A_j \to A$ by $T_j(x) = (x_1, \dots, x_{j-1}, x_j - 1, \dots, x_M - 1) = x'$. Clearly, $T_j : A_j \to T_j(A_j)$ is a bijection. Write

$$L_M^\alpha(x) = \Delta_M(x)^2 \prod_{i=1}^{M} \frac{\alpha^{x_i}}{x_i!^2} f(x_i + K - M).$$

For $x \in A_j$,

$$\left(\frac{\Delta_M(x)}{\Delta_M(x')}\right)^2 \prod_{i=1}^{M} \frac{(x_i'!)^2}{(x_i!)^2} \alpha^{x_i - x_i'} = \alpha^{M-j+1} \left(\frac{\Delta_M(x)}{\Delta_M(x')}\right)^2 \prod_{i=j}^{M} \frac{1}{x_i^2}.$$

Since

$$\prod_{i=j}^{M} \frac{1}{x_i^2} \le \prod_{i=j}^{M} \frac{1}{i^2} = \left(\frac{(j-1)!}{M!}\right)^2$$

and

$$\frac{\Delta_M(x)}{\Delta_M(x')} = \prod_{k=j}^{M} \frac{x_k}{x_k - (j-1)} \le \prod_{k=j}^{M} \frac{k}{k - (j-1)} = \binom{M}{j-1}$$

if $x \in A_j$, we obtain, using our assumption on f,

$$L_M^\alpha(x) \le \frac{(C_0\alpha)^{M-j+1}}{(M-j+1)!^2} L_M^\alpha(T_j(x)).$$

Inserting this into (3.14) yields

$$\begin{aligned} e^{-\alpha} \sum_{\lambda \in \Omega^*_{M,K}(j)} g(\lambda) V_M(\lambda)^2 W_M(\lambda)^2 \prod_{i=1}^{M} \alpha^{\lambda_i} &= e^{-\alpha} \sum_{x \in A_j} L_M^\alpha(x) \\ &\le \frac{(C_0\alpha)^{M-j+1}}{(M-j+1)!^2} e^{-\alpha} \sum_{x \in A_j} L_M^\alpha(T_j(x)) \le \frac{(C_0\alpha)^{M-j+1}}{(M-j+1)!^2} F_M^\alpha[g], \end{aligned}$$

and the lemma is proved. □

LEMMA 3.4. *Let $g \in \mathcal{G}_L$ be generated by f which satisfies* (3.12). *Assume that $M > K \ge \max(L, e\sqrt{2C_0\alpha})$. Then,*

$$\left|F_M^\alpha[g] - F_K^\alpha[g]\right| \le 2\left(\frac{C_0\alpha e^2}{(K+1)^2}\right)^{K+1} F_M^\alpha[g]. \tag{3.15}$$

Proof. If $\lambda \in \Omega_{M,K}$ then $\ell(\lambda) \le K < M$ and hence by Lemma 3.1, (1.11) and the fact that $\Omega_{M,K}$ and Ω_K can be identified we obtain

$$e^{-\alpha} \sum_{\lambda \in \Omega_{M,K}} g(\lambda) V_M(\lambda)^2 W_M(\lambda)^2 \prod_{i=1}^{M} \alpha^{\lambda_i} = F_K^\alpha[g].$$

The left-hand side of (3.15) is

$$
\begin{aligned}
&\leq \sum_{j=1}^{M-K} e^{-\alpha} \sum_{\lambda \in \Omega^*_{M,K}(j)} g(\lambda) V_M(\lambda)^2 W_M(\lambda)^2 \prod_{i=1}^{M} \alpha^{\lambda_i} \\
&\leq \sum_{j=1}^{M-K} \frac{(C_0\alpha)^{M-j+1}}{(M-j+1)!^2} F_M^\alpha[g] \\
&\leq \sum_{j=K+1}^{\infty} (C_0\alpha)^j \frac{1}{j!^2} F_M^\alpha[g] \leq 2\left(\frac{C_0 \alpha e^2}{(K+1)^2}\right)^{K+1} F_M^\alpha[g]
\end{aligned}
$$

by (3.13). This completes the proof of the lemma. □

We can now demonstrate how the Plancherel measure can be approximated by a Coulomb gas, (compare with the discussion in the Appendix in [BDJ1]).

PROPOSITION 3.5. *Assume that $g \in \mathcal{G}_K$ is generated by f which satisfies* (3.12). *Let $K = [r\sqrt{\alpha}]$, $r > \sqrt{2C_0e^2}$. Then,*

$$
\mathbb{E}^\alpha_{\text{Plan}}[g] = (1 + O(r^{-K})) \frac{1}{Z_K^\alpha} \sum_{h \in \mathbb{N}^K} \Delta_K(h)^2 \prod_{i=1}^{K} \frac{\alpha^{h_i}}{h_i!^2} \prod_{i=1}^{K} f(h_i), \tag{3.16}
$$

where

$$
Z_K^\alpha = \sum_{h \in \mathbb{N}^K} \Delta_K(h)^2 \prod_{i=1}^{K} \frac{\alpha^{h_i}}{h_i!^2}.
$$

Proof. Write

$$
\mathbb{E}^\alpha_{\text{Plan}}[g] = \mathbb{E}^\alpha_{\text{Plan}}[g] - F_M^\alpha[g] + \frac{F_M^\alpha[1]}{\mathbb{E}^\alpha_{\text{Plan}}[1]} \frac{F_M^\alpha[g]}{F_K^\alpha[g]} \frac{F_K^\alpha[1]}{F_M^\alpha[1]} \frac{F_K^\alpha[g]}{F_K^\alpha[1]}. \tag{3.17}
$$

By Lemma 3.2

$$
\lim_{M \to \infty} F_M^\alpha[g] = \mathbb{E}^\alpha_{\text{Plan}}[g]. \tag{3.18}
$$

By Lemma 3.4 and the choice of K and r

$$
\frac{F_K^\alpha[g]}{F_M^\alpha[g]} = 1 + O(r^{-K}), \tag{3.19}
$$

for any $M > K$, and similarly with g replaced by 1. Using (3.18) and (3.19) in (3.17) and letting $M \to \infty$ we obtain

$$
\mathbb{E}^\alpha_{\text{Plan}}[g] = (1 + O(r^{-K})) \frac{F_K^\alpha[g]}{F_K^\alpha[1]},
$$

which is exactly (3.16). The proposition is proved. □

Thus we have an approximate Coulomb gas picture of the (shifted) rows of λ under the Plancherel measure analogous to Dyson's Coulomb gas picture for the eigenvalues of a random matrix.

Remark 3.6. The confining potential for the discrete Coulomb gas in (3.16) is

$$V_K^\alpha[h_i] = -\frac{1}{K}\log(\alpha^{h_i}/(h_i!)^2)$$

with limit

$$\lim_{\alpha\to\infty} V_K^\alpha[Kx] = 2[x\log x + (\log r - 1)x] = V(x).$$

We can now use general techniques for Coulomb gases, see e.g. [Jo1], [Jo3], to deduce asymptotic distribution properties. The potential V has the (constrained) equilibrium measure; compare with Section 2 in [Jo3], $u(t)dt$, where

$$u(t) = \begin{cases} 1, & \text{if } 0 \le t \le 1 - 2/r \\ \frac{1}{2} - \frac{1}{\pi}\arcsin(\frac{r}{2}(t-1)), & \text{if } 1 - 2/r \le t \le 1 + 2/r \\ 0, & \text{if } t \ge 1 + 2/r. \end{cases}$$

Pick $f(t) = \exp(\phi(t/[r\sqrt{\alpha}]))$ with $\phi : \mathbb{R} \to \mathbb{R}$ continuous, bounded together with its derivative and $\phi(t) = 0$ if $t \le 0$. Then,

$$g(\lambda) = \prod_{i=1}^{\infty} \exp(\phi(\frac{\lambda_i + [r\sqrt{\alpha}] - i}{[r\sqrt{\alpha}]})).$$

If we pick r sufficiently large (depending on ϕ) we can use (3.19) and (3.20) to show that

$$\lim_{\alpha\to\infty} \frac{1}{[r\sqrt{\alpha}]} \log \mathbb{E}^\alpha_{\text{Plan}}[g(\lambda)] = \int_0^{1+2/r} \phi(t)u(t)dt. \tag{3.20}$$

From the limit (3.20) it is possible to deduce Vershik and Kerov's Ω-law for the asymptotic shape of the Young diagram, [VK], see also [AD], where an outline of the argument using the hook-integral is given. (The r-dependence in the formulas above goes away after appropriate rescaling.) From what has been said above we see that the Ω-law is directly related to an equilibrium measure for a discrete Coulomb gas. Using the general results in [Jo3] we can use (3.16) to show upper- and lower-tail large deviation formulas for λ_1 (= the length of the longest increasing subsequence in a random permutation) under the Poissonized Plancherel measure. These formulas have been proved in [Se1], [Jo2] and [DZ] by other methods and we will not give the details of the new proof.

