Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 1, 75-86.

Immersions into Spheres with Parallel Higher Fundamental Forms

Antonella Carfagna D'Andrea
Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate, Università degli Studi di Roma "La Sapienza", Via A. Scarpa, 16, 00161 Roma

Abstract. In this note we prove helical minimal immersions of a compact rank one symmetric space into spheres are 2-symmetric submanifolds.

We give conditions under which a nicely-curved 2-symmetric submanifold of a space of constant curvature $c \geq 0$ is a helical immersion and vice versa.

MSC 1991: 53C42, 53A07

Introduction

The notion of k-symmetric submanifold of the Euclidean space was given by Kowalski and Külich [5]. In the case $k = 2$, 2-symmetric submanifolds are a generalization of the symmetric submanifolds of the Euclidean space introduced and classified by Ferus [2].

In [1] a local extension of the notion of 2-symmetric submanifolds is given considering immersions into a space of constant curvature. Moreover it is proved that a nicely curved submanifold [see definition 1.2] of a space of constant curvature $c \geq 0$ is a 2-symmetric submanifold if and only if all higher fundamental forms $\overset{k}{S}$ are parallel, i.e., $\overset{k+1}{\nabla} \overset{k}{S} = 0$ where $\overset{k+1}{\nabla}$ is the derivative of $\overset{k}{S}$ as a multilinear form with values in the $(k+1)$-th normal bundle.

In this note we prove that the minimal immersions of a compact rank one symmetric space into a unit sphere S_1^n with higher fundamental forms isotropic [10] have all higher fundamental forms parallel, so they are 2-symmetric submanifolds of S_1^n. As a consequence helical minimal geodesic immersions into spheres (see for the definition [7]) and standard minimal immersions into spheres (see [3]) are examples of 2-symmetric submanifolds.

In general, when the immersions are not minimal, we give conditions under which a nicely curved 2-symmetric submanifold of a space of constant curvature $c \geq 0$ is a helical geodesic immersion and vice versa, applying the Frenet theory of immersed curves in Riemannian manifolds.

In section 1 the notions of the higher fundamental forms, and some results about the isotropic fundamental forms are recalled.

Section 2 deals with the minimal immersions of a compact rank one symmetric space into a unit sphere and helical geodesic immersions.

Sections 3 and 4 are devoted to the study of the relations between the isotropic minimal immersions, helical geodesic immersions and 2-symmetric submanifolds.

1. Riemannian submanifolds and higher fundamental forms

Let: $\varphi : M \to \tilde{M}$ be an isometric immersion of a m-dimensional Riemannian manifold M into a n-dimensional Riemannian manifold $\tilde{M}$ of constant curvature.

For any $p \in M$ denote by T_pM the tangent space of M at p and by N_pM the normal space of the immersion φ at p.

In a small neighbourhood U of any $p \in M$, φ is an imbedding, thus locally we can identify a vector field Y on M with its image $d\varphi(Y)$ on $\varphi(M) \subset \tilde{M}$.

Denote by $\tilde{\nabla}$ the Riemannian Levi-Civita connection of $\tilde{M}$ and by ∇ the induced connection on M via the projection on the tangent space TM of M, the Gauss formula is

$$\tilde{\nabla}_{X_p} Y = \nabla_{X_p} Y + \overset{0}{s}(X_p, Y_p)$$

where $X_p, Y_p \in T_pM$, Y is an extension of Y_p and $\overset{0}{s}(X_p, Y_p) = (\tilde{\nabla}_{X_p} Y)^{\perp}$ is the second fundamental form of φ at p ($\perp$ means taking the orthogonal complement in $T_{\varphi(p)}\tilde{M}$).

We define the first normal space $\overset{1}{\mathrm{N}}_pM$ of M at p to be the linear span of the image of $\overset{0}{s}$ in the normal space N_pM.

For convenience, we put $\overset{0}{\mathrm{N}}_pM = T_pM$.

