Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 1, 17-29.

Σ–Chamber Systems, Coloured Graphs and Orbifolds

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Abstract. We define the notion of covering projections for Σ–chamber systems in purely algebraic terms and construct a presentation for the group of decktransformations of such a covering. Translating this concept into the language of orbifolds yields a presentation of the fundamental group of an orbifold.
MSC 1991: 51M20, 57M07, 20F32

0. Introduction

In the literature, one can find at least two different approaches to the problem of encoding manifolds in an efficient way, by means of some combinatorial structure. One method uses coloured graphs (as reference see e.g. [5]). The other approach employs Σ–chamber systems (cf. [4]). The latter concept was developed by Dress, as a method of describing tilings of a simply connected manifold which are equivariant with respect to some group, acting on that manifold. His so–called Delaney symbol, which is associated with such an equivariant tiling, contains the complete information about the orbifold which one obtains by passing to the orbit space of the group action.

In the language of coloured graphs, there also exists a notion for encoding orbifold–structures: the concept of coverings, or folded coverings (cf. [2]).

It is the aim of this paper to compare these two approaches. This discussion establishes a close connection between combinatorics, algebra and topology. In particular, it yields an algebraic treatment of coverings and their group of decktransformations. We obtain a presentation of this group by purely algebraic considerations. On the other hand, this group can occur as the fundamental group of an orbifold, under suitable conditions.

This paper is organized as follows: In the first section, we recall some basic definitions and state the almost obvious equivalence between coloured graphs and Σ-chamber systems. The next section develops the notion of a covering projection between sets on which a fixed group operates. In the third section, we treat the special case, where this group is a free Coxeter group. We then define and discuss the notion of Delaney symbols and its connection to the concept of folded coverings.

In the last section, the results are applied to determine the fundamental group of an orbifold in terms of a triangulation.

1. Σ–Chamber systems and coloured graphs

The notion of a Σ-chamber system can be seen as a particular case of the more general notion of a chamber system. Following Tits, [8] 2.1, we define this notion as follows.

Definition 1.1 *A chamber system over I is a pair $(\mathcal{C}, (\overset{i}{\sim})_{i\in I})$, where $\mathcal{C}$ is some set and $(\overset{i}{\sim})_{i\in I}$ is a family of equivalence relations. By abuse of language, we also call $\mathcal{C}$ a chamber system, if the equivalence relations are unterstood. The elements of $\mathcal{C}$ are called* chambers *und two chambers C, C' are called* i–adjacent*, if the relation $C \overset{i}{\sim} C'$ holds.*

A chamber complex is called quasi-thin*, if each equivalence class of $\overset{i}{\sim}$ contains at most two elements.*

Let Σ be the free Coxeter group:

$$\Sigma = \Sigma_I := \langle \sigma_i \mid \sigma_i^2 = 1, i \in I \rangle$$

where $I = \{0, 1, ..., n\}$ is the index set.

Definition 1.2 *Let $\mathcal{C}$ be a Σ–set, i.e. the group Σ is acting from the right on $\mathcal{C}$. The orbits of $\langle\sigma_i\rangle$ are the classes of an equivalence relation $\overset{i}{\sim}$. The pair of $\mathcal{C}$ and this family of equivalence relations is obviously a quasi–thin chamber system. If moreover Σ acts transitively on $\mathcal{C}$, we call $\mathcal{C}$ a* Σ–chamber system.

Definition 1.3 *Let $\mathcal{C}$ and $\mathcal{C}'$ be two Σ_I-chamber systems. A map $\Phi : \mathcal{C} \to \mathcal{C}'$ is a* morphism of Σ-chamber systems *(Σ-morphism) if Φ is onto and $\Phi(C)\sigma_i = \Phi(C\sigma_i)$ for each $C \in \mathcal{C}$ and $i \in I$.*

We shall use the term graph in the usual sense, allowing multiple edges and loops.

Definition 1.4 *Let Γ be a connected graph. Let $V(\Gamma)$ and $E(\Gamma)$ denote the sets of vertices and edges of Γ, respectively. Let $I := \{0, 1, ..., n\}$. A map $\gamma : E(\Gamma) \to I$ is an* edge-colouration *on Γ, if $\gamma(e_1) \neq \gamma(e_2)$ for any two edges e_1, e_2 which are incident to the same vertex. We* *call I the colour set. If Γ isconnected, regular of degree $n+1$ (that is, there are $n+1$ edges incident to each vertex of Γ) and γ is an edge colouration on Γ, then the pair (Γ, γ) is called an I-coloured graph. Two vertices of the graph are called i–adjacent, if there exists an edge e incident to these vertices, such that $\gamma(e) = i$ holds.*

Definition 1.5 *Let (Γ, γ) and (Γ', γ') be I-coloured graphs. A morphism $\Phi : (\Gamma, \gamma) \to (\Gamma', \gamma')$ is a map $\Phi : V(\Gamma) \to V(\Gamma')$ which is onto and preserves i-adjacency for all $i \in I$.*

Proposition 1.6 *There is a natural one–to–one correspondence between Σ_I-chamber systems and I-coloured graphs, which also carries morphisms of Σ_I-chamber systems to morphism of I-coloured graphs.*

Proof: Let $\mathcal{C}$ be a Σ_I-chamber system. We want to construct the I-coloured graph $(\Gamma_\mathcal{C}, \gamma_\mathcal{C})$ associated with $\mathcal{C}$.
We take a vertex for each chamber $C \in \mathcal{C}$, i.e. $V(\Gamma_\mathcal{C}) = \mathcal{C}$, and we take as the set of edges:

$$E(\Gamma) = \{(C\langle\sigma_i\rangle, i) \mid i \in I, C \in \mathcal{C}\}.$$

That means, that $(C\langle\sigma_i\rangle, i)$ is the edge of $\Gamma_\mathcal{C}$ joining the vertex C and $C\sigma_i$.

