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(n+1)–Families of Sets in General Position

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Abstract. A family of sets in $\mathbf{R}^n$ is said to be in general position if any m–flat, $0 \le m \le n-1$, intersects at most $m+1$ members of the family. Using the Fan–Glicksberg–Kakutani fixed point theorem, we prove that if $\{A_1, A_2, \ldots, A_{n+1}\}$ is a family of compact convexly connected sets then for any proper subset I of $\{1, 2, \ldots, n+1\}$, there exists exactly one hyperplane H which separates strictly the sets $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \{1, 2, \cdots, n+1\} \setminus I\}$ and satisfies $d(A_1, H) = d(A_2, H) = \cdots = d(A_{n+1}, H)$.

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1. Introduction

Denote by $\mathbf{R}^n$, the real n-dimensional Euclidean space, and by $\|\cdot\|$, the norm in this space. If A is a compact set and H is a hyperplane in $\mathbf{R}^n$, we set $d(A, H) = \min\{\|x - y\| : x \in A,\, y \in H\}$. The standard abbreviations aff A, conv A, card A are used to define the affine hull, convex hull and cardinality of A. For every subset I of $\{1, 2, \ldots, n+1\}$, $\overline{I}$ denotes the complement of I in $\{1, 2, \ldots, n+1\}$. If $H = \{x \in \mathbf{R}^n :< x, b >= \lambda\}$ is a hyperplane, the corresponding halfspaces $\{x \in \mathbf{R}^n :< x, b >< \lambda\}$, $\{x \in \mathbf{R}^n :< x, b > \le \lambda\}$, $\{x \in \mathbf{R}^n :< x, b >> \lambda\}$, $\{x \in \mathbf{R}^n :< x, b > \ge \lambda\}$, are denoted respectively by $H^<$, $H^\le$, $H^>$, $H^\ge$.

Definition 1. *A family $\mathcal{A}$ of sets in $\mathbf{R}^n$ is said to be in general position if any m-flat, $0 \le m \le n-1$, intersects at most $m+1$ members of $\mathcal{A}$.*

A family of sets in general position is called in [8], [9] an independent family, and in [1] and [4] a separated family.

Remark. Let $m = \min\{n+1, \operatorname{card} \mathcal{A}\}$. It is easy to see that the family $\mathcal{A}$ is in general position if and only if for every choice of sets $A_1, A_2, \ldots, A_m \in \mathcal{A}$ and every choice of points $a_1 \in A_1$, $a_2 \in A_2, \ldots, a_m \in A_m$, the set $\{a_1, a_2, \ldots, a_m\}$ is affinely independent.

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Definition 2. *A set A is called convexly connected if there is no hyperplane H such that $H \cap A = \emptyset$ and A contains points in both the open halfspaces determined by H.* (cf. [5] and [13], p.174)

2. A characterization of (n+1)-families of convexly connected sets in general position

Let us begin by recalling two types of separation of two sets (see [6]). Let A and B be to sets in a vector space separated by a hyperplane H. We say that the sets A and B are nicely separated by H if the separating hyperplane H is disjoint from A or from B. The sets A and B are strongly separated by H if the separating hyperplane is at positive distance from both A and B.

Theorem 1. *Let $\mathcal{A}$ be a $(n+1)$-family of convexly connected sets in $\mathbf{R}^n$. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ is in general position.

(ii) For any hyperplane H, there exist two members of $\mathcal{A}$ which are nicely separated by a translate of H.

Proof. (i) implies (ii). Let H be a hyperplane in $\mathbf{R}^n$ and $b \neq 0$ be a normal vector to H. We define a continuous function $f : \mathbf{R}^n \to \mathbf{R}$, $f(x) = <x, b>$. Since each $A \in \mathcal{A}$ is convexly connected, $f(A)$ is an interval and hence, a convex set in $\mathbf{R}$.

We claim that there exist two members A_1, A_2 of $\mathcal{A}$ such that $f(A_1) \cap f(A_2) = \emptyset$. Otherwise, it follows by Helly's theorem that there exists an element $\lambda \in \cap\{f(A) : A \in \mathcal{A}\}$ and the hyperplane $\{x \in \mathbf{R}^n : <x, b> = \lambda\}$ intersects all members of $\mathcal{A}$. But this contradicts (i).