We can define the third fundamental form $\overset{1}{S}_p$ of φ at p by

$$\overset{1}{S}_p(X_p, Y_p, Z_p) = \left(\tilde{\nabla}_{X_p} \overset{0}{s}(Y, Z)\right)^{\perp_2} \tag{1}$$

for Y, Z vector fields arbitrarily extensions of Y_p, $Z_p \in T_pM$, where $(,)^{\perp_2}$ is the nomal projection in N_pM on the space orthogonal to $\bigoplus_{i=0}^{1} \overset{i}{\mathrm{N}}_pM = T_p \oplus \overset{1}{\mathrm{N}}_pM$.

Denote by $\overset{2}{\mathrm{N}}_pM$ the linear span of the image of $\overset{1}{S}_p$, and call it the second normal space of M at p.

From the fact $\tilde{M}$ has constant curvature, $\overset{1}{S}$ is symmetric and moreover induces a linear map from $T_pM \times T_pM \times T_pM$ into $\overset{2}{\mathrm{N}}_pM$.

If k is any integer, the fundamental form $\overset{k-1}{S}_p$ and the normal space at p of order k are defined by recursion:

$$\overset{k-1}{S}_p(X_p, X_1, \dots, X_k) = \left(\tilde{\nabla}_{X_p} \overset{k-2}{S}(\tilde{X}_1, \dots, \tilde{X}_k)\right)^{\perp_k} \tag{2}$$

for $X_p, X_1, \ldots, X_k \in T_pM$ arbitrarily extended to vector fields $\tilde{X}_1, \ldots, \tilde{X}_k$ on M and $(,)^{\perp_k}$ is the orthogonal projection on the space normal to $\bigoplus_{i=0}^{k-1} \overset{i}{\mathrm{N}}_pM$.

Clearly the process must stop since $\dim(T_pM \oplus \overset{1}{\mathrm{N}}_pM \oplus \ldots \oplus \overset{k}{\mathrm{N}}_pM) \leq \dim T_{\varphi(p)}\tilde{M}$.

For each k, $\overset{k}{S}_p$ is a symmetric $k+2$-linear form on T_pM with values in $\overset{k+1}{\mathrm{N}}_pM$, because $\tilde{M}$ is of constant curvature. For more details on higher fundamental forms see [8], [3].

Definition 1.1 *Let m be the first integer, $m \geq 1$ such that $\overset{m}{S}_p \neq 0$ but $\overset{m+1}{S}_p \equiv 0$. We call m the degree of the immersion φ at p.*

Definition 1.2 *The submanifold $\varphi(M) \subset \tilde{M}$ is called nicely curved if the dimension of each normal space $\overset{k}{\mathrm{N}}_pM$ does not depend on $p \in M$.*

Definition 1.3 *We say that an immersion $\varphi : M \to \tilde{M}$ is full if $\varphi(M)$ is not contained in a totally geodesic submanifold of $\tilde{M}$.*

It is known that if $\tilde{M}$ is a space of constant curvature and $\varphi(M)$ is a connected, nicely curved submanifold of $\tilde{M}$, φ is full in a totally geodesic submanifold of $\tilde{M}$ of dimension equal to dimension of $\bigoplus_{i=0}^{m} \overset{i}{\mathrm{N}}_pM$. In particular $N_pM = \bigoplus_{i=1}^{m} \overset{i}{\mathrm{N}}_pM$ [8].

Clearly, for a nicely curved submanifold we have a vector bundle $\overset{k}{\mathrm{N}}M$ whose fiber at p is $\overset{k}{\mathrm{N}}_pM$, for each k.

The connection $\tilde{\nabla}$ induces a connection $\overset{k}{\nabla}$ on the vector bundle $\overset{k}{\mathrm{N}}M$ by the formula

$$\overset{k}{\nabla}_{X_p} \overset{k}{\xi} = \perp_k \left(\tilde{\nabla}_{X_p} \overset{k}{\xi} \right) \qquad X_p \in T_pM, \quad \overset{k}{\xi} \in \Gamma(\overset{k}{\mathrm{N}}M)$$

where $\Gamma(\overset{k}{\mathrm{N}}M)$ are the sections of $\overset{k}{\mathrm{N}}M$, and $\perp_k$ is the orthogonal projection on $\overset{k}{\mathrm{N}}_pM$.