We define the map $\gamma_\mathcal{C} : E(\Gamma_\mathcal{C}) \to I$ such that $\gamma(e) = i$ if and only if $e = (\langle\sigma_i\rangle C, i)$ for some $C \in \mathcal{C}$.

Since Σ acts on $\mathcal{C}$ transitively, $\Gamma_\mathcal{C}$ is connected and regular of degree $n+1$. Thus the pair $(\Gamma_\mathcal{C}, \gamma_\mathcal{C})$ is an I-coloured graph.

Conversely, given (Γ, γ) an I-coloured graph, we have to construct a Σ_I- chamber system $\mathcal{C}_{(\Gamma,\gamma)}$. We define $\mathcal{C}_{(\Gamma,\gamma)} := V(\Gamma)$ and Σ acts on $\mathcal{C}_{(\Gamma,\gamma)}$ in the following way: set $v\sigma_i = v'$, if and only if v and v' are joined by an edge e and $\gamma(e) = i$.

Since Γ is regular of degree $n+1$ and γ is an edge-colouration, this action is well defined, and also the identity $\sigma_i^2 = 1$ holds for each $i \in I$. Furthermore this action is transitive, because Γ is connected.

It is obvious, that $\mathcal{C} = \mathcal{C}_{(\Gamma_\mathcal{C}, \gamma_\mathcal{C})}$. Moreover, the two constructions described above map Σ_I–morphisms to morphisms of I–coloured graphs and vice versa. These mappings are inverse to each other. Thus we have established a natural one–to–one correspondence between these two categories. □

Definition 1.7 *Let (Γ, γ) be an I-coloured graph. For each $J \subset I$, let Γ_J be the graph with vertices $V(\Gamma)$ and edges $\gamma^{-1}(J)$, i.e., the graph Γ_J is obtained from (Γ, γ) by cancelling the edges with colour in $I \backslash J$. Each connected component H of Γ_J is called a J-residue.*

Let $\mathcal{C}$ be the Σ-chamber system associated with (Γ, γ). A J-residue containing $v \in V(\Gamma)$ corresponds to the Σ_J–orbit $C_v\langle\sigma_j | j \in J\rangle$ in $\mathcal{C}$, considered as a Σ_J–set. Note, that for different $J, K \subseteq I$ the corresponding orbits of C may coincide as sets, but they are different, if we consider them as Σ_J–, respectively Σ_K–sets, since the groups are different subgroups of Σ_I.

According to our construction which we used in the proof above, the vertex v and the chamber C corresponding to that vertex coincide. But since we denote vertices and chambers by different letters, we indicate this correspondence by $C = C_v$.

Definition 1.8 *Let $\mathcal{C}$ be a Σ_I-chamber system. Let $\Delta = \Delta(I)$ denote the standard simplex whose vertices are labelled with the set I. The topological realization $\|\mathcal{C}\|$ of $\mathcal{C}$ is defined as the set*

$$\bigcup_{C \in \mathcal{C}} (C \times \Delta)$$

with the identifications:

$$(C, \partial_i \Delta) \equiv (C\sigma_i, \partial_i \Delta) \text{ for each } i \in I,$$

where $\partial_i \Delta$ is the codimension–1 face of type $I \backslash \{i\}$ of Δ.

Definition 1.9 *Let (Γ, γ) be an I-coloured graph, and $\Delta = \Delta(I)$ as above. The topological realization $K(\Gamma, \gamma)$ of (Γ, γ) is defined as the set*

$$\bigcup_{v \in V(\Gamma)} (v \times \Delta)$$

with the identifications:

$$(v_h, \partial_i \Delta) \equiv (v_m, \partial_i \Delta) \Longleftrightarrow \left\{ \begin{array}{l} v_h \text{ and } v_m \text{ are joined by} \\ \text{an edge } e, \text{ and } \gamma(e) = i. \end{array} \right.$$

Obviously a Σ–chamber system and a coloured graph, which are mapped to each other by the above constructed one–to–one correspondence, have identical topological realizations. (See [4] and [1]).

2. Morphisms of G-sets as covering projections

There is a close analogy between covering projections of topological spaces[1] and morphisms between group actions: If one replaces 'fundamental group' by 'stabilizer of a point' and 'covering-projection' by 'G-morphism', then the basic theorems about covering projections translate verbatim to theorems about group actions. In this section, we state some theorems one can get in this way and show that they are in some sense corollaries to a very basic fact about G-sets stated in lemma 2.2. The results will be applied to Σ-chamber systems in the next section.

Let us fix some group G for this section. A *G–set* is understood to be a set M together with an action of G from the right. This action is denoted in the usual manner: the image of $m \in M$ under g is the element mg of M.

A map $\phi : M \longrightarrow N$ between two G–sets is a morphism of G–sets, if it commutes with the action of G on both sets, i.e. if the identity $\phi(mg) = \phi(m)g$ always holds.