3.3. *Proof of Theorem* 1.3. We will now use Theorem 1.2 to prove Theorem 1.3, but before we can do this we need certain asymptotic results for

Charlier polynomials and Bessel functions. Let

$$w_a(x) = e^{-a}\frac{a^x}{x!}, \quad x \in \mathbb{N}, \quad a > 0.$$

The normalized Charlier polynomials, $c_n(x; a)$, $n \geq 0$ are orthonormal on $\mathbb{N}$ with respect to this weight. The relevant value of the parameter a for us will be $a = \alpha/M$, and we define the *Charlier kernel*

(3.21)

$$K^{\alpha}_{\mathrm{Ch},M}(x,y) = \sqrt{\alpha}\frac{c_M(x;\frac{\alpha}{M})c_{M-1}(y;\frac{\alpha}{M}) - c_{M-1}(x;\frac{\alpha}{M})c_M(y;\frac{\alpha}{M})}{x-y} \times w_{\alpha/M}(x)^{1/2}w_{\alpha/M}(y)^{1/2},$$

for $x \neq y$ and

(3.22) $K^{\alpha}_{\mathrm{Ch},M}(x,x)$

$$= \sqrt{\alpha}w_{\alpha/M}(x)\left[c'_M\left(x;\frac{\alpha}{M}\right)c_{M-1}\left(x;\frac{\alpha}{M}\right) - c_{M-1}\left(x;\frac{\alpha}{M}\right)c'_M\left(x;\frac{\alpha}{M}\right)\right].$$

The polynomials $c_n(x; \alpha/M)$, $n \geq 0$, have the generating function

$$\sum_{n=0}^{\infty}\left(\frac{\alpha}{M}\right)^{n/2}\frac{1}{\sqrt{n!}}c_n\left(x;\frac{\alpha}{M}\right)w^n = e^{-\alpha w/M}(1+w)^x.$$

It follows from this formula that we have the following integral representations. If $0 < r \leq \sqrt{\alpha}/M$, then

$$c_n(x;\frac{\alpha}{M}) = \sqrt{\frac{n!}{M^n}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-\sqrt{\alpha}re^{i\theta}}\left(1+\frac{Mre^{i\theta}}{\sqrt{\alpha}}\right)^x\frac{1}{(re^{i\theta})^n}d\theta \tag{3.23}$$

and if $\sqrt{\alpha}/M < r$, then

(3.24)

$$c_n(x;\frac{\alpha}{M}) = \sqrt{\frac{n!}{M^n}}\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-\sqrt{\alpha}re^{i\theta}}\left(1+\frac{Mre^{i\theta}}{\sqrt{\alpha}}\right)^x\frac{1}{(re^{i\theta})^n}d\theta - (-1)^n\frac{\sin \pi x}{\pi}\int_{\sqrt{\alpha}/M}^{r}e^{\sqrt{\alpha}s}\left(\frac{Ms}{\sqrt{\alpha}}-1\right)^x s^{-n}\frac{ds}{s},$$

for any $x \in \mathbb{R}$, where the powers are defined using the principal branch of the logarithm.

We want to write the Charlier kernel in a form that will be convenient for later asymptotic analysis. Define, for a given $r > 0$, $x \in \mathbb{Z}$,

$$A^{\alpha}_M(x) = \sqrt{\alpha}\frac{M!}{M^M}w_{\alpha/M}(x)\left(1+\frac{M}{\sqrt{\alpha}}\right)^{2x}e^{-2\sqrt{\alpha}},$$

$$D^{\alpha,r}_M(x,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi}g(re^{i\theta})e^{\sqrt{\alpha}(1-re^{i\theta})}\left(\frac{\sqrt{\alpha}+Mre^{i\theta}}{\sqrt{\alpha}+M}\right)^x\frac{1}{(re^{i\theta})^M}d\theta,$$

$$F_M^{\alpha,r}(x,g)=(-1)^{x+M+1}\int_{\sqrt{\alpha}/M}^{r} g(s)e^{\sqrt{\alpha}(1+s)}\left|\frac{\sqrt{\alpha}-Ms}{\sqrt{\alpha}+M}\right|^{x}s^{-M}\frac{ds}{s},$$

if $r>\sqrt{\alpha}/M$, and if $r\le\sqrt{\alpha}/M$, then $F_M^{\alpha,r}(x,g)=0$. Then, some computation shows that when x is an *integer* (the case we are interested in),

(3.25)
$$K_{\mathrm{Ch},M}^{\alpha}(x,y)=\sqrt{A_M^{\alpha}(x)A_M^{\alpha}(y)}\frac{D_M^{\alpha,r}(x,g_1)D_M^{\alpha,r}(y,g_2)-D_M^{\alpha,r}(x,g_2)D_M^{\alpha,r}(y,g_1)}{x-y},$$

when $x\neq y$, and

(3.26)
$$\begin{aligned}K_{\mathrm{Ch},M}^{\alpha}(x,x)&=A_M^{\alpha}(x)\big[D_M^{\alpha,r}(x,g_2)D_M^{\alpha,r}(x-1,g_3)-D_M^{\alpha,r}(x,g_1)D_M^{\alpha,r}(x-1,g_4)\big]\\&\quad+A_M^{\alpha}(x)\big[F_M^{\alpha,r}(x,g_1)D_M^{\alpha,r}(x,g_2)-F_M^{\alpha,r}(x,g_2)D_M^{\alpha,r}(x,g_1)\big],\end{aligned}$$

where $g_1(z)\equiv 1$, $g_2(z)=z-1$,

$$g_3(z)=\left(\frac{\sqrt{\alpha}+Mz}{\sqrt{\alpha}+M}\right)\log\left(\frac{\sqrt{\alpha}+Mz}{\sqrt{\alpha}+M}\right),$$

and $g_4(z)=g_2(z)g_3(z)$. Note that all the g_i's are bounded on $|z|=r$.

The discrete Bessel kernel is defined by (1.10) for $x\neq y$ and

$$B^{\alpha}(x,x)=\sqrt{\alpha}[L_x(2\sqrt{\alpha})J_{x+1}(2\sqrt{\alpha})-J_x(2\sqrt{\alpha})L_{x+1}(2\sqrt{\alpha})] \tag{3.27}$$

for $x=y$, where $L_x(t)=\frac{d}{dx}J_x(t)$. The Bessel function has the integral representation

(3.28)
$$J_x(2\sqrt{\alpha})=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{\sqrt{\alpha}(\frac{1}{r}e^{-i\theta}-re^{i\theta})+ix\theta}r^x d\theta-\frac{\sin\pi x}{\pi}\int_0^r e^{\sqrt{\alpha}(-1/s+s)}s^x\frac{ds}{s},$$

for $x\in\mathbb{R}$, $r>0$. Differentiation shows that for *integer* x,

$$\begin{aligned}L_x(2\sqrt{\alpha})&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(re^{i\theta})e^{\sqrt{\alpha}(\frac{1}{r}e^{-i\theta}-re^{i\theta})+ix\theta}r^x d\theta\\&\quad-(-1)^x\int_0^r e^{\sqrt{\alpha}(-1/s+s)}s^x\frac{ds}{s}.\end{aligned} \tag{3.29}$$

The next lemma shows that the discrete Bessel kernel is the $M\to\infty$ limit of the Charlier kernel and establishes some technical estimates. (We will only consider the case when x,y are integers but this restriction can be removed.)