Setting

$$\perp_{k-1} \left(\tilde{\nabla}_{X_p} \overset{k}{\xi} \right) = - \overset{k}{\mathrm{A}} \left(X_p, \overset{k}{\xi}_p \right)$$

$$\perp_{k+1} \left(\tilde{\nabla}_{X_p} \overset{k}{\xi} \right) = \overset{k}{s} \left(X_p, \overset{k}{\xi}_p \right)$$

we have the Frenet formulas [8]

$$\tilde{\nabla}_{X_p} \overset{k}{\xi} = - \overset{k}{\mathrm{A}}(X_p, \overset{k}{\xi}_p) + \overset{k}{\nabla}_{X_p} \overset{k}{\xi} + \overset{k}{s}(X_p, \overset{k}{\xi}_p) \tag{3}$$

It can easily be seen that $\overset{1}{\mathrm{A}}$ is the restriction of classical Weingarten map to $\overset{1}{\mathrm{N}}M$ and this justifies the notion *Weingarten map of order k* for $\overset{k}{\mathrm{A}}$.

By (1) and (2) we have

$$\overset{0}{s}(X_1, X_2) = \overset{0}{S}(X_1, X_2)$$

$$\overset{1}{s}(X_1, \overset{0}{s}(X_2, X_3)) = \overset{1}{S}(X_1, X_2, X_3)$$

and in general

$$\overset{k}{S}(X_1, \dots, X_{k+2}) = \overset{k}{s}(X_1, \overset{k-1}{s}(X_2, \dots, \overset{0}{s}(X_{k+1}, X_{k+2}) .$$

If we denote by $\langle\, , \,\rangle$ the inner product on the normal bundles, then

$$\langle \overset{k}{\mathrm{A}}(X_p, \overset{k}{\xi}), \overset{k-1}{\eta} \rangle = \langle \overset{k}{\xi}, \overset{k-1}{s}(X_p, \overset{k-1}{\eta}) \rangle$$

where $\overset{k}{\xi} \in \overset{k}{\mathrm{N}}M$, $\overset{k-1}{\eta} \in \overset{k-1}{\mathrm{N}}\, M$, see [8].

For any $\overset{k}{\mathrm{N}}$-valued symmetric k-linear form Q on T_pM, we can define the derivative by the formula

$$(4) \qquad (\overset{k}{\nabla} Q)(X, X_1, \dots, X_k) = \overset{k}{\nabla}_X Q(\tilde{X}_1, \dots, \tilde{X}_k) - \sum_{i=1}^{k} Q(X_1, \dots, \nabla_X \tilde{X}_i, \dots, X_k)$$

for $X, X_1, \dots, X_k \in T_pM$ arbitrarily extended to vector fields $\tilde{X}_1, \dots, \tilde{X}_k$ on M.

O'Neill [6] has given the following

Definition 1.4 *The immersion φ is isotropic at a point p if the norm $\| \overset{0}{s}(X, X) \|$ of the vector $\overset{0}{s}(X, X)$ is equal to a constant for any unit vector $X \in T_pM$.*

These three conditions are equivalent :

1) φ is isotropic at p;
2) $\sum \overset{1}{\mathrm{A}}(X, \overset{0}{s}(Y, Z)) = \lambda^2 \sum \langle X, Y \rangle Z$
   for every $X, Y, Z \in T_pM$ where $\sum$ denotes the cyclic sum with respect to X, Y, Z, [4];
3) $\langle \overset{0}{s}(X, X), \overset{0}{s}(X, Y) \rangle = 0$ for any two orthonormal vectors X and Y in T_pM, [6].