Definition 2.1 *A morphism $\pi : M \to N$ between G–sets is called a G–covering, if G operates on both sets transitively. (Therefore G–coverings are always onto.)*

[1]To be precise we have to restrict to pathwise connected, semilocally 1-connected spaces (cf. [6], ch. 2, sec. 6)

Lemma 2.2 *Let M, N be transitive G-sets, $m \in M$ and $n \in N$. Furthermore let G_x denote the stabilizer group of a point x. There exists a G-covering $\pi : M \to N$ with $\pi(m) = n$ if and only if $G_m \subseteq G_n$ holds. If π exists it is uniquely determined by the condition $\pi(m) = n$.*

Proof: The only possible way to define π is by setting $\pi(mg) = ng$ for all $g \in G$. This is well-defined, if and only if $G_m \subseteq G_n$. One easily checks that this map is a G-morphism. □

The group G itself is a transitive G–set via multiplication from the right. Because the stabilizer of any point is trivial, we get as corollary to the lemma.

Corollary 2.3 (Existence of universal covering) *Let N be a transitive G-set, $n \in N$. The G-covering $\pi : G \to N$ defined by $\pi(1) = n$ (cf. lemma 2.2) is universal in the following sense: For every G-covering $\psi : M \to N$ and $m \in M$ with $\psi(m) = n$ there exists exactly one G-covering $\varphi : G \to M$ with $\varphi(1) = m$ and $\pi = \psi \circ \varphi$.*

Definition 2.4 *Let $\pi : M \to N$ be a G-covering. A* decktransformation, *or* covering transformation, *of π is an automorphism γ of M with $\pi = \pi \circ \gamma$. The group of all covering transformations is denoted by $Aut(\pi)$, i. e.*

$$Aut(\pi) = \{\gamma \in Aut(M) \mid \pi = \pi \circ \gamma\}$$

As corollary to lemma 2.2 we get the following description of Aut (π).

Corollary 2.5 *Let $\pi : M \to N$ be a G-covering and $m \in M$. By $\overline{G}_m$ denote the normalizer of G_m in $G_{\pi(m)}$. For every $g \in \overline{G}_m$ there exists exactly one covering transformation γ_g with $\gamma_g(m) = mg$. This gives rise to the following exact sequence*

$$1 \longrightarrow G_m \longrightarrow \overline{G}_m \longrightarrow \mathrm{Aut}(\pi) \longrightarrow 1$$

$$g \longmapsto g \longmapsto \gamma_g$$

Proof:

i) Existence of γ_g: One has to show: $G_{mg} = G_m$ if $g \in \overline{G_m}$. But we know: $G_{mg} = g^{-1}G_m g$. If $g \in \overline{G_m}$, this implies $g^{-1}G_m g = G_m$ and we are done. That γ_g is a decktransformation is implied by $g \in G_{\pi(m)}$.

ii) Surjectivity of $g \mapsto \gamma_g$: Assume $\gamma(m) = mg$ for a $\gamma \in \mathrm{Aut}(\pi)$. One has to show: $g \in \overline{G_m}$. But due to the fact that γ is an automorphism, the lemma implies:

$$G_m = G_{\gamma(m)} = g^{-1}G_m g \ ,$$

i.e. g normalizes G_m. Moreover $g \in G_{\pi(m)}$, because $\gamma \in \mathrm{Aut}(\pi)$.

iii) $g \mapsto \gamma_g$ is a homomorphism:

$$\begin{aligned} \gamma_{gh}(m) &= m(gh) \\ \gamma_g \circ \gamma_h(m) &= \gamma_g(mh) \\ &= \gamma_g(m)h = (mg)h \end{aligned}$$

iv) The exactness at $\overline{G_m}$ follows from the definition of γ_g, because

$$\gamma_g(m) = m \iff m = mg \iff g \in G_m.$$

□

Finally, we translate the one–to–one correspondence between isomorphism classes of covering spaces over a suitable space (cf. [6], ch. 2, sec. 6) and conjugation classes of subgroups of the fundamental group of this space into a theorem about G-coverings.

Definition 2.6 *Two G-coverings $\pi_i : M_i \to N$ $(i = 1, 2)$ of the same base space are said to be isomomorphic, if there exists an isomorphism of G-sets $\psi : M_1 \to M_2$ with $\pi_1 = \pi_2 \circ \psi$.*

Definition 2.7 *Let $\pi : M \to N$ be a G-covering, $n \in N$ and let $F := \pi^{-1}(n)$ be the fibre over n. Further, denote by $S(F)$ the group of all permutations of F. The* monodromy-representation *of π with respect to n is the following anti-morphism between groups:*

$$\begin{aligned} \mu_{\pi,n} : G_n &\longrightarrow S(F) \\ g &\longmapsto [\, m \longmapsto mg \,] \end{aligned}$$

In other words: the monodromy-representation is the induced action of G_n on the fibre of n.

Corollary 2.8 *Let N be a G-set and $n \in N$. There are canonical one–to–one correspondences between each pair of the following three sets:*

A) *isomorphism classes of G-coverings of N*

B) *isomorphism classes of G_n-sets*

C) *conjugation classes of subgroups of G_n*

These one–to–one correspondences are induced by the following maps:

$A \to B :$ $\pi \longmapsto \mu_{\pi,n}$

$B \to C :$ *choose for each G_n-set F an element $f \in F$ and map F to G_f.*

$C \to A :$ *map a subgroup H of G_n to the following G-covering $\pi : M \to N$: let $M := H\backslash G$ with multiplication from the right as G-action and set $\pi(Hg) = ng$.*

Proof: One easily checks that the maps defined above induce maps on the level of equivalence classes and that the composition of these induced maps gives the identity on the set A).