LEMMA 3.7. *For any $x,y\in\mathbb{Z}$,*

(i)

$$\lim_{M\to\infty}K_{\mathrm{Ch},M}^{\alpha}(M+x,M+y)=B^{\alpha}(x,y). \tag{3.30}$$

(ii)

$$B^{\alpha}(x,y)=\sum_{k=1}^{\infty}J_{x+k}(2\sqrt{\alpha})J_{y+k}(2\sqrt{\alpha}). \tag{3.31}$$

Furthermore, there is a constant $C = C(\alpha, L)$, *such that*
(iii)

$$\sum_{x=-L}^{\infty} K^{\alpha}_{\mathrm{Ch},M}(M+x, M+x) \leq C \tag{3.32}$$

if M *is large enough, and*
(iv)

$$\sum_{x=-L}^{\infty} B^{\alpha}(x,x) \leq C. \tag{3.33}$$

(*In* (3.33) *we can take* $C(\alpha, L) = \alpha/\sqrt{2} + L$.)

Proof. We have to show that (3.25) and (3.26) converge to (1.10) and (3.27) respectively. Using Stirling's formula we see that $A^{\alpha}_M(M+x) \to \sqrt{\alpha}$ as $M \to \infty$. The result then follows from the integral formulas above, the fact that

$$\lim_{M\to\infty} e^{\sqrt{\alpha}(1-z)} \left(\frac{\sqrt{\alpha} + Mz}{\sqrt{\alpha} + M} \right)^{x+M} \frac{1}{z^M} = e^{\sqrt{\alpha}(1/z - z)} z^y,$$

and $g_3(z) \to z \log z$ as $M \to \infty$. This establishes (3.30). The identity (3.31) follows from the recursion relation $J_{x+1}(t) = 2xJ_x(t)/t - J_{x-1}(t)$, which implies

$$B^{\alpha}(x,y) = J_{x+1}(t)J_{y+1}(t) + B^{\alpha}(x+1, y+1),$$

and (3.31) follows by using the decay properties of the Bessel function; see Lemma 3.9 below.

The estimate (3.32) is proved using the formula (3.26). Stirling's formula can be used to show that $A^{\alpha}_M(x+M) \leq 2\sqrt{\alpha}$ for all $x \geq 0$. We have

$$\begin{aligned} \left| \frac{\sqrt{\alpha} + Mz}{\sqrt{\alpha} + M} \right|^M \frac{1}{|z|^M} \left| \frac{\sqrt{\alpha} + Mz}{\sqrt{\alpha} + M} \right|^y &\leq \left(1 + \frac{\sqrt{\alpha}}{M|z|} \right)^M \left(|z| + \frac{\sqrt{\alpha}}{M} \right)^y \\ &\leq \exp((1-\delta)^{-1}\sqrt{\alpha})(1-\delta/2)^y \end{aligned}$$

if $|z| = r = 1 - \delta$ and $M \geq 2\sqrt{\alpha}/\delta$. This estimate can be used in the integral formulas for $D^{\alpha,r}_M$ and $F^{\alpha,r}_M$ and we obtain

$$|D^{\alpha,1/2}_M(M+x; g_i)|, |F^{\alpha,1/2}_M(M+x; g_i)| \leq Ce^{4\sqrt{\alpha}} (\frac{3}{4})^x.$$

Thus,

$$\sum_{x=-L}^{\infty} K^{\alpha}_{\mathrm{Ch},M}(M+x, M+x) \leq C\sqrt{\alpha} e^{4\sqrt{\alpha}} \sum_{x=-L}^{\infty} (\frac{3}{4})^x.$$

The estimate (iv) can be proved in a similar way but we can also proceed as follows. Using the generating function for the Bessel functions $J_n(t)$, $n \in \mathbb{Z}$, one can show that $\sum_{n=1}^{\infty} n^2 J_n(t)^2 = t^2/4$, see [Wa, 2.72(3)], and $\sum_{n=1}^{\infty} J_n(t)^2 = \frac{1}{2}(1 - J_0(t)^2) \le 1/2$, so by (ii) and the fact that $B(x,x) \le 1$,

$$\sum_{x=-L}^{\infty} B^\alpha(x,x) \le L + \sum_{n=1}^{\infty} nJ_n(2\sqrt{\alpha})^2 \le \alpha/\sqrt{2} + L,$$

where we have used the Cauchy-Schwarz inequality. The lemma is proved. □

We are now ready for the

Proof of Theorem 1.3. We have

$$\mathbb{E}^\alpha_{\mathrm{Ch},M}[g] = \prod_{j=1}^{M-1} \frac{1}{j!} \sum_{\lambda \in \Omega_M} \prod_{i=1}^{M} f(\lambda_i + L - i) V_M(\lambda)^2 W_M(\lambda) \prod_{i=1}^{M} \left(\frac{\alpha}{M}\right)^{\lambda_i} e^{-\alpha/M}.$$

If we make the change of variables $h_i = \lambda_i + M - i$, this can be written

$$\mathbb{E}^\alpha_{\mathrm{Ch},M}[g] = \frac{1}{Z^\alpha_M} \sum_{h \in \mathbb{N}^M} \prod_{i=1}^{M} (1 + \phi(h_i - M + L)) \Delta_M(h)^2 \prod_{i=1}^{M} w_{\alpha/M}(h_i).$$

Now, using a standard computation from random matrix theory, see [Me], [TW2], we can write this as

$$\mathbb{E}^\alpha_{\mathrm{Ch},M}[g] = \sum_{k=0}^{M-1} \sum_{h \in \mathbb{N}^k} \prod_{i=1}^{k} \phi(h_i) \det(K^\alpha_{\mathrm{Ch},M}(h_i + M - L, h_j + M - L))_{i,j=1}^k \tag{3.34}$$

since $\phi(t) = 0$ if $t < 0$. The Charlier kernel is positive definite, so we have the estimate

$$\left| \det(K^\alpha_{\mathrm{Ch},M}(x_i, x_j))_{i,j=1}^k \right| \le \prod_{j=1}^{k} K^\alpha_{\mathrm{Ch},M}(x_j, x_j).$$

Thus, by Lemma 3.7(iii),

$$\left| \sum_{h \in \mathbb{N}^k} \prod_{i=1}^{k} \phi(h_i) \det(K^\alpha_{\mathrm{Ch},M}(h_i + M - L, h_j + M - L))_{i,j=1}^k \right| \le ||\phi||_\infty^k \left(\sum_{x=-L}^{\infty} K^\alpha_{\mathrm{Ch},M}(M + x, M + x) \right)^k \le (C||\phi||_\infty)^k.$$

The analogous estimate for the Bessel kernel follows from Lemma 3.7(iv). These estimates and Lemma 3.7(i) allow us to take the $M \to \infty$ limit in (3.34). By Theorem 1.2 this gives (1.18). The theorem is proved. □

Note that we could just as well use Theorem 1.1 and the Meixner ensemble to prove Theorem 1.3. The proof would be the same and we just have to prove (3.30) and (3.32) for the Meixner ensemble instead.

3.4. *Asymptotics of the Plancherel measure.* Theorem 1.3 can be used to analyze the asymptotic properties of the Plancherel measure in different regions. One can distinguish three cases corresponding to three different scaling limits of the Bessel kernel. First we have the *edge scaling limit*,

$$\lim_{\alpha\to\infty}\alpha^{1/6}B^\alpha(2\sqrt{\alpha}+\xi\alpha^{1/6},2\sqrt{\alpha}+\eta\alpha^{1/6})=A(\xi,\eta), \tag{3.35}$$

where A is the *Airy kernel* defined in (1.4). This is the case that is considered in Theorem 1.4. Secondly we have the *bulk scaling limit*,

$$\lim_{\alpha\to\infty}B^\alpha(r\sqrt{\alpha},r\sqrt{\alpha}+u)=\frac{\sin(uR)}{u\pi}, \tag{3.36}$$

$u\in\mathbb{Z}$, $-2<r<2$, where $R=\arccos(r/2)$; the right-hand side is the *discrete sine kernel.* We will not discuss the local behavior in the bulk of the Young diagram; see [BOO]. Thirdly we have an *intermediate region*,

$$\lim_{\alpha\to\infty}\pi\alpha^{1/4-\delta/2}B^\alpha(2\sqrt{\alpha}-\alpha^\delta+\pi\xi\alpha^{1/4-\delta/2},2\sqrt{\alpha}-\alpha^\delta+\pi\eta\alpha^{1/4-\delta/2}) =\frac{\sin\pi(\xi-\eta)}{\pi(\xi-\eta)}, \tag{3.37}$$

if $1/6<\delta<1/2$, the ordinary *sine kernel.* Thus in this region the local behavior is the same as that in the bulk in a random Hermitian matrix. The limits (3.35) to (3.37) can be proved using the saddle-point method on the integral formula for the Bessel function. From the point of view of the Coulomb gas picture of the Young diagram, the cases one and three are similar to the random matrix case since at the edge a discrete Coulomb gas approximates a continuous Coulomb gas. Case two is different however, since in the bulk the discrete nature is manifest; the charges sit close to each other.