Choosing a convenient sense for the vector b, it follows that

$$<x, b> \; < \; <y, b> \quad \text{for every } x \in A_1 \text{ and } y \in A_2 \tag{1}$$

and

$$\sup\{<x, b>: x \in A_1\} \leq \inf\{<y, b>: y \in A_2\}; \tag{2}$$

that is, the sets A_1 and A_2 are separated by a translate of H. The relation (1) implies that A_1 and A_2 are in fact nicely separated.

(ii) implies (i). By way of contradiction, suppose that there exists a hyperplane $H = \{x \in \mathbf{R}^n : <x, b> = \lambda\}$ which intersects all members of $\mathcal{A}$. From the hypothesis, there exists two members A_1 and A_2 of $\mathcal{A}$ nicely separated by a translate of H. If $x \in A_1 \cap H$ and $y \in A_2 \cap H$ then we obtain the contradiction

$$\lambda = <x, b> \; < \; <y, b> = \lambda. \qquad \square$$

Corollary 2. *Let $\mathcal{A}$ be a $(n+1)$-family of compact convexly connected sets in $\mathbf{R}^n$. Then the following assertions are equivalent:*

(i) $\mathcal{A}$ is in general position.

(ii) For any hyperplane H, there exist two members of $\mathcal{A}$ which are strongly separated by a translate of H.

Proof. (i) implies (ii). Since the sets $A_i \in \mathcal{A}$ are compact, the infimum and the supremum in (2) are attained values and from (1), we deduce that A_1 and A_2 are strongly separated by H.

Since the strong separation implies the nice separation, the implication $(ii) \to (i)$ follows from the similar implication in Theorem 1. □

3. Equally spaced hyperplanes for (n+1)-families of compact sets

In our proof of Theorem 5 the following result is crucial:

Lemma 3. *Let $[x_1, x_2, \dots, x_{n+1}]$ be an n-simplex in $\mathbf{R}^n$. Then*

(i) for each proper subset I of $\{1, 2, \dots, n+1\}$ there exists exactly one hyperplane H such that

$$\text{the sets } \{x_i : i \in I\} \text{ and } \{x_j : j \in \overline{I}\} \text{ are strictly separated by } H, \text{ and} \tag{3}$$

$$d(x_1, H) = d(x_2, H) = \cdots = d(x_{n+1}, H). \tag{4}$$

(ii) There exist exactly $2^n - 1$ hyperplanes H which satisfy (4).

Proof. (i) For the sake of simplicity, let $I = \{1, 2, \dots, k\}$, with $1 \le k \le n$. The distance from an arbitrary point x_0 to a hyperplane

$$H = \{x \in \mathbf{R}^n :< x, b >= \lambda\} \tag{5}$$

is given by the Ascoli formula (see [11] p.91)

$$d(x_0, H) = \frac{| < x_0, b > -\lambda|}{||b||} \tag{6}$$

Since the pair $(b, \lambda) \in (\mathbf{R}^n \setminus \{0\}) \times \mathbf{R}$ for which the hyperplane H admits the representation (5) is uniquely determined apart from a multiplicative constant, the conditions (3) and (4) are equivalent with the conditions

$$\begin{cases} < x_1, b > -\lambda & = \cdots = & < x_k, b > -\lambda & = & 1 \\ < x_{k+1}, b > -\lambda & = \cdots = & < x_{n+1}, b > -\lambda & = & -1 \end{cases} \tag{7}$$

Denoting by $(x_{i1}, x_{i2}, \dots, x_{in})$ the coordinates of x_i, $1 \le i \le n+1$, and by $(b_1, b_2, \dots, b_n)$ the coordinates of b, we are lead to an $(n+1) \times (n+1)$ linear system

$$x_{i1}b_1 + x_{i2}b_2 + \cdots + x_{in}b_n - \lambda = \varepsilon_i, \quad 1 \le i \le n+1, \tag{8}$$

where

$$\varepsilon_i = \begin{cases} 1 & \text{if} \quad 1 \le i \le k \\ -1 & \text{if} \quad k+1 \le i \le n+1. \end{cases}$$