3. Universal coverings associated with Delaney symbols and their automorphism group

In this section, we apply the theory of G-coverings to the special case $G = \Sigma = \Sigma_I$ for some fixed I, i.e. to morphisms between Σ-chamber systems as defined in section 1. The result can be summarized as follows: one can associate with every Delaney symbol (as defined in [4]) an universal Σ-covering. By corollary 2.5, we obtain a presentation of the group of covering transformations of this group. For the geometrical meaning of this constructions see [4]. In the next section, we shall give an application to orbifolds.

3.1. Delaney symbols and universal coverings

First, we recall some definitions associated with the notion of a Delaney symbol. Our approach will be slightly different to the one given in [4], but this difference is not essential — the purpose is to shorten notation.

Definition 3.1 (Coxeter Matrix) *Let $\mathcal{C}$ be a Σ-set. The Coxeter matrix of $\mathcal{C}$ is the map $M : \mathcal{C} \to (\mathbf{N} \cup \{\infty\})^{I\times I}$ defined by*

$$\begin{aligned} M(C) &= (m_{ij}(C))_{i,j\in I} \quad \textit{where} \\ m_{ij}(C) &:= \min\{m \in \mathbf{N}\cup\{\infty\} \mid m \neq 0,\ C(\sigma_i\sigma_j)^m = C\} \\ &= [\langle\sigma_i\sigma_j\rangle : (\langle\sigma_i\sigma_j\rangle \cap \Sigma_C)] \end{aligned}$$

for every $C \in \mathcal{C}$.

Obviously, the function m_{ij} is constant on $\langle\sigma_i, \sigma_j\rangle$–orbits and $M(C)$ is symmetric with diagonal entries equal to 1. Now let $\pi : \mathcal{C}_1 \to \mathcal{C}_2$ be a Σ-covering and M^1, M^2 the respective Coxeter matrices. By the characterization of m_{ij} as the index of $\langle\sigma_i\sigma_j\rangle \cap \Sigma_C$ in $\langle\sigma_i\sigma_j\rangle$, it follows immediately: for all $i, j \in I, C \in \mathcal{C}_1$ there exists a number $b_{ij}(C) \in \mathbf{N}\cup\{\infty\}$ such that

$$m^1_{ij}(C) = m^2_{ij}(\pi(C))b_{ij}(C).$$

Adopting the usual conventions about multiplication with '∞' and setting $b_{ij}(C) = 1$, if $m^1_{ij}(C) = m^2_{ij}(C) = \infty$, it is obvious that the numbers b_{ij} are uniquely determined by π.

Definition 3.2 *Using the notation above, the map $B^\pi : \mathcal{C}_1 \to (\mathbf{N}\cup\{\infty\})^{I\times I}$ is defined by $B^\pi(C) = (b_{ij}(C))_{i,j\in I}$ for $C \in \mathcal{C}_1$ and is called the branching index of π.*
A map $L : \mathcal{C}_2 \to \mathbf{N}$ is called a branching limit for π if it satisfies the following two conditions:

i) $L(C) := (\ell_{ij}(C))_{i,j\in I}$ is symmetric with diagonal entries equal to 1 and l_{ij} is constant on $\langle\sigma_i, \sigma_j\rangle$-orbits.

ii) $b_{ij}(C)$ divides $l_{ij}(\pi(C))$ for all $C \in \mathcal{C}_1,\ i, j \in I$.

A Delaney symbol is a pair $(\mathcal{C}, L)$ consisting of a transitive Σ-set $\mathcal{C}$ and a branching limit for $\mathrm{id}_{\mathcal{C}}$.

With every Delaney symbol one can associate a Σ-covering in a canonical way.

Theorem 3.3 *Let $(\mathcal{C}, L)$ be a Delaney symbol and $C_0 \in \mathcal{C}$. There is a Σ-covering $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ and $\tilde{C}_0 \in \tilde{\mathcal{C}}$ satisfiying following universal properties:*

i) L is a branching limit for π

ii) For every Σ-covering $\psi : \mathcal{C}_1 \to \mathcal{C}$ with L a branching limit for ψ and $C_1 \in \mathcal{C}_1$ with $\psi(C_1) = C_0$ there exists one and only one Σ-covering $\varphi : \tilde{\mathcal{C}} \to \mathcal{C}_1$ with $\varphi(\tilde{C}_0) = C_1$ and $\pi = \psi \circ \varphi$.

By this universality π is well defined up to isomorphism.