Before turning to the proof of Theorem 1.4 we have to say something about de-Poissonization, the joint distribution of the first k rows (k largest eigenvalues) and the asymptotics of the Bessel kernel.

We have the following generalization of a lemma in [Jo2].

LEMMA 3.8. *Let $\mu_N=N+4\sqrt{N\log N}$ and $\nu_N=N-4\sqrt{N\log N}$. Then there is a constant C such that, for $0\le x_i\le N$,*

$$\mathbb{P}^{\mu_N}_{\text{Plan}}[\lambda_1\le x_1,\dots,\lambda_k\le x_k]-\frac{C}{N^2}\le\mathbb{P}_{\text{Plan},N}[\lambda_1\le x_1,\dots,\lambda_k\le x_k] \le\mathbb{P}^{\nu_N}_{\text{Plan}}[\lambda_1\le x_1,\dots,\lambda_k\le x_k]+\frac{C}{N^2}. \tag{3.38}$$

Proof. This is proved as Lemmas 2.4 and 2.5 in [Jo2]. Denote a permutation in S_N by $\pi^{(N)}$ and let $S_{N+1}(j)$ denote the set of all $\pi^{(N+1)}$ such that $\pi^{(N+1)}(N+1)=j$. Each $\pi^{(N+1)}$ in $S_{N+1}(j)$ is mapped to a permutation $F_j(\pi^{(N+1)})$ in S_N by replacing each $\pi^{(N+1)}(i)>j$ by $\pi^{(N+1)}(i)-1$. The map

F_j is a bijection from $S_{N+1}(j)$ to S_N. Apply the Robinson-Schensted correspondence to $F_j(\pi^{(N+1)})$ to obtain the P-tableau. Replace the entries i by $i+1$ for $i=j,\dots,N$ and then insert j. This insertion can only increase the length of any row and we obtain the P-tableau for $\pi^{(N+1)}$. Thus,

$$\lambda_i(F_j(\pi^{(N+1)})) \le \lambda_i(\pi^{(N+1)}),$$

for all rows. If we define $g(\pi^{(N)})$ to be 1 if $\lambda_i(\pi^N) \le x_i$ for $i=1,\dots,k$ and 0 otherwise, we see that

$$g(F_j(\pi^{(N+1)})) \ge g(\pi^{(N+1)}),$$

and we can proceed exactly as in [Jo2] using the fact that the Plancherel measure on $\mathcal{P}^{(N)}$ is the push-forward of the uniform distribution on S_N. □

For $x \in \mathbb{R}^M$, $n \in \mathbb{N}^k$ and a sequence $\mathcal{I} = (I_1,\dots,I_k)$ of intervals in $\mathbb{R}$ we let $\chi(\mathcal{I},n,x)$ denote the characteristic function for the set of all $x \in \mathbb{R}^M$ such that exactly n_j of the x_i's belong to I_j, $j=1,\dots,k$. A computation shows that for a single interval

$$\chi(I_j,n_j,x) = \frac{1}{n_j!}\frac{\partial^{n_j}}{\partial z_j^{n_j}}\prod_{i=1}^{M}(1+z_j\chi_{I_j}(x_i))\Big|_{z_j=-1}$$

and hence

$$\chi(\mathcal{I},n,x) = \frac{1}{n_1!\dots n_k!}\frac{\partial^{n_1+\dots+n_k}}{\partial z_j^{n_1}\dots\partial z_j^{n_k}}\prod_{i=1}^{M}\prod_{j=1}^{k}(1+z_j\chi_{I_j}(x_i))\Big|_{z_1=\dots=z_k=-1}. \tag{3.39}$$

Note that if the intervals are pairwise disjoint, then $\prod_{j=1}^k(1+z_j\chi_{I_j}(x_i)) = 1+\sum_{j=1}^k z_j\chi_{I_j}(x_i)$; compare with [TW2]. Let $\mathbb{P}$ be a probability measure on $\mathbb{R}^M$ and let $a_1 \ge \dots \ge a_k$. Set $I_{j+1} = (a_{j+1},a_j]$, $j=1,\dots,k-1$ and $I_1 = (a_1,\infty)$. Let

$$\mathbb{L}_k = \{n \in \mathbb{N}^k\,;\,\sum_{j=1}^{r} n_j \le r-1,\, r=1,\dots,k\}.$$

Define $x^{(j)}$ to be the j^{th} largest of the x_i's. Then,

$$\mathbb{P}[x^{(1)} \le a_1,\dots,x^{(k)} \le a_k] = \sum_{n\in\mathbb{L}_k}\mathbb{E}[\chi(\mathcal{I},n,x)]. \tag{3.40}$$

Hence, the problem of investigating the distribution function in (3.40) reduces to investigating expectations of the right-hand side of (3.39).

In the proof of Theorem 1.4 we will need some asymptotic results for Bessel functions.

LEMMA 3.9. *Let $M_0 > 0$ be given. Then there exists a constant $C = C(M_0)$ such that if we write $x = 2\sqrt{\alpha} + \xi\alpha^{1/6}$, then*

$$|J_x(2\sqrt{\alpha})| \le C\alpha^{-1/6} \exp\left[-\frac{1}{4}\min\left(\frac{1}{4}\alpha^{1/6}, |\xi|^{1/2}\right)|\xi|\right] \tag{3.41}$$

for $\xi \in [-M_0, \infty)$. Furthermore

$$\lim_{\alpha\to\infty} \alpha^{1/6} J_x(2\sqrt{\alpha}) = Ai(\xi), \tag{3.42}$$

uniformly for $\xi \in [-M_0, M_0]$.

This can be deduced from classical asymptotic results, [Wa] and it is also rather straightforward to proceed as in Section 5 of [Jo3] using the integral formula (3.28).

We are now ready for the

Proof of Theorem 1.4. We will prove (1.20). The proof of (1.19) is analogous using the Hermite kernel instead. From Lemma 3.8, the fact that a distribution function is increasing in its arguments, that the distribution function $F(t_1, \dots, t_k)$ is continuous and $\sqrt{\mu_N} - \sqrt{N} \approx 2\sqrt{\log N}$, $\sqrt{\nu_N} - \sqrt{N} \approx 2\sqrt{\log N}$, we see that it suffices to prove that

$$\lim_{\alpha\to\infty} \mathbb{P}^{\alpha}_{\text{Plan}}[\lambda_1 - 1 \le 2\sqrt{\alpha} + t_1\alpha^{1/6}, \dots \lambda_k - k \le 2\sqrt{\alpha} + t_k\alpha^{1/6}] = F(t_1, \dots, t_k), \tag{3.43}$$

for any fixed $(t_1, \dots, t_k) \in \mathbb{R}^k$, $t_1 \ge \dots \ge t_k$. Set

$$I_{j+1} = (2\sqrt{\alpha} + t_{j+1}\alpha^{1/6}, 2\sqrt{\alpha} + t_j\alpha^{1/6}], \qquad j = 1, \dots, k-1$$

and $I_1 = (2\sqrt{\alpha} + t_1\alpha^{1/6}, \infty)$. By (3.39) and (3.40) it is enough to consider the expectations

$$\mathbb{E}^{\alpha}_{\text{Plan}}\left[\prod_{i=1}^{\infty}\prod_{j=1}^{k}(1 + z_j\chi_{I_j}(\lambda_j - j))\right]. \tag{3.44}$$

If we write $\phi_\alpha(s) = \prod_{j=1}^k (1 + z_j\chi_{I_j}(s)) - 1$ it follows from Theorem 1.3, with $L = 0$, that the expectation (3.44) can be written as

$$F_\alpha(z, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{h\in\mathbb{N}^k} \prod_{j=1}^{k} \phi_\alpha(h_j) \det[B^\alpha(h_i, h_j)]_{i,j=1}^k. \tag{3.45}$$