From the afine independence of the points $x_1, x_2, \ldots, x_{n+1}$, it follows that the determinant D of order $n+1$ having the general row $(x_{i1}, x_{i2}, \ldots, x_{in}, 1)$ is different from zero. This proves that the system (8) possesses the unique solution

$$\text{(9)} \qquad \begin{cases} b_j &= -\frac{D_j}{D}, \qquad 1 \le j \le n \\ \lambda &= -\frac{D_{n+1}}{D} \end{cases}$$

where D_j is the determinant of order $n+1$ having the general row $(x_{i1}, x_{i2}, \ldots, x_{i,j-1}, \varepsilon_i, x_{i\,j+1}, \ldots, x_{in}, 1)$ and D_{n+1} is the determinant of the same order, with the general row $(x_{i1}, \ldots, x_{in}, \varepsilon_i)$. This shows that there exists a unique hyperplane H which satisfies the conditions (3) and (4). Note that $d(x_i, H) = \frac{1}{\|b\|}$, $1 \le i \le n+1$ and that the points x_i, $i \in I$, lie in the open halfspace $H^{>}$, while the points x_j, $j \in \overline{I}$ lie in $H^{<}$.

Obviously, if H is a hyperplane equally spaced from the vertices of a simplex then these vertices can not be all situated in the same part of H. The assertion (ii) follows now from (i). □

Let X and Y be two topological spaces. Denote by 2^Y the family of all subsets of Y. A map $f : X \to 2^Y$ is said to be upper semicontinous if $f^{-1}(F) = \{x \in X : f(x) \cap F \neq \emptyset\}$ is closed in X for any closed subset F of Y. The following result can be found in [12], p.69.

Lemma 4. *Let X and Y be two compact metric spaces and $f : X \to 2^Y$ a map such that $f(x)$ is closed for each $x \in X$. Then the following statements are equivalent:*

(i) f is upper semicontinuous.

(ii) For any two sequences $(x_m)_{m\in\mathbf{N}}$, $(y_m)_{m\in\mathbf{N}}$, the relations $x_m \to x$, $y_m \in f(x_m)$, $y_m \to y$ imply $y \in f(x)$.

Theorem 5. *Let $\{A_1, A_2, \ldots, A_{n+1}\}$ be a family of compact convex sets in general position in $\mathbf{R}^n$. Then*

(i) for each proper subset I of $\{1, 2, \ldots, n+1\}$, there exists exactly one hyperplane H such that

$$\text{(10)} \qquad H \text{ separates strictly the sets } \cup\{A_i : i \in I\} \text{ and } \cup\{A_j : j \in \overline{I}\},$$

and

$$\text{(11)} \qquad d(A_1, H) = d(A_2, H) = \cdots = d(A_{n+1}, H).$$

(ii) There exist exactly $2^n - 1$ hyperplanes satisfying (11).

Proof. The assertion (ii) follows immediately from (i).

Let $A = A_1 \times A_2 \times \cdots \times A_{n+1}$. The elements of A are denoted by $\overline{x}, \overline{y}, \ldots$ Let I be a nonempty set of indices, arbitrarily fixed. By Lemma 3, for each $\overline{x} = (x_1, x_2, \ldots, x_{n+1}) \in A$ $(x_i \in A_i,\ 1 \le i \le n+1)$, there exists a unique hyperplane, denoted by $H(\overline{x})$, such that

$$\text{(12)} \qquad d(x_1, H(\overline{x})) = d(x_2, H(\overline{x})) = \cdots = d(x_{n+1}, H(\overline{x}))$$

and the set $\{x_i : i \in I\}$ is situated in the open halfspace $H^>(\overline{x})$, while the set $\{x_j : j \in \overline{I}\}$ is situated in the opposite halfspace $H^<(\overline{x})$. The equation of $H(\overline{x})$ is $< x, b >= \lambda$ where $b = (b_1, b_2, \ldots, b_n) \in \mathbf{R}^n \setminus \{0\}$ and $\lambda \in \mathbf{R}$ are given by the formulas (9).