In general we have

Definition 1.5 *The higher fundamental form $\overset{k}{S}$ is λ-isotropic at p if $\| \overset{k}{S}(X^{k+2}) \| =$ constant $= \lambda$, for any unit vector $X \in T_pM$ [11], where $\overset{k}{S}(X^{k+2}) = \overset{k}{S}(X, \dots, X)$.*

These three conditions are equivalent:

1) $\overset{k}{S}$ is λ-isotropic;

2) $\sum_{2k+4}\langle \overset{k}{S}(X_1, X_2, \ldots, X_{k+2}), \overset{k}{S}(X_{k+3}, \ldots, X_{2k+4}) =$
$= \sum_{2k+4}\langle X_1, X_2\rangle \ldots \langle X_{2k+3}, X_{2k+4}\rangle$
where $\sum_{2k+4}$ denotes the symmetrized of order $2k+4$, [11];

3) $\langle \overset{k}{S}(X^{k+2}), \overset{k}{S}(X^{k+1}, Y)\rangle = 0$
for any unit vector Y which is orthogonal to X, [4].

2. Minimal immersion of a compact rank one symmetric space into a unit sphere

Let $M = G/K$ be a n-dimensional compact homogeneous Riemannian manifold with an irreducible linear isotropy group and V^s the s-th eigenspace of the Laplacian Δ_M corresponding to the s-th eigenvalue λ_s. We define an inner product $\langle\, ,\, \rangle$ in V^s by

$$\langle f, h\rangle = \int_M fh\, dv \qquad f, h \in V^s\,.$$

where dv is the volume element of (M, g).

For simplicity, we normalize the canonical measure dv of (M, g) in such a way that $\int_M dv = \dim V^s = m_s + 1$.
Let $\{\underline{\varphi^i, i = 0, \ldots, m_s}\}$ be an orthonormal basis for V^s and define a map $\varphi : M \to R^{m_s+1}$ by $\varphi(p) = \{\varphi^0(p), \ldots, \varphi^{m_s}(p)\}$. The transitive action of G on M induces a natural action on V^s by $(g \circ f)(p) = f \circ (g^{-1}p)$, $g \in G$, $p \in M$.

It is easily seen that $\varphi(M)$ is contained into a sphere $S_1^{m_s}$ of curvature 1 and the metric on M induced by φ coincides with g up to a constant. Moreover φ is an immersion of (M, g) into $S_1^{m_s}$.

By a theorem of Takahashi [9], φ is then a minimal immersion of (M, cg) into $S_1^{m_s}$ with $c = \dfrac{\lambda_s}{n}$, called the *s-th standard minimal immersion of* M *into a sphere associated to the eigenvalue* λ_s.

All standard minimal immersions are full and equivariant, i.e. for each $g \in G$ there exists an element $\rho(g)$ of the isometry group of $S_1^{m_s}$ such that $\varphi(gp) = \rho(g)\varphi(p)$, $p \in M$. As a consequence of the invariance, we have that $\varphi(M)$ is nicely curved. Moreover the higher fundamental forms $\overset{k}{S}$ of a standard minimal immersion are isotropic [10].

Do Carmo and Wallace showed that the degree of s-th standard minimal immersion is s when M is a sphere $S_k^{m_s}$ of curvature k.

We recall that two such immersions φ_1 and φ_2 are equivalent if there exists an isometry σ of $S_1^{m_s}$ such that $\varphi_2 = \sigma \circ \varphi_1$.

Tsukada [10] proved the following

Theorem 2.1 *Let* $\varphi : M \to S_1^m$ *be a minimal immersion of a n-dimensional compact rank one symmetric space into a unit sphere corresponding to the s-th eigenvalue. If we assume that*

1) φ *is full,*

2) *in the case* $M = S^n$, *the degree of* φ *is* $\geq \left[\frac{s}{2}\right]$, *where* $\left[\frac{s}{2}\right]$ *is the largest integer less than or equal to* $\frac{s}{2}$, *in the other cases the degree of* φ *is* $\geq s$,

3) *the higher fundamental forms are isotropic for all* $p \in M$,

then φ *is equivalent to the standard minimal immersion corresponding to the same eigenvalue* s.

It is known that the standard minimal immersions of compact rank one symmetric space are helical geodesic. We recall the definition of a helical geodesic immersion given by Sakamoto [7].