Note that $F_\alpha(z, t)$ is an entire function of z. Set $J_{j+1} = (t_{j+1}, t_j]$, $j = 1, \dots, k-1$, $J_1 = (t_1, \infty)$ and write $\psi(s) = \prod_{j=1}^k (1 + z_j\chi_{J_j}(s)) - 1$. Define

$$F(z, t) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\mathbb{R}^k} \prod_{j=1}^{k} \psi(\xi_j) \det[A(\xi_i, \xi_j)]_{i,j=1}^k. \tag{3.46}$$

We want to show that

$$\lim_{\alpha\to\infty} F_\alpha(z,t) = F(z,t), \tag{3.47}$$

uniformly for z in a compact subset of $\mathbb{C}^k$. Then also derivatives of $F_\alpha(z,t)$ converge to the corresponding derivatives of $F(z,t)$. The limit (3.43) then follows with

$$F(t_1,\dots,t_k) = \sum_{n\in\mathbb{L}_k} \frac{1}{n_1!\dots n_k!}\frac{\partial^{n_1+\dots+n_k}}{\partial z_1^{n_1}\dots\partial z_k^{n_k}} F(z,t)\Big|_{z_1=\dots z_k=-1}. \tag{3.48}$$

So it remains to prove (3.47). Note that $\phi_\alpha(s)=0$ if $s<2\sqrt{\alpha}+t_k\alpha^{1/6}$ and that $\phi_\alpha(s)=\psi(\alpha^{-1/6}(s-2\sqrt{\alpha}))$. Given $r\in\mathbb{R}$ we set $\mathbb{A}(r)=\{r,r+1,r+2,\dots\}$. Then,

$$F_\alpha(z,t) = \sum_{l=0}^{\infty}\frac{1}{l!}\sum_{h\in\mathbb{A}(t_k\alpha^{1/6})^l}\prod_{j=1}^{l}\psi\left(\frac{h_j}{\alpha^{1/6}}\right)\det[\tilde{B}^\alpha(\xi,\eta)]\frac{1}{(\alpha^{1/6})^l}, \tag{3.49}$$

where $\tilde{B}^\alpha(\xi,\eta) = \alpha^{1/6}B^\alpha(2\sqrt{\alpha}+\xi\alpha^{1/6}, 2\sqrt{\alpha}+\eta\alpha^{1/6})$. We can now prove that (3.47) holds pointwise in z by the same argument as was used in the proof of the analogous statement in Section 3 of [Jo3]. That proof depends on the following properties of the kernel; compare with Lemma 3.1 in [Jo3] and Lemma 4.1 below.

(i) For any $M_0>0$ there is a constant $C=C(M_0)$ such that for all $\xi\ge -M_0$

$$\sum_{m=1}^{\infty} B^\alpha(2\sqrt{\alpha}+\xi\alpha^{1/6}+m, 2\sqrt{\alpha}+\xi\alpha^{1/6}+m)\le C.$$

(ii) For any $\varepsilon>0$, there is an $L>0$ such that

$$\sum_{m=1}^{\infty} B^\alpha(2\sqrt{\alpha}+L\alpha^{1/6}+m, 2\sqrt{\alpha}+L\alpha^{1/6}+m)\le\varepsilon,$$

for all sufficiently large α.

(iii) For any $M_0>0$ and any $\varepsilon>0$

$$\left|\tilde{B}^\alpha\left(\frac{n}{\alpha^{1/6}},\frac{m}{\alpha^{1/6}}\right) - A\left(\frac{n}{\alpha^{1/6}},\frac{m}{\alpha^{1/6}}\right)\right|\le\varepsilon$$

for all integers $m,n\in[-M_0\alpha^{1/6}, M_0\alpha^{1/6}]$ provided α is sufficiently large.

The estimate (i) is used to estimate the tail in the k-summation in (3.49), (ii) is used to limit the h-summation and (iii) is used to prove that the Riemann sums converge to integrals.

If z belongs to a compact set K there is a constant C, independent of z, such that $||\psi||_\infty\le C$. Together with (i) this shows that the family $\{F_\alpha(z,t)\}$ is uniformly bounded for $\alpha>0$, $z\in K$ and hence (3.47) holds uniformly by a normal family argument.

The properties (i) to (iii) above are straightforward to prove using the representation (3.31) and Lemma 3.9. To prove (i) and (ii) we use

$$\sum_{m=1}^{\infty} B^{\alpha}(x+m,x+m) = \sum_{n=1}^{\infty} nJ^2_{x+n+1}(2\sqrt{\alpha}),$$

which can be estimated using (3.41) (we get a Riemann sum). Similarly, $\tilde{B}^{\alpha}(\frac{n}{\alpha^{1/6}}, \frac{m}{\alpha^{1/6}})$ can be written as a Riemann sum, using (3.31), which is controlled using (3.41) and (3.42). This Riemann sum can be compared with the corresponding Riemann sum for the following representation of the Airy kernel, [TW1],

$$A(\xi,\eta) = \int_0^{\infty} \mathrm{Ai}\,(\xi+t)\mathrm{Ai}\,(\eta+t)dt$$

and in this way we obtain (iii).

4. Random words and the Charlier ensemble

In this section we will prove our results on random words.

Proof of Proposition 1.5. Let $L(M,N,\lambda)$ denote the number of pairs (P,Q) of tableaux of shape $\lambda \in \Omega_M^{(N)}$ with P semistandard with elements in $\{1,\dots,M\}$ and Q standard with elements in $\{1,\dots,N\}$. Then

$$\mathbb{P}_{\mathrm{W},M,N}[S^{-1}(\lambda)] = \frac{1}{M^N} L(M,N,\lambda). \tag{4.1}$$

The number of possible P's is, by [Fu],

$$d_{\lambda}(M) = \prod_{1\le i<j\le M} \frac{\lambda_i-\lambda_j+j-i}{j-i} = \left(\prod_{j=1}^{M-1}\frac{1}{j!}\right) V_M(\lambda), \tag{4.2}$$

and the number of possible Q's is f^{λ} given by (1.13). By (4.2), (4.3) and Lemma 3.1 we obtain

$$L(M,N,\lambda) = N!\left(\prod_{j=1}^{M-1}\frac{1}{j!}\right) V_M(\lambda)^2 W_M(\lambda). \tag{4.3}$$

Inserting the formula (4.3) into (4.1) yields the desired result (1.21). The formulas (1.22) and (1.23) are immediate consequences. The proposition is proved. □

Next, we give the

Proof of Theorem 1.6. We will prove (1.24); the proof of (1.25) is analogous. Both are straightforward asymptotic computations using Stirling's formula and we will indicate the main steps. Set

$$x_j = \frac{\lambda_j - N/M}{\sqrt{2N/M}}, \quad j = 1, \dots, M.$$

Note that $\sum_{j=1}^{M} x_j = 0$, since $\sum_{j=1}^{M} \lambda_j = N$. Then,

$$(\lambda_j + M - j)! = \sqrt{\frac{2\pi N}{M}} \left(\frac{N}{M}\right)^{N/M+M-j} e^{x_j^2 - N/M + o(1)}$$

as $N \to \infty$, and hence

$$W_M(\lambda) \sim \left(\frac{2\pi N}{M}\right)^{-M/2} e^N \left(\frac{M}{N}\right)^{N+M(M-1)/2} \prod_{j=1}^{M} e^{-x_j^2}.$$

Furthermore,

$$V_M(\lambda)^2 = \left(\frac{2N}{M}\right)^{M(M-1)/2} \prod_{1 \le i < j \le M} \left(x_i - x_j + \frac{i-j}{\sqrt{2N/M}}\right),$$

and consequently

(4.4)
$$\begin{aligned}\mathbb{P}_{\mathrm{Ch,M,N}}[\{\lambda\}] &\sim \sqrt{\pi M}(2\pi)^{-M/2} 2^{M^2/2} \prod_{j=1}^{M-1} \frac{1}{j!} \Delta_M(x)^2 \prod_{j=1}^{M} e^{-x_j^2} \left(\frac{2N}{M}\right)^{-(M-1)/2} \\ &= \sqrt{\pi M} M! \phi_{\mathrm{GUE},M}(x).\end{aligned}$$

From this we see that the left-hand side of (1.24) is approximately a Riemann sum for the right-hand side, which in the limit $N \to \infty$ converges to the right-hand side. The factor $M!$ in the last expression in (4.4) comes from the fact that in (4.4) the variables are ordered. This completes the proof. □

For the proof of Theorem 1.7 we need asymptotic results for the Charlier kernel analogous to those for the Bessel kernel in the proof of Theorem 1.4.