Proof: Define $\Sigma(C_0, L)$ to be the following normal subgroup of Σ_{C_o}:

$$\Sigma(C_0, L) := \langle \tau(\sigma_i\sigma_j)^{m_{ij}(C\tau)\ell_{ij}(C\tau)}\tau^{-1} \mid \tau \in \Sigma,\ i, j \in I\rangle$$

Now define $\tilde{\mathcal{C}} := \Sigma(C_0, L)\backslash\Sigma$ with Σ operating by multiplication from the right and

$$\widetilde{C}_0 := \Sigma(C_0, L)1.$$

One observes that a G-covering $\psi : \mathcal{C}_1 \to \mathcal{C}$ with $\psi(C_1) = C_0$ has L as a branching limit, if and only if $\Sigma(C_0, L) \subseteq \Sigma_{C_1}$. Therefore $\pi : \tilde{\mathcal{C}} \to \mathcal{C}$ is defined by $\pi(\widetilde{C}_0) = C_0$ and has the desired properties due to lemma 2.2. □

3.2. Presentation of Aut (π)

By Corollary 2.5, we obtain the following infinite presentation of Aut (π):

$$1 \longrightarrow \Sigma(C_0, L) \longrightarrow \Sigma_{C_0} \longrightarrow \mathrm{Aut}\,(\pi) \longrightarrow 1$$

From this we want to construct a finite presentation of Aut (π). In order to do this, we need some technical definitions:
A *path* in $\mathcal{C}$ is an element $w = (\tau, C) \in \Sigma \times \mathcal{C}$. Its *origin* is $\alpha(w) := C$ and its endpoint is $\omega(w) := C\tau$. If $w_i = (\tau_i, C_i)$ $(i = 1, 2)$ are two paths with $\alpha(w_2) = \omega(w_1)$, then their concatenation is defined to be

$$w_1 * w_2 := (\tau_1\tau_2, C_1).$$

Obviously $\alpha(w_1 * w_2) = \alpha(w_1)$ and $\omega(w_1 * w_2) = \omega(w_2)$. For a path $w = (\tau, C)$, its *reduced presentation* is w itself, if $\tau = 1$, otherwise it is the sequence

$$(\sigma_{i_1}, C_1), \ldots, (\sigma_{i_k}, C_k)$$

defined by the following two conditions:
 i) $w = (\sigma_{i_1}, C_1) * \cdots * (\sigma_{i_k}, C_k)$
 ii) $i_j \neq i_{j+1}$ for $j = 1, \ldots, k - 1$
The unique existence of the reduced presentation is implied by the unique existence of the reduced presentation for the free Coxeter group Σ. A *section* is a map $s : \mathcal{C} \to \Sigma$ with $s(C_0) = 1$ and $C_0 s(C) = C$. For the Delaney symbol $(\mathcal{C}, L)$ and a section s, we define the presentation of a group $G := G(\mathcal{C}, L, s)$, which is shown to be isomorphic to Aut (π). The generators of G are the elements of $I \times \mathcal{C}$. In order to obtain short notation for writing down the relation, we adopt the following convention: Let w be a path. By $\langle w\rangle$ we denote the following product of generators:

$$\langle w\rangle := \begin{cases} 1 & \text{, if } w = (1, C) \\ (i_1, C_1)\ldots(i_k, C_k) & \text{, if } (\sigma_{i_1}, C_1), \ldots, (\sigma_{i_k}, C_k) \text{ is the} \\ & \text{ reduced presentation of } w. \end{cases}$$

There are three kinds of relations:

$$\begin{array}{lrcll} \mathrm{I} & (i, C)\,(i, C\sigma_i) & = & 1 & \text{for all } i \in I,\ C \in \mathcal{C} \\ \mathrm{II} & \langle(s(C), C_0)\rangle & = & 1 & \text{for all } C \in \mathcal{C} \\ \mathrm{III} & \langle(z_{ij}(C), (C_0)\rangle^{\ell_{ij}(C)} & = & 1 & \text{for all } i, j \in I,\ i \neq j,\ C \in \mathcal{C}, \\ & & & & \text{here } z_{ij}(C) := (\sigma_i\sigma_j)^{m_{ij}(C)} \end{array}$$

Theorem 3.4 *The map* $h : G \to \Sigma_{C_o}/\Sigma(C_0, L)$ *defined by*

$$h(i, C) := s(C)\sigma_i\, s(C\sigma_i)^{-1}\Sigma(C_0, L)$$

is an isomorphism. Therefore it induces an isomorphism between G *and* $Aut(\pi)$.

Proof: In a first step, we are considering the case that $L = L_\infty$ is constant and equals the matrix with all non–diagonal entries equal to ∞. In this case, $\Sigma(C_0, L)$ is trivial and there is no relation of type III. In order to show that h is well defined, we have to show that h is compatible with I, II:

I $s(C)\sigma_i s(C\sigma_i)^{-1}s(C\sigma_i)\sigma_i s(C) = 1$

II The case $C = C_0$ is trivial. Let $C \neq C_0$ and $(\sigma_{i_1}, C_1), \ldots, (\sigma_{i_k}, C_k)$ be the reduced presentation of $(s(C), C_0)$. One has to show:

$$s(C_1)\sigma_{i_1}s(C_1\sigma_{i_1})^{-1}\ldots s(C_k)\sigma_{i_k}s(C_k\sigma_{i_k})^{-1} = 1$$

But we know:

$$\begin{array}{rcll} s(C_{j+1}) & = & s(C_j\sigma_{i_j}) & , j = 1, \ldots, k-1 \\ C_1 & = & C & , \text{ i.e. } s(C_1) = 1 \\ C & = & C_k\sigma_{i_k} & , \text{ i.e. } s(C_k\sigma_{i_k}) = \sigma_{i_1}\ldots\sigma_{i_k} \end{array}$$