Let $\gamma : I \to \tilde{M}$ be a curve parametrized by the arc length s. Let $\gamma_1(s) = X(s)$ the unit tangent vector and put $k_1 = \|\tilde{\nabla}_X X\|$. If k_1 is not identically zero, then we define γ_2 by $k_1\gamma_2 = \tilde{\nabla}_X X$ on the set $I_1 = \{s \in I \,|\, k_1(s) \neq 0\}$. Set $k_2 = \|\tilde{\nabla}_X \gamma_2 + k_1 X\|$. If k_2 vanishes on I_1 then γ is said of order 2. If k_2 is not identically zero on I_1, then we can define γ_3 by $\tilde{\nabla}_X \gamma_2 = -k_1 X + k_2 \gamma_3$.

Inductively, we can define γ_d and $k_d = \|\tilde{\nabla}_X \gamma_d + k_{d-1}\gamma_{d-1}\|$ and if $k_d = 0$ identically on $I_{d-1} = \{s \in I \,|\, k_{d-1}(s) \neq 0\}$ then γ is said of order d. $\gamma_1, \gamma_2, \ldots, \gamma_d$ are called Frenet vectors and $k_1, k_2, \ldots, k_{d-1}$ Frenet curvatures.

Definition 2.2 *Let* $\varphi : M \to \tilde{M}$ *be an isometric immersion of a connected Riemannian manifold* M *into a Riemannian manifold* $\tilde{M}$ *and* $\sigma : I \to M$ *be an arbitrary geodesic in* M *parametrized by the arc length if the curve* $\gamma = \varphi \circ \sigma$ *in* $\tilde{M}$ *is of order* d *and has constant curvatures* $k_1, k_2, \ldots, k_{d-1}$ *which do not depend on* σ, *then* φ *is called a helical immersion of order* d.

It is known [11]

Theorem 2.3 *Let* $\varphi : M \to S_1^m$ *be an immersion of a compact rank one symmetric space* M *into a unit sphere.* φ *is a helical geodesic immersion if and only if the* i*-th fundamental form is isotropic for* $0 \leq i \leq$ *degree of* φ.

3. Isotropic minimal immersions and 2-symmetric submanifolds

In the paper [1] the notion of 2-symmetric submanifold of a space $\tilde{M}$ of constant curvature is given.

Definition 3.1 $f : M \to \tilde{M}$ *is a* 2*-symmetric submanifold (or immersion) if, for every point* $p \in M$, *there exist neighbourhoods* U *of* p *in* M *and* $\tilde{U}$ *of* $f(p)$ *in* $\tilde{M}$ *and an isometry* $\sigma : \tilde{U} \to \tilde{U}$ *such that*

i) $\sigma^2 = Id_{\tilde{U}}$

ii) $f(U) \subseteq \tilde{U}$, *and* $\sigma(f(U)) \subseteq f(U)$

iii) $f(p)$ *is an isolated fixed point of* σ.

It is proved [1] that

Theorem 3.2 *$f : M \to \tilde{M}$ is a nicely curved 2-symmetric submanifold of a space $\tilde{M}$ of constant curvature $c \geq 0$ if and only if $\overset{h+1}{\nabla}\overset{h}{s} = 0$ or equivalently $\overset{h+1}{\nabla}\overset{h}{S} = 0$ for any h.*

Theorem 3.3 *Let $\varphi : M \to S_1^l$ be a minimal immersion of a compact rank one symmetric space into a unit sphere S_1^l of R^{l+1}. We assume φ is full, and the higher fundamental forms $\overset{k}{S}$ are isotropic for each k. Then we see that $\varphi(M)$ is a minimal 2-symmetric submanifold of S_1^l.*

Proof. We denote by $(\overline{\nabla}\overset{k}{S})(X_1, X_2, \ldots, X_{k+3})$ the derivative of $\overset{k}{S}$ defined by the formula

$$(5)\qquad \begin{aligned}&(\overline{\nabla}\overset{k}{S})(X_1, X_2, \ldots, X_{k+3}) =\\ &= \nabla^{\perp}_{X_1}\overset{k}{S}(\tilde{X}_2, \ldots, \tilde{X}_{k+3}) - \sum_{i=2}^{k+3}\overset{k}{S}(X_2, \ldots, \nabla_{X_1}\tilde{X}_i, \ldots, X_{k+3})\end{aligned}$$

for $X_1, \ldots, X_{k+3} \in T_pM$ arbitrarily extended to vector fields $\tilde{X}_1, \ldots, \tilde{X}_{k+3}$ on M, and $\nabla^{\perp}$ is the connection on the normal bundle NM. □

In the above hypothesis, it is proved that, for any point $p \in M$, the dimension of $\overset{k}{\mathrm{N}}_pM$ does not depend on p, therefore $\varphi(M)$ is nicely curved [11].