LEMMA 4.1. *Let* $\nu = M + \alpha/M + 2\sqrt{\alpha}$ *and* $\sigma = (1 + \sqrt{\alpha}/M)^{2/3} \alpha^{1/6}$.

(i) *For any* $M_0 > 0$ *there is a constant* $C = C(M_0)$ *such that, for all integers* $n \ge -M_0 \sigma$,

(4.5)
$$\sum_{m=1}^{\infty} K_{\mathrm{Ch},M}^{\alpha}([\nu] + n + m, [\nu] + n + m) \le C.$$

(ii) *For any* $\varepsilon > 0$ *there is an* $L > 0$ *such that*

(4.6)
$$\sum_{m=1}^{\infty} K_{\mathrm{Ch},M}^{\alpha}([\nu] + [\sigma L] + m, [\nu] + [\sigma L] + m) \le \varepsilon$$

if M, α *are sufficiently large.*

(iii) *For any* $M_0 > 0$ *and any* $\varepsilon > 0$,

$$\left|\sigma K^{\alpha}_{\mathrm{Ch},M}([\nu]+m,[\nu]+n) - A\left(\frac{m}{\sigma},\frac{n}{\sigma}\right)\right| \le \varepsilon \tag{4.7}$$

for all integers $m, n \in [-M_0\sigma, M_0\sigma]$ *provided* α *and* M *are sufficiently large.*

Proof. The proof is based on the formulas (3.25) and (3.26) for the Charlier kernel. The proof is completely analogous to the proof of the corresponding result for the Meixner kernel in Lemma 3.2 in [Jo3, §5], so we will not give the details here. Asymptotic formulas for Charlier polynomials with fixed $a = \alpha/M$ have been obtained in [Go]. □

Proof of Theorem 1.7. By (1.16) and (1.23)

$$\mathbb{P}^{\alpha}_{\mathrm{W},M}[L(w)\le s] = \prod_{j=1}^{M}\frac{1}{j!}\sum_{\substack{h\in\mathbb{N}^M\\ \max h_j \le s+M-1}} \Delta_M(h)^2\prod_{j=1}^{M} w_{\alpha/M}(h_i),$$

where we have made the substitution $h_i = \lambda_i + M - i$. Using Lemma 4.1 this can be analyzed exactly as the analogous problem involving the Meixner weight in Section 3 in [Jo3]. Lemma 3.1 in [Jo3] gives

$$\mathbb{P}^{\alpha}_{\mathrm{W},M}\left[L(w)\le \frac{\alpha}{M} + 2\sqrt{\alpha} + \left(1+\frac{\sqrt{\alpha}}{M}\right)^{2/3}\alpha^{1/6}\xi\right] \to F(\xi), \tag{4.8}$$

as $\alpha, M \to \infty$ with $F(\xi)$ given by (1.5). This proves (1.26). Next, we observe that for fixed M, $\mathbb{P}_{\mathrm{W},M,N}[L(w)\le s]$ is a decreasing function of N, which can be proved as the corresponding result for permutations in [Jo2]. Thus, with μ_N and ν_N as in Lemma 3.8, we have

$$\mathbb{P}^{\mu_N}_{\mathrm{W},M}[L(w)\le s] - \frac{C}{N^2} \le \mathbb{P}_{\mathrm{W},M,N}[L(w)\le s] \le \mathbb{P}^{\nu_N}_{\mathrm{W},M}[L(w)\le s] + \frac{C}{N^2}. \tag{4.9}$$

Set $s(\alpha, M, \xi) = \frac{\alpha}{M} + 2\sqrt{\alpha} + \left(1+\frac{\sqrt{\alpha}}{M}\right)^{2/3}\alpha^{1/6}\xi$. Then, $s(N, M, \xi) = s(\mu_N, M, \xi+\delta)$ and $s(N, M, \xi) = s(\nu_N, M, \xi+\delta')$, where $\delta, \delta' \to 0$ as $M, N \to \infty$ if $M^{-1}(\log N)^{1/6}$ converges to 0 as $M, N \to \infty$. Thus, (1.27) follows from (4.8) and (4.9) and the theorem is proved. □

5. Applications of the Krawtchouk ensemble

5.1. *Seppäläinen's first passage percolation model.* The *Krawtchouk ensemble* is defined by (1.7) with the weight $w(x) = \binom{K}{x}p^x q^{K-x}$, $0 \le x \le K$, i.e. we consider the probability measure

$$\mathbb{P}_{\mathrm{Kr},N,K,p}[h] = \frac{1}{Z_{N,K,p}}\Delta_N(h)^2\prod_{j=1}^{N}\binom{K}{h_j}p^{h_j}q^{K-h_j}$$

on $\{0,\dots,K\}^N$, where $Z_{N,K,p} = N!\Big(\prod_{j=0}^{N-1}\frac{j!}{(K-j)!}\Big)(K!)^N(pq)^{N(N-1)/2}$. The first problem where the Krawtchouk ensemble appears is in the simplified first-passage percolation model introduced by Seppäläinen in [Se2]. Consider the lattice $\mathbb{N}^2$ and attach a *passage time* $\tau(e)$ to each nearest neighbour edge. If e is vertical $\tau(e) = \tau_0 > 0$, and if e is horizontal then $\tau(e)$ is random with $P[\tau(e) = \lambda] = p$ and $P[\tau(e) = \kappa] = q = 1-p$, where $\kappa > \lambda \geq 0$, $0 < p < 1$. All passage times assigned to horizontal edges are independent random variables. Hence, all randomness sits in the horizontal edges. The *minimal passage time* from $(0,0)$ to (k,l) along nearest neighbour paths is defined by

$$T(k,l) = \min_p \sum_{e\in p} \tau(e) \tag{5.1}$$

where the minimum is over all non-decreasing nearest neighbour paths p from $(0,0)$ to (k,l). The *time constant* is defined by $\mu(x,y) = \lim_{n\to\infty}\frac{1}{n}T([nx],[ny])$. (The existence of the limit follows from subadditivity.) In [Se2] it is proved, using a certain associated stochastic process, that

$$\mu(x,y) = \begin{cases} \lambda x + \tau_0 y, & \text{if } py > qx \\ \lambda x + \tau_0 y + (\kappa-\lambda)(\sqrt{qx}-\sqrt{py})^2, & \text{if } py \leq qx. \end{cases} \tag{5.2}$$

We will show that the distribution of the random variable $T(k,l)$ relates to the distribution of the rightmost charge ("largest eigenvalue") in a Krawtchouk ensemble.

Write $M = k$, $N = l+1$ and consider an $M\times N$ matrix W whose elements, $w(i,j)$, are independent Bernoulli random variables, $P[w(i,j) = 0] = q$ and $P[w(i,j) = 1] = p = 1-q$. Let $\Pi_{M,N}$ be the set of all sequences $\pi = \{(k,j_k)\}_{k=1}^M$ such that $1 \leq j_1 \leq \dots \leq j_M \leq N$, i.e. up/right paths in W with exactly one element in each row. Introduce the random variable

$$L(W) = \max\left\{\sum_{(i,j)\in\pi} w(i,j)\,;\,\pi\in\Pi_{M,N}\right\}. \tag{5.3}$$

Write $\rho = 1/q-1$, so that $q = (1+\rho)^{-1}$ and $p = \rho(1+\rho)^{-1}$. It is straightforward to show that

$$T(k,l) = l\tau_0 + k\kappa - (\kappa-\lambda)L(W). \tag{5.4}$$

Proposition 5.1. *Let $L(W)$ be defined by* (5.3) *with W an $M\times N$ $0-1$-matrix with independent Bernoulli elements $w(i,j)$, the probability of* 1 *being p. Then,*

$$P[L(W)\leq n] = \mathbb{P}_{\mathrm{Kr},N,N+M-1,p}\left[\max_{1\leq j\leq N} h_j \leq n+N-1\right]. \tag{5.5}$$