Lemma 3.5 *i)* $\langle w_1 * w_2\rangle = \langle w_1\rangle\langle w_2\rangle$

ii) $\forall g \in G\ \exists \tau \in \Sigma_{C_o} : g = \langle(\tau, C_0)\rangle$

iii) $h(\langle\tau, C\rangle) = s(C)\tau s(C\tau)^{-1}$

With the help of this lemma, it is not difficult to prove that h is an isomorphism onto Σ_{C_o}:

- *surjectivity*: Let $\tau \in \Sigma_{C_0}$. Due to iii), one has $h(\langle\tau, C_0\rangle) = \tau$.
- *injectivity*: Let $g \in G$ with $h(g) = 1$. Choose $\tau \in \Sigma_{C_o}$ according to ii) with $g = \langle(\tau, C_0)\rangle$. Due to iii), one has $1 = h(g) = h(\langle\tau, C_0\rangle) = \tau$ and therefore $g = \langle(1, C_0)\rangle = 1$.

Before proving the lemma, we consider the general case: If L is any branching limit, then $G(\mathcal{C}, L, s)$ is $G(\mathcal{C}, L_\infty, s)/N$, with N the normal hull of elements of the form III. On the other hand: the normal hull of the image of these elements under h is equal to $\Sigma(C_0, L)$. Therefore h induces an isomorphism $\overline{h} : G(\mathcal{C}, L, s) \to \Sigma_{C_o}/\Sigma(C_0, L)$.

Proof of lemma 3.5:

i) Without loss of generality, we assume that w_1, w_2 are non–trivial paths.
Let $(\sigma_{i_1}, C_1), \ldots, (\sigma_{i_k}, C_k)$ and $(\sigma_{j_1}, D_1), \ldots, (\sigma_{jm}, D_m)$ be the respective reduced presentations. The reduced presentation of $w_1 * w_2$ is obtained by deleting subsequences of the form $(\sigma_i, C)\,(\sigma_i, C\sigma_i)$ in

$$(\sigma_{i_1}C_1), \ldots, (\sigma_{i_k}, C_k), (\sigma_{j_1}, D_1), \ldots, (\sigma_{jm}, D_m).$$

Therefore by relation I equality i) holds.

ii) Due to i), one only has to consider the case $w = (i, C)$. Relation II and i) now imply:

$$\begin{aligned} g &= \langle s(C)\rangle (i, C) \langle s(C\sigma_i)^{-1}, C\sigma_i\rangle \\ &= \langle s(C)\sigma_i \, s(C\sigma_i)^{-1}, C_0\rangle. \end{aligned}$$

And due to the fact that s is a section, one has

$$\tau := s(C)\sigma_i \, s(C\sigma_i)^{-1} \in \Sigma_{C_o}.$$

iii) Due to i), one only has to consider the case $(\tau, C) = (\sigma_i, C)$. But in that case, the assertion iii) restates the definition of h.

□

If one analyzes the definition of $G(\mathcal{C}, L, s)$, one can give the following algorithm for constructing a presentation of Aut (π), using the coloured graph (Γ, γ) associated with $\mathcal{C}$ in section 1. Choose a spanning tree T of Γ and give every edge an orientation. As generators of the group G', which we want to construct, take the set $E(\Gamma)$ of the oriented edges of Γ. The inverse of an edge is the edge with the inverse orientation. In order to describe the defining relations of the group, we adopt the following notation:

Let B denote the subset of $E(\Gamma)$ consisting of all loops. If p is a path in Γ, we denote by

$$\langle p\rangle := e_{i_1}^{\varepsilon_1} \ldots e_{i_k}^{\varepsilon_k}$$

the result of 'reading' the edges in p according to their orientation.

Recall, that if $J = \{i, j\} \subset I$, an J–residue containing the vertex v is an J–coloured graph which corresponds to $C\langle \sigma_i, \sigma_j\rangle$ in $\mathcal{C}$, where $C = C_v$. The relations for G' are the following:

I $e^2 = 1$, if $e \in B$.

II $e = 1$, if $e \in T$.

III,a) For each closed $\{i, j\}$–residue, choose a vertex v and a simple closed path p starting at v in this 2-residue, and take as relation:

$$\langle p\rangle^{\ell_{ij}(C_v)} = 1$$

III,b) Each open $\{i, j\}$–residue contains exactly two loops b_1, b_2 at the vertices v_1, v_2. Take the simple path p joining v_1 and v_2 and take as relation $(C = C_{v_1})$:

$$[b_1\langle p\rangle b_2 \langle p\rangle^{-1}]^{\ell_{ij}(C)} = 1$$

The following considerations show that this algorithm is correct: The tree T defines a section S_T, namely: $S_T(C) = \sigma_{i_1} \ldots \sigma_{i_k}$, if $i_1, \ldots, i_k$ is the sequence of colours of edges of the unique path in T from C_0 to C. Define the map $h : E(\Gamma) \to I \times \mathcal{C}$ by $h(e) = (\sigma_i, C)$, if $e \in E(\Gamma), \gamma(e) = i$ and $C = C_v$ with v origin of e. It is now an easy exercise to verify that the map h induces an isomorphism $\widetilde{h} : G' \longrightarrow G(\mathcal{C}, L, S_T)$.

4. Applications to orbifolds

In this section, we are going to associate with any given orbifold a corresponding Delaney symbol. Then we will apply the results obtained in the previous section to obtain a finite presentation for the fundamental group of the orbifold, alternative to the one given in [2].