Moreover $\overline{\nabla}\overset{k}{S}$ has component only on the normal spaces $\overset{k}{\mathrm{N}}_pM$ and $\overset{k+2}{\mathrm{N}}_pM$ [11].

By the Frenet formulas (3), we have for $X_1, X_2, \ldots, X_{k+3} \in T_pM$ arbitrarily extended to vector fields $\tilde{X}_1, \tilde{X}_2, \ldots, \tilde{X}_{k+3}$

$$\begin{aligned}&(\overline{\nabla}\overset{k}{S})(X_1, X_2, \ldots, X_{k+3}) =\\ &= -\overset{k+1}{\mathrm{A}}(X_1, \overset{k}{S}(\tilde{X}_2, \ldots, \tilde{X}_{k+3})) + \overset{k+1}{\nabla}_{X_1}\overset{k}{S}(\tilde{X}_2, \ldots, \tilde{X}_{k+3}) + \overset{k+1}{S}(X_1, \ldots, X_{k+3})\\ &-\sum_{i=2}^{k+3}\overset{k}{S}(X_2, \ldots, \nabla_{X_1}\tilde{X}_i, \ldots, X_{k+3})\,.\end{aligned}$$

Since $\overset{k+1}{\mathrm{A}}(X_1, \overset{k}{S}) \in \overset{k}{\mathrm{N}}M$ and $\overset{k+1}{S} \in \overset{k+2}{\mathrm{N}}M$, we have for (4)

$$(\overset{k+1}{\nabla}\overset{k}{S})(X_1, \ldots, X_{k+3}) = 0\,.$$

By the theorems 2.3 and 2.1 we have the following corollaries:

Corollary 3.4 *Let $\varphi : M \to S_1^l$ be a helical minimal immersion of a compact rank one symmetric space into a unit sphere. We assume φ is full. Then φ is a 2-symmetric submanifold of S_1^l.*

Corollary 3.5 *Let $\varphi : M \to S^l_1$ be a standard minimal immersion of compact rank one symmetric space into the unit sphere S^l_1. Then φ is a 2-symmetric submanifold of S^l_1.*

Observation. For a helical immersion $\varphi : M \to S^l_1$ of a compact rank one symmetric space into a unit sphere, the Frenet vectors $\gamma_1, \gamma_2, \dots, \gamma_d$ satisfy the conditions:

$$\gamma_1 \in T_pM, \gamma_2 \in \overset{1}{\mathrm{N}}_pM, \dots, \gamma_d \in \overset{d-1}{\mathrm{N}}_pM\,.$$

In fact, by theorem 2.3 $\overset{k}{S}$ are isotropic, so

$$\gamma_2 = \frac{\overset{0}{S}(X,X)}{k_1}\,, \quad \gamma_3 = \frac{1}{k_1k_2}\overset{1}{S}(X,X,X)\,, \quad \dots \quad \gamma_d = \frac{1}{k_1k_2,\dots,k_{d-1}}\overset{d-2}{S}(X^d)\,,$$

where k_i are the normal curvatures, [11].

Moreover, since $\overset{k}{S}$ are isotropic, we have $\overset{1}{\mathrm{A}}(X,\overset{0}{s}(X,X)) = k_1^2X$ and in general

$$\overset{i}{\mathrm{A}}(X,\overset{i-1}{S}(X^{i+1}) = k_i^2\overset{i-2}{S}(X^i)$$

for $i = 2,\dots,d-1$.