We define the map $f : A \to 2^A$ by $f(\overline{x}) = (P_1(\overline{x}), P_2(\overline{x}), \ldots, P_{n+1}(\overline{x}))$, $\overline{x} \in A$ and $P_i(\overline{x})$ defined by

$$\begin{cases} (a) \quad P_i(\overline{x}) = \{x \in A_i :< x, b >= \min\{< y, b >: y \in A_i\}\}, & i \in I \\ (b) \quad P_i(\overline{x}) = \{x \in A_i :< x, b >= \max\{< y, b >: y \in A_i\}\}, & i \in \overline{I} \end{cases} \tag{13}$$

Since the sets A_i are compact, the sets $P_i(\overline{x})$ are nonempty. If $y_i \in P_i(\overline{x})$, $1 \leq i \leq n+1$, then $P_i(\overline{x})$ coincides with the intersection of the set A_i with the hyperplane parallel to $H(\overline{x})$ which contains the point y_i. Thus each $P_i(\overline{x})$ is a compact convex set, and $f(\overline{x})$ is a nonempty compact convex set for each $\overline{x} \in A$.

Using Lemma 4, we prove that f is upper semicontinuous. Let $(\overline{x}^m)_{m\in\mathbf{N}}$, $(\overline{y}^m)_{m\in\mathbf{N}}$ be two sequences in A and $\overline{x}, \overline{y} \in A$ such that $\overline{x}^m \to \overline{x}$, $\overline{y}^m \to \overline{y}$, $\overline{y}^m \in f(\overline{x}^m)$, $m \in \mathbf{N}$. If the hyperplanes $H(\overline{x}^m)$, $m \in \mathbf{N}$, have the equations $< x, b^m > -\lambda^m = 0$ with $b^m = (b_1^m, b_2^m, \ldots, b_n^m)$ and λ^m given by the formulas (9) then from $\overline{x}^m \to \overline{x}$, it follows that $b^m \to b$. Let $i \in I$ arbitrarily fixed. Since $\overline{y}^m \in f(\overline{x}^m)$, we have $< {y_i}^m, b^m > \leq < z_i, b^m >$ whenever $z_i \in A_i$. Letting $m \to \infty$, we obtain $< y_i, b > \leq < z_i, b >$; that is $y_i \in P_i(\overline{x})$. Using an appropriate argument, we obtain $y_i \in P_i(\overline{x})$ for each $i \in \overline{I}$, and $\overline{y} \in f(\overline{x})$. According to Lemma 4, it follows that f is upper semicontinuous.

By the Fan-Glicksberg-Kakutani fixed point theorem (see [2] and [3]), there is a point $\overline{z} = (z_1, z_2, \ldots, z_{n+1}) \in A$ such that $\overline{z} \in f(\overline{z})$. Let $< x, b^0 > -\lambda^0 = 0$ be the equation of the hyperplane $H(\overline{z})$, with $b^0 = (b_1^0, b_2^0, \ldots, b_n^0)$ and λ^0 given by the formulas (9). For each $i \in I$, $z_i \in H^>(\overline{z})$ and by the definition of f, $z_i \in P_i(\overline{z})$. Thus, we infer from (13a) that $A_i \subset H^>(\overline{z})$ for all $i \in I$.

Then, for each $i \in I$, we have

$$d(A_i, H(\overline{z})) = \min\left\{\frac{< x, b^0 > -\lambda^0}{\|b^0\|} : x \in A_i\right\} =$$

$$= \frac{< z_i, b^0 > -\lambda^0}{\|b^0\|} = d(z_i, H(\overline{z})).$$

In a similar manner, we obtain that $A_j \subset H^<(\overline{z})$ and $d(A_j, H(\overline{z})) = d(z_j, H(\overline{z}))$ for all $j \in \overline{I}$.

Therefore, $H(\overline{z})$ separates strictly the sets $\cup\{A_i : i \in I\}$ and $\cup\{A_j : j \in \overline{I}\}$ and taking into account (12), we obtain that

$$d(A_1, H(\overline{z})) = d(z_1, H(\overline{z})) = d(z_i, H(\overline{z})) = d(A_i, H(\overline{z}))$$

for each i, $2 \leq i \leq n+1$.