The notion of an orbifold was introduced by Thurston [7]. We recall the basic definitions and facts, for more details see the cited reference.

An *orbifold* $\mathcal{O}$ is a paracompact, Hausdorff space $|\mathcal{O}|$ together with (compatible) modellings of neighbourhoods of each point in $|\mathcal{O}|$ on $\mathbf{R}^n/G$, where G is some finite group acting (compatible) on $\mathbf{R}^n$. This atlas is closed under intersection. If $U = \tilde{U}/G \subseteq V = \tilde{V}/H$ are two such modellings, then $\tilde{U}$ can be embedded in $\tilde{V}$ and G can be considered as subgroup of H, such that the embeddings are equivariant with respect to G.

Depending on whether the embedding is topological, piecewise linear or differentiable, we speak of a topological, piecewise linear or differentiable orbifold. Since our considerations apply to all these cases, we shall speak in the sequel simply of orbifolds, without specifying a type. Of course, the atlas of modellings is not an intrinsic part of the orbifold structure.

Each point x in an orbifold $\mathcal{O}$ is associated with a group G_x, well defined up to isomorphism: in a local coordinate system $U = \tilde{U}/G$, the group G_x is just the stabilizer group of an arbitrary point in $\tilde{U}$ corresponding to x. For each x, we obviously can always choose a modelling of the form $U = \tilde{U}/G_x$, where only one point $\tilde{x}$ in $\tilde{U}$ corresponds to x.

The *singular set of* $\mathcal{O}$, denoted by $S_{\mathcal{O}}$, consists of all such points which have a non–trivial stabilizer group, i.e., $S_{\mathcal{O}} = \{x \in |\mathcal{O}| \mid G_x \neq \{1\}\}$.

A *covering orbifold* of an orbifold $\mathcal{O}$ is an orbifold $\tilde{\mathcal{O}}$ together with a projection $p : |\tilde{\mathcal{O}}| \to |\mathcal{O}|$, such that each point $x \in |\mathcal{O}|$ has a neighbourhood $U = \tilde{U}/G$ and each component V of $|p^{-1}|(U)$ is homeomorphic to $\tilde{U}/H$, where H is a subgroup of G and the projection p restricted to V is just the obvious projection from $\tilde{U}/H$ to $\tilde{U}/G$. For simplicity we shall denote a covering orbifold by $p : \tilde{\mathcal{O}} \to \mathcal{O}$.

There exists a unique orbifold $\tilde{\mathcal{O}}$, which is the *universal covering of* $\mathcal{O}$. It has the property, that for any other covering orbifold $q : \mathcal{O}' \to \mathcal{O}$, the covering $p : \tilde{\mathcal{O}} \to \mathcal{O}$ factors through q, i.e., there is a covering $r : \tilde{\mathcal{O}} \to \mathcal{O}'$ such that $q \circ r = p$.

The *fundamental group* $\pi_1^{orb}(\mathcal{O})$ of an orbifold $\mathcal{O}$ is the group of covering transformations of the universal covering $\tilde{\mathcal{O}}$.

These are all basic facts about orbifolds we need. Assume now that $\mathcal{O}$ is an n-dimensional orbifold with a numbered triangulation K, such that the singular set $S_{\mathcal{O}}$ of $\mathcal{O}$ is a subcomplex of $|K|$.

Let $\mathcal{C}$ be the Σ_I-chamber system associated with K, such that $\|\mathcal{C}\| = |\mathcal{O}|$. For each $C \in \mathcal{C}$, the n-simplex of $\|\mathcal{C}\|$ corresponding to the chamber C is denoted by Δ_C. Its vertices are labelled with the index set $I = \{0, 1, \ldots, n\}$. Let $C \in \mathcal{C}$ and λ denote a $(n-2)$–dimensional face of Δ_C. Then the vertices of λ are labelled with the colours in $I \backslash \{i, j\}$ for some $i, j \in I$.

Since the singular set is a subcomplex of $\|\mathcal{C}\|$, the type of G_x does not change, when x runs through λ: If λ is not contained in the boundary of $|\mathcal{O}|$, then consider some neighbourhood $U = \tilde{U}/G_x$ of x in the open star of λ. The open star is the set of all points which are contained in the interior of some simplex which contains λ. The canonical projection from $\tilde{U}$ onto U is a topological covering outside the preimage of λ. If U is suitable choosen, then the complement of λ in U has the homotopy type of a circle. This also holds true for the preimage in $\tilde{U}$. Therefore the projection is a branched covering, i.e. G_x is generated by a rotation of order b around an axis of codimension 2. If λ is contained in the boundary of $|\mathcal{O}|$, then G_x contains elements which do not preserve orientation. If we look at $\tilde{U}/G_x^0$, where G_x^0 denotes the subgroup of orientation preserving elements of G_x, then we are in the latter case, since we can obtain $\tilde{U}/G_x^0$ by just gluing two disjoint copies of U along their respective intersection with the boundary of $|K|$. Hence G_x is a dihedral group of order $2b$, generated by one or two reflections at hyperplanes.

Set $l_{ij}(C) := b$. Thus we have defined a function $L : \mathcal{C} \to (\mathbf{N} \cup \{\infty\})$, such that $(\mathcal{C}, L)$ is a Delaney symbol.