4. Helical immersions and 2-symmetric submanifolds

In this section we show the following

Theorem 4.1 *Let $\varphi : M \to \tilde{M}$ be a nicely curved, 2-symmetric submanifold of a space $\tilde{M}$ of constant curvature $c \geq 0$. Denote by $\sigma : I \to M$ a geodesic of M and by $\gamma = \varphi \circ \sigma : I \to \tilde{M}$ its image curve in $\tilde{M}$. We assume that the Frenet vectors $\gamma_1,\dots,\gamma_{d-1}$ of every curve γ of $\tilde{M}$ satisfy the conditions:*

$$\gamma_h \in \overset{h-1}{\mathrm{N}}_pM\,, \quad 1 \leq h \leq d-1\,. \tag{6}$$

Then φ is a helical geodesic immersion.

Before the proof, we prepare the following lemmas.

Lemma 4.2 *Let $\varphi : M \to \tilde{M}$ be a nicely curved, 2-symmetric submanifold of a space $\tilde{M}$ of constant curvature $c \geq 0$. Then all the curvatures of the geodesics of M are constant along the geodesics.*

Proof. It is an easy consequence of the symmetry property of the 2-symmetric submanifolds, (see [1]).

Lemma 4.3 *In the hypothesis of theorem 4.1, we have*

$$\gamma_h = \frac{1}{k_1 k_2, \ldots, k_{h-1}} \overset{h-2}{S}(X^h) \quad \text{for any} \quad h. \tag{7}$$

Proof. By Frenet formulas, put

$$\gamma_2 = \frac{\overset{0}{s}(X,X)}{k_1}\,, \text{if } k_1 \neq 0\,, \gamma_3 \text{ is defined by } \tilde{\nabla}_X \frac{\overset{0}{s}(X,X)}{k_1} + k_1 X = k_2 \gamma_3\,,$$

so, by the hypothesis (6): $\gamma_3 \in \overset{2}{\mathrm{N}}_p M$, we have

$$\overset{1}{\mathrm{A}}(X, \overset{0}{s}(X,X)) = k_1^2 X\,, \tag{8}$$

for any X and $k_2\gamma_3 = \frac{1}{k_1}\overset{1}{S}(X^3)$.
Since $k_2 = \frac{1}{k_1}\Big\| \overset{1}{S}(X^3)\Big\|$, we have that k_2 is constant along γ.
If $k_2 \neq 0$ then $\gamma_3 = \frac{1}{k_1 k_2}\overset{1}{S}(X^3)$.
Now we apply the mathematical induction. For this we suppose that $k_1, k_2, \ldots, k_{\rho-1}$ are constant, different from zero, along γ and $\gamma_\rho = \frac{1}{k_1 k_2, \ldots, k_{\rho-1}} \overset{\rho-2}{S}(X^\rho)$. By Frenet formulas (3)

$$\frac{1}{k_1 k_2 \ldots k_{\rho-1}} \tilde{\nabla}_X \overset{\rho-2}{S}(X^\rho) + \frac{k_{\rho-1}}{k_1 k_2 \ldots k_{\rho-2}} \overset{\rho-3}{S}(X^{\rho-1}) = k_\rho \gamma_{\rho+1}$$

$$\frac{1}{k_1 k_2 \ldots k_{\rho-1}} \Big[- \overset{\rho-1}{\mathrm{A}}(X, \overset{\rho-2}{S}(X^\rho)) + \overset{\rho-1}{S}(X^{\rho+1})\Big] +$$

$$+ \frac{k_{\rho-1}}{k_1 k_2 \ldots k_{\rho-2}} \overset{\rho-3}{S}(X^{\rho-1}) = k_\rho \gamma_{\rho+1}\,.$$

By $\gamma_{\rho+1} \in \overset{\rho}{\mathrm{N}} M$, we have

$$\overset{\rho-1}{\mathrm{A}}(X, \overset{\rho-2}{S}(X^\rho) = k_{\rho-1}^2 \overset{\rho-3}{S}(X^{\rho-1}) \tag{9}$$

and $k_\rho \gamma_{\rho+1} = \frac{1}{k_1 k_2 \ldots k_{\rho-1}} \overset{\rho-1}{S}(X^{\rho+1})$. □

Now we prove

Lemma 4.4 *In the hypothesis of theorem 4.1, we have that the j-th fundamental forms are isotropic for $0 \leq j \leq d$.*

Proof. The second fundamental form is isotropic if and only if $\langle \overset{0}{s}(X,X), \overset{0}{s}(X,Y)\rangle = 0$ for any orthonormal vector fields X and Y of M (see section 1).