Proof. Interpreting the formula (7.30) in Theorem 7.1 in [BR1] in the appropriate case, we get

$$P[L(W) \le n] = (1+\rho)^{-MN} \sum_{\substack{\lambda \in \mathcal{P} \\ \ell(\lambda) \le n}} d_\lambda(M) d_{\lambda'}(N) \prod_{i=1}^{\ell(\lambda)} \rho_{\lambda_i}, \tag{5.6}$$

where λ' is the partition conjugate to λ, λ'_k is the length of the k^{th} column in λ, and $d_\lambda(M)$ is the number of semi-standard tableaux of shape λ with elements in $\{1, \dots, M\}$; if $\ell(\lambda) \le M$, $d_\lambda(M)$ is given by (4.2). The proof of (5.6) is based on the RSK-correspondence between 0-1 matrices and pairs of semistandard Young tableaux (P, Q) where P has shape λ and Q has shape λ', see [Fu], [St]. Set

$$\Omega_M(N) = \{\lambda \in \Omega_M\,;\, N \ge \lambda_1 \ge \dots \lambda_M \ge 0\}.$$

Since $d_\lambda(M) = 0$ if $\ell(\lambda) > M$ and $d_{\lambda'}(N) = 0$ if $\lambda_1 > N$, (5.6) can be written as

$$\begin{aligned} &P[L(W) \le n] \\ &\quad = (1+\rho)^{-MN} \left(\prod_{j=1}^{M-1} \frac{1}{j!} \right) \left(\prod_{j=1}^{N-1} \frac{1}{j!} \right) \sum_{\substack{\lambda \in \Omega_M(N) \\ \ell(\lambda) \le n}} V_M(\lambda) V_N(\lambda') \prod_{i=1}^{M} \rho^{\lambda_i}. \end{aligned} \tag{5.7}$$

Note that $\lambda \in \Omega_M(N)$ if and only if $\lambda' \in \Omega_N(M)$ and $\ell(\lambda) = \lambda'_1$.

Lemma 5.2. *If $\mu \in \Omega_N(M)$, then*

$$V_M(\mu') = \left(\prod_{j=1}^{N+M-1} j! \right) V_N(\mu) W_N(\mu) \prod_{j=1}^{N} \frac{1}{(M+j-1-\mu_j)!}. \tag{5.8}$$

Proof. One way to prove (5.8) is to use the fact that $V_M(\mu')W_M(\mu') = V_M(\mu)W_M(\mu)$ by the hook formula for f^μ; compare with (1.13) and Lemma 3.1. We will give another proof. Set $s_i = \mu_i + N + 1 - i$, $1 \le i \le N$ and $r_j = N + j - \mu'_j$, $1 \le j \le M$. Then,

$$\{s_1, \dots, s_N\} \cup \{r_1, \dots, r_M\} = \{1, \dots, N+M\}. \tag{5.9}$$

To see this, notice that since $1 \le s_i, r_j \le N + M$ it suffices to show that $s_i \ne r_j$ for all i, j. Looking at the μ-diagram one sees that $\mu_i + \mu'_j \le i + j - 2$ or $\mu_i + \mu'_j \ge i + j$, which implies $s_i \ne r_j$.

Let $n_k = 1$ if $k \in \{s_1, \dots, s_N\}$ and $n_k = 0$ if $k \in \{r_1, \dots, r_M\}$, $k = 1, \dots, N+M$. Then, by (5.9),

$$V_M(\mu') = \prod_{1 \le k < l \le N+M} (l-k)^{(1-n_k)(1-n_l)}. \tag{5.10}$$

Now,

$$\prod_{1\le k<l\le N+M}(l-k)^{n_k n_l}=V_N(\mu),$$

$$\prod_{1\le k<l\le N+M}(l-k)^{n_k}=\prod_{j=1}^{N}\prod_{l=s_j+1}^{N+M}(l-s_j)=\prod_{j=1}^{N}(N+M-s_j)!$$

and

$$\prod_{1\le k<l\le N+M}(l-k)^{n_l}=\prod_{j=1}^{N}\prod_{k=1}^{s_j-1}(s_j-k)=\prod_{j=1}^{N}(s_j-1)!.$$

Inserting this into (5.10) gives the formula (5.8). The lemma is proved. □

We can now finish the proof of the proposition. If we write $\mu=\lambda'$, we see from (5.8) that (5.7) can be written as

(5.11)
$$P[L(W)\le n]=(1+\rho)^{-MN}\prod_{j=0}^{N-1}\frac{(j+M)!}{j!}\times\sum_{\substack{\mu\in\Omega_N(M)\\ \mu_1\le n}}V_N(\mu)^2W_N(\mu)\prod_{j=1}^{N}\frac{\rho^{\mu_j}}{(M+j-1-\mu_j)!}.$$

As usual we introduce the new coordinates $h_j=\mu_j+N-j$. Then, using $\rho=1/q-1$, we obtain

$$P[L(W)\le n]=\frac{1}{N!}\prod_{j=0}^{N-1}\frac{(j+M)!}{j!}\frac{(pq)^{N(N-1)/2}}{((N+M-1)!)^N}\times\sum_{\substack{h\in\mathbb{N}^N\\ \max(h_j)\le n+N-1}}\Delta_N(h)^2\prod_{j=1}^{N}\binom{N+M-1}{h_j}p^{h_j}q^{N+M-1-h_j},$$

which completes the proof. □

Using Proposition 5.1 we can prove a limit theorem for the first passage time $T(k,l)$. The result should be compared with Remark 1.8 and Conjecture 1.9 in [Jo3].

THEOREM 5.3. *If* $\mu(x,y)$ *is given by* (5.2),

$$\sigma(x,y)=\frac{(pq)^{1/6}}{(xy)^{1/6}}(\sqrt{px}+\sqrt{qy})^{2/3}(\sqrt{qx}-\sqrt{py})^{2/3}$$

and $py < qx$, *then*

$$\lim_{n\to\infty} \mathbb{P}\big[\frac{T([nx],[ny]) - n\mu(x,y)}{\sigma(x,y)n^{1/3}} \le \xi\big] = 1 - F(-\xi),$$

where $F(t)$ *is the Tracy-Widom distribution* (1.5).

Proof. The proof uses (5.4) and Proposition 5.1 and is analogous to the proof of Theorem 1.7, the difference being that we now need the analogue of Lemma 4.1 for the Krawtchouk polynomials. This can be obtained from a steepest descent analysis of the integral formula for these polynomials in much the same way as in the analysis of the Meixner polynomials in Section 5 of [Jo3]; see [Jo4] for some more details. The time constant is related to the right endpoint of the support of the equilibrium measure associated with the Krawtchouk ensemble, and the constant $\sigma(x,y)$ comes out of the steepest descent argument. We can also get large deviation results by using the general results of Section 4 in [Jo3]. □

5.2. *The Aztec diamond.* We turn now to the relation between the Krawtchouk ensemble and domino tilings of the Aztec diamond introduced by Elkies, Kuperberg, Larsen and Propp in [EKLP]. The definitions are taken from that paper and the papers [JPS] and [CEP] where more details and pictures can be found. A *domino* is a closed 1×2 or 2×1 rectangle in $\mathbb{R}^2$ with corners in $\mathbb{Z}^2$, and a *tiling* of a region $R \subseteq \mathbb{R}^2$ by dominoes is a set of dominoes whose interiors are disjoint and whose union is R. The *Aztec diamond*, A_n, of order n is the union of all lattice squares $[m, m+1]\times[l, l+1]$, $m, l \in \mathbb{Z}$, that lie inside the region $\{(x,y)\,;\,|x|+|y| \le n+1\}$. It is proved in [EKLP] that the number of possible domino tilings of A_n equals $2^{n(n+1)/2}$. Color the Aztec diamond in a checkerboard fashion so that the leftmost square in each row in the top half is white. A horizontal domino is *north-going* if its leftmost square is white, otherwise it is *south-going*. Similarly, a vertical domino is *west-going* if its upper square is white, otherwise it is *east-going*. Two dominoes are *adjacent* if they share an edge, and a domino is adjacent to the boundary if it shares an edge with the boundary of the Aztec diamond. The *north polar region* is defined to be the union of those north-going dominoes that are connected to the boundary by a sequence of adjacent north-going dominoes. The south, west and east polar regions are defined analogously. In this way a domino tiling partitions the Aztec diamond into four polar regions, where we have a regular brick wall pattern, and a fifth central region, the *temperate zone*, where the tiling pattern is irregular.

Consider the diagonal of white squares with opposite corners Q_k^r, $k = 0,\dots,n+1$, where $Q_k^r = (-r+k, n+1-k-r)$, $r = 1,\dots,n$. A *zig-zag* path Z_r in A_n from Q_0^k to Q_{n+1}^r is a path of edges going around these white squares.