In the second part of the proof, we verify the uniqueness of the hyperplane H which satisfies (10) and (11) for an arbitrary fixed set of indices I. Note that the existance of a such hyperplane H is equivalent to the existance of two hyperplanes H_1 and H_2 parallel to H such that

$$(14)\qquad \begin{cases} H_1 \text{ is a common supporting hyperplane to all sets } A_i,\ i \in I, \\ H_2 \text{ is a common supporting hyperplane to all sets } A_j,\ j \in \overline{I}, \\ \text{the open strip determinated by } H_1 \text{ and } H_2 \text{ is disjoint from } \bigcup_{i=1}^{n+1} A_i. \end{cases}$$

By way of contradiction, suppose that there exist two hyperplanes $H^{'}$ and $H^{''}$ which satisfy (10) and (11). Then there exist two pairs of parallel hyperplanes (H_1', H_2') and (H_1'', H_2''), each satisfying (14).

Obviously the sets A_i, $i \in I$, are contained in $S_1 = H_1^{'\geq} \cap H_1^{''\geq}$. The intersection points of the sets A_i, $i \in I$, with $H_1^{'}$ are all situated on a face of the dihedral angle S_1 while the intersection points of these sets with $H_1^{''}$ are all situated on the other face of S_1.

Let $L_1 = H_1^{'} \cap H_1^{''}$ and $L_2 = H_2^{'} \cap H_2^{''}$. Then L_1 and L_2 are two parallel $(n-2)$-flats which determine together a hyperplane H_0. Since H_0 separates the faces of the dihedral angle S_1, the convexity of A_i implies that H_0 intersects all sets A_i, $i \in I$. Using a similar argument we obtain that H_0 intersects all sets A_j, $\in \overline{I}$. Therefore H_0 intersects each member of the family $\{A_1, A_2, \ldots, A_{n+1}\}$ which is in general position; a contradiction. □

We extend the previous theorem to a family of convexly connected sets using the following result.

Lemma 6. (see [7], Th.1). *Let $\mathcal{A} = \{A_i : i \in I\}$ be a family of compact convexly connected sets in $\mathbf{R}^n$. If $\mathcal{A}$ is in general position then $\{conv\ A_i :\ i \in I\}$ is a family of compact sets also in general position.*

Corollary 7. *The assertion of Theorem 5 holds if $\{A_1, A_2, \ldots, A_{n+1}\}$ is a family of compact convexly connected sets in general position in $\mathbf{R}^n$.*

Proof. By the previous lemma, Theorem 5 is applicable to the family $\{\text{conv}\, A_1, \text{conv}\, A_2, \ldots,$ $\text{conv}\, A_{n+1}\}$.

Let I be a proper subset of $\{1, 2, \ldots, n+1\}$ and let H be the unique hyperplane which satisfies the conditions

$$(10')\qquad \begin{array}{l} H \text{ separates strictly the sets} \\ \cup\{\text{conv}\, A_i : i \in I\} \text{ and } \cup\{\text{conv}\, A_j : j \in \overline{I}\} \end{array}$$

and

$$(11')\qquad d(\text{conv}\, A_1, H) = d(\text{conv}\, A_2, H) = \cdots = d(\text{conv}\, A_{n+1}, H).$$

Since two sets can be strictly separated if and only if their convex hulls can be strictly separated (see [10] p.102), the conditions $(10')$ and (10) are equivalent.

The inclusion $A_i \subset \operatorname{conv} A_i$ implies $d(A_i, H) \geq d(\operatorname{conv} A_i, H)$, $1 \leq i \leq n+1$. Suppose that there exists an index $i \in \{1, 2, \dots, n+1\}$ such that

$$d(A_i, H) > d(\operatorname{conv} A_i, H). \tag{15}$$

Let $x \in \operatorname{conv} A_i$ such that $d(x, H) = d(\operatorname{conv} A_i, H)$. By (15), the hyperplane H_0, parallel to H and passing through x, does not intersect the set A_i. Let S be the open halfspace bounded by H_0 which does not contain H. Since A_i is convexly connected it is contained in S. Now, the convexity of S implies $\operatorname{conv} A_i \subset S$. We obtain in this way the contradiction:

$$x \in \operatorname{conv} A_i \cap H_0 \subset S \cap H_0 = \emptyset.$$

Therefore $d(A_i, H) = d(\operatorname{conv} A_i, H)$, $1 \leq i \leq n+1$, and the conditions (11) and $(11^{'})$ are equivalent.

From (10) $\iff$ $(10')$ and (11) $\iff$ $(11')$ it follows that H is the unique hyperplane which satisfies (10) and (11). $\square$

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