Proposition 4.1 *If $p : \mathcal{O}' \to \mathcal{O}$ is an orbifold covering, then $p = |\pi|$ for a suitable Σ–covering $\pi : \mathcal{C}' \to \mathcal{C}$ with branching limit L. On the other hand: If π is given, then $\|\mathcal{C}'\|$ can be given an orbifold structure, such that $|\pi|$ is an orbifold covering.*

Before we prove this statement, we draw an immediate consequence:

Corollary 4.2 *The fundamental group of the orbifold $\|\mathcal{O}\|$ can be identified with the group of decktransformations of the universal Σ–covering of $(\mathcal{C}, L)$. In particular, we obtain a presentation of the fundamental group by the procedure described in the previous section.*

Proof: The first implication: We have to find a chamber system $\mathcal{C}'$ together with a Σ–covering π, such that $\|\mathcal{C}\|$ can be identified with $|\mathcal{O}|$ in a manner which identifies $|\pi|$ with the orbifold covering. In order to do this, we just lift the triangulation of $\|\mathcal{C}\| = |\mathcal{O}|$ along p:

The chambers of $\mathcal{C}'$ are the connected components of the preimage of the set of all points in $\|\mathcal{C}\|$ which are contained in the interior of some Δ_C. Due to the fact, that the singular set is a subcomplex of $\|\mathcal{C}\|$, the projection maps each such component C' linearly onto the interior of some $\Delta_{\pi(C')}$. Thus we have constructed a map $\pi : \mathcal{C}' \to \mathcal{C}$.

Moreover: if x is some point in $|\mathcal{O}'|$, such that $p(x)$ is contained in the interior of some simplex of codimension 1, then the stabilizer $G_{p(x)}$ can consist only of the identity and at most one reflection. Hence at most two elements of $\mathcal{C}'$ have that point in their closure, and we can thus define a Σ–action on $\mathcal{C}'$, such that π becomes a Σ–covering.

Furthermore, we can identify the union W of the subsets $C' \in \mathcal{C}'$ of $|\mathcal{O}'|$ with the union X of the interiors of $\Delta_{C'}$ in $\|C'\|$ in such a way, that $|\pi|$ can be identified with p, if we restrict both maps to W and X, respectively. That we can identify $\|\mathcal{C}'\|$ with $|\mathcal{O}'|$ and $|\pi|$ with p follows now from the fact, that $\|\mathcal{C}'\|$ is the metrical completion of X and $|\mathcal{O}'|$ the metrical completion of W with respect to some metric compatible with the linear structure of $|\mathcal{O}| = \|\mathcal{C}\|$ and $\|\mathcal{C}'\|$ respectively. (Here we need the assumption, that all considered spaces are locally compact).

Note, that these considerations are local in nature. We can prove in the same way: Let $x \in |\mathcal{O}'|$ with neighbourhood $U = \tilde{U}/G_x$. There exists a Σ–chamber system $\mathcal{C}_U$, some point $y \in \|\mathcal{C}_U\|$ and some neighbourhood V of y, such that the following holds:
The neighbourhood V can be identified with U, such that x and y are identified. If we look at the set $\mathcal{C}_x$ of simplices Δ_C which contain $x = y$, then V is contained in the union of these simplices. The set $\mathcal{C}_x$ is some suborbit Σ_J of $\mathcal{C}_U$ and the projection p restricted to V induces a Σ_J–covering between $\mathcal{C}_x$ and $\mathcal{C}_{p(x)}$. The latter set is defined as above.

In particular, we can identify $\tilde{U}$ itself with an open set in some Σ_J–orbit $\tilde{\mathcal{C}}_x$ of a suitable Σ–chamber system $\tilde{\mathcal{C}}_{\tilde{U}}$ and G_x acts by automorphisms of $\tilde{\mathcal{C}}_x$. Moreover, the projection onto U defines a Σ_J–covering π_x of $\mathcal{C}_{p(x)}$. Due to the construction of the branching limit given above, this covering has the restriction of L as branching limit. Furthermore, it is universal with respect to this branching limit L_x for $\mathcal{C}_x$. This is a consequence of the following considerations: The case that J is of cardinality less or equal to 2 is obvious. In the other case, the complement Y of the singular set in U is simply connected, since it is homotopically equivalent to a sphere of dimension $\#J - 1$. On the other hand, a Σ_J–covering of $\tilde{\mathcal{C}}_x$ with the same branching limit over $\mathcal{C}_{p(x)}$ would yield a topological covering of Y. Hence this covering is a homeomorphism and the corresponding Σ_J–covering an isomorphism. Therefore $\tilde{\mathcal{C}}_x$ is universal with respect to the restriction of L to $\mathcal{C}_{p(x)}$ and J.

This fact implies: If $\pi' : \mathcal{C}' \to \mathcal{C}$ is any Σ–covering of $\mathcal{C}$ with branching limit L, then the neighbourhood U' of some point x', which is mapped by $|\pi'|$ to $p(x)$ can be identified with $\tilde{U}/G$ for some subgroup G of $G_{p(x)}$ according to the above considerations. This proves the second implication. □

Acknowledgements: We express our thanks to A.W.M. Dress for his encouragement to write this article and to Daniel Huson and Olaf Delgado Friedrichs for helpful discussions. Both authors were partially supported by the Center for Interdisciplinary Research (ZIF), Bielefeld (Germany).

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Received September 23, 1994