When going from Q_k^r to Q_{k+1}^r we can go either first one step east and then one step south, or first one step south and then one step east. A domino tiling on A_n defines a unique zig-zag path Z_r from Q_0^r to Q_{n+1}^r if we require that the zig-zag path does not intersect the dominoes. Similarly, we can define zig-zag paths from $P_0^r=(-r,n-r)$ to $P_n^r=(n-r,-r)$ going around black squares.

We consider random tilings of the Aztec diamond, where each of the $2^{n(n+1)/2}$ possible tilings have the same probability. This induces a probability measure on the zig-zag paths. Consider a zig-zag path in A_n from Q_0^k to Q_{n+1}^r around white squares. Let $h_r<\dots<h_1$ be those k for which we go first east and then south when we go from Q_k^r to Q_{k+1}^r, $k=0,\dots,n$; there are exactly r such k if the zig-zag path comes from a domino tiling, [EKLP]. Call this zig-zag path $Z_r(h)$.

Proposition 5.4. *Let $\{h_1,\dots,h_r\}\subseteq\{0,\dots,n\}$ be the positions of the east/south turns in a zig-zag path $Z_r(h)$ in the Aztec diamond A_n from $(-r,n+1-r)$ to $(n+1-r,-r)$ around white squares. Then, the probability for this particular zig-zag path is*

$$P[Z_r(h)]=\mathbb{P}_{\mathrm{Kr},r,n,1/2}[h]. \tag{5.12}$$

If $\{h_1,\dots,h_r\}\subseteq\{0,\dots,n-1\}$ are the positions of the south/east turns in a zig-zag path $Z_r'(h)$ in A_n from $(-r,n-r)$ to $(n-r,-r)$ around black squares, then

$$P[Z_r'(h)]=\mathbb{P}_{\mathrm{Kr},r,n-1,1/2}[h]. \tag{5.13}$$

Proof. Let $\mathcal{U}_r(h)$ be the number of possible domino tilings above $Z_r(h)$ in the Aztec diamond. From the arguments in [EKLP], see also [PS], it follows that

$$\mathcal{U}_r(h)=2^{r(r-1)/2}\prod_{1\le i<j\le r}\frac{h_i-h_j}{j-i}. \tag{5.14}$$

Let $k_1<\dots<k_{n+1-r}$ be defined by

$$\{k_1,\dots,k_{n+1-r}\}=\{0,\dots,n\}\setminus\{h_1,\dots,h_r\}.$$

If $\mathcal{L}_r(h)$ is the number of domino tilings of the region below $Z_r(h)$ in A_n, then, using the symmetry of the Aztec diamond, we see that

$$\mathcal{L}_r(h)=2^{(n+1-r)(n-r)/2}\prod_{1\le i<j\le n+1-r}\frac{k_j-k_i}{j-i}. \tag{5.15}$$

Thus, the probability for a certain zig-zag path $Z_r=Z_r(h)$, specified by h, is

$$P[Z_r(h)]=\frac{2^{(n+1-r)(n-r)/2+r(r-1)/2}}{2^{n(n+1)/2}}\prod_{1\le i<j\le r}\frac{h_i-h_j}{j-i}\prod_{1\le i<j\le n+1-r}\frac{k_j-k_i}{j-i}. \tag{5.16}$$

If we let $h_i = \mu_i + r - i$, $1 \le i \le r$ and $k_j = r + j - 1 - \mu'_j$, $1 \le j \le n+1-r$, then μ and μ' are conjugate partitions; compare with the proof of Lemma 5.1 ($N = r$, $M = n+1-r$, $s_i = h_i + 1$, $r_j = k_j + 1$). We see from (5.16) that

$$P[Z_r(h)] = 2^{-(n+1-r)r}\left(\prod_{j=1}^{r-1}\frac{1}{j!}\right)\left(\prod_{j=1}^{n-r}\frac{1}{j!}\right)V_r(\mu)V_{n+1-r}(\mu'),$$

where $\mu \in \Omega_r(n+1-r)$. We can now apply Lemma 5.1, which gives

$$P[Z_r(h)] = \frac{2^{r(r-1)}}{(n!)^r}\prod_{j=1}^{r-1}\frac{(n-j)!}{j!}V_r(\mu)^2\prod_{j=1}^{r}\binom{n}{\mu_j + r - j}\frac{1}{2^n}.$$

Now, $h_i = \mu_i + r - i$, so we obtain

$$P[Z_r(h)] = \frac{2^{r(r-1)}}{(n!)^r}\prod_{j=1}^{r-1}\frac{(n-j)!}{j!}\Delta_r(h)^2\prod_{j=1}^{r}\binom{n}{h_j}\frac{1}{2^n}, \tag{5.17}$$

which is the Krawtchouk ensemble. Note that in (5.17) the order of the h_i's is unimportant, so we can let $\{h_1, \dots, h_r\} \subseteq \{0, \dots, n\}$ be the (unordered) positions of the east/south turns. A completely analogous argument applies to the zig-zag paths in A_n from P_0^r to P_n^r around black squares. This completes the proof. □

It is proved in [JPS] that, with probability tending to 1 as $n \to \infty$, the asymptotic shape of the temperate zone is a circle centered at the origin and tangent to the boundary of the Aztec diamond (*the arctic circle* theorem). This can be deduced from Proposition 5.4 and the general results in Section 4 of [Jo3]. The arctic circle is determined by the endpoints of the support of the equilibrium measure (or the points where it saturates). Also, from Theorem 5.3, we see that the fluctuations of the temperate zone around the arctic circle is described by the Tracy-Widom distribution. This can also be deduced from the fact, derived in [JPS], that the shape of a polar region is related to the shape of a randomly growing Young diagram. The growth model obtained is exactly the discrete time growth model studied in [Jo3], and we can apply the results of that paper. See [Jo4] for more details.

Finally, we will shortly discuss another random tiling problem related to plane partitions using the combinatorial analysis by Cohn, Larsen and Propp in [CLP]. For more details and pictures see the paper [CLP]. Plane partitions in an $a \times b \times c$ box can be seen to be in one-to-one correspondence with tilings of an a, b, c-hexagon with unit rhombi with angles $\pi/3$ and $2\pi/3$, called *lozenges*. An a, b, c-hexagon has sides of length a, b, c, a, b, c (in clockwise order), equal angles and the length of the horizontal sides is b. If the major diagonal of the lozenge is vertical we talk about a *vertical lozenge*. Consider the uniform distribution on the set of all possible tilings of the a, b, c-hexagon with lozenges, which corresponds to the uniform distribution on all plane partitions in the $a \times b \times c$ box. For simplicity we will now restrict ourselves to the case $a = b = c$. A horizontal line k steps from the top, $k = 0, \dots, a$ will intersect the vertical lozenges at positions $h_1 + 1, \dots, h_k + 1$, $0 \le h_1 < \dots < h_k \le a + k - 1$, otherwise it passes through sides of the lozenges. A random tiling induces a probability measure on the sequences $h = (h_1, \dots, h_k)$. Interpreting the formulas in Theorem 2.2 in [CLP] we see that the probability for h is

$$P[h] = \frac{1}{Z_{k,a}} \Delta_k(h)^2 \prod_{j=1}^{k} \binom{h_j + a - k}{h_j} \binom{2a - 1 - h_j}{a + k - 1 - h_j}, \tag{5.18}$$

where $Z_{k,a}$ is a constant that can be computed explicitly. Note that the measure is symmetric in the h_i's so we can regard (5.18) as a measure on $\{0, \dots, a + k - 1\}^k$. Thus, again we get a discrete orthogonal polynomial ensemble, this time with the weight

$$w(x) = \binom{x + \alpha}{x} \binom{N + \beta - x}{N - x} \tag{5.19}$$

on $\{0, \dots, N\}$, with $\alpha = \beta = a - k$ and $N = a + k - 1$. The orthogonal polynomials for this weight are the Hahn polynomials, [NSU], so (5.18) should be called the *Hahn ensemble*. If we do not have $a = b = c$ we will again get a weight function of the form (5.19) but with different values of α, β and N and with a different number of particles. This model is further discussed in [Jo4], but to obtain the Tracy-Widom distribution in this model is more complicated due to the fact that it is less straightforward to compute the asymptotics of the Hahn polynomials.

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