In our case, by property (8)

$$\langle \overset{0}{s}(X,X), \overset{0}{s}(X,Y)\rangle = \langle \overset{1}{\mathrm{A}}(X, \overset{0}{s}(X,X)), Y\rangle = k_1^2\langle X, Y\rangle = 0\,.$$

In general we have, by property (9)

$$\langle \overset{\rho}{S}(X^{\rho+2}), \overset{\rho}{S}(X^{\rho+1}, Y)\rangle = \langle \overset{\rho+1}{\mathrm{A}}(X, \overset{\rho}{S}(X^{\rho+2}), \overset{\rho-1}{S}(Y, X^{\rho})\rangle =$$

$$= k_{\rho+1}^2 \langle \overset{\rho-1}{S}(X^{\rho+1}), \overset{\rho-1}{S}(Y, X^{\rho})\rangle\,.$$

So the statement follows by induction. If $\overset{\rho-1}{S}$ is isotropic $\langle \overset{\rho-1}{S}(X^{\rho+1}), \overset{\rho-1}{S}(Y, X^{\rho})\rangle = 0$ (see section 1) and then $\overset{\rho}{S}$ is isotropic. □

Now the proof of the theorem 4.1 is an easy consequence of these lemmas. In fact, we proved that all normal curvatures are constant along γ and since all higher fundamental forms are isotropic at p, by (7) all curves emanating from a point p have the same curvatures. □

We can also prove the following

Theorem 4.5 *Let $\varphi : M \to S_1^l$ be a helical immersion of order d. We assume that, for any geodesic σ in M, the Frenet vectors of the curve $\varphi \circ \sigma$ satisfy the following conditions:*

$$\text{(10)} \qquad \gamma_h \in \bigoplus_{j=0}^{q} \overset{2j+1}{\mathrm{N}} M\,, \text{ if h is even}\,, \qquad \gamma_h \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M\,, \text{ if } h \text{ is odd}\,,$$

where $q = \left[\dfrac{d-1}{2}\right]$ and $p = \left[\dfrac{d}{2}\right]$, then $\varphi(M)$ is a 2-symmetric submanifold.

Proof. Put $X(s) = \dot{\sigma}(s)$. We note that for $\gamma_1 = X$ and $\gamma_2 = \dfrac{\overset{0}{s}(X,X)}{k_1}$ the conditions (10) are always satisfied.

Sakamoto [7] calculated the Frenet vectors of the curve γ.

Let $\overline{\nabla}^i \overset{0}{s}$ be denote the i-th order derivative of the second fundamental form (see (5)) we have that

$$\text{(11)} \qquad \gamma_j = (k_1 k_2 \dots k_{j-1})^{-1} \sum_i a_{ji} (\overline{\nabla}^{i-2} \overset{0}{s})(X^i)$$

for $j = 2, \dots, d$ where the index i runs over the range $\{2, 4, \dots, j\}$ if j is even, and $\{3, 5, \dots, j\}$ if j is odd. The coefficients a_{ji} are polynomials of the $k_1^2, \dots, k_{j-2}^2$ [7].

We will show that the conditions (10) are equivalent to

$$(12)\qquad \begin{cases} (\overline{\nabla}^i \overset{0}{s})(X^{i+2}) \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M\,, \quad \text{if } i \text{ is odd,} \\ (\overline{\nabla}^i \overset{0}{s})(X^{i+2}) \in \bigoplus_{j=0}^{q} \overset{2j+1}{\mathrm{N}} M\,, \quad \text{if } i \text{ is even,} \\ \text{for } 1 \le i \le h\,. \end{cases}$$

In fact, for $h = 3$ the equivalence is true since $\gamma_3 = (k_1k_2)^{-1}(\overline{\nabla}\,\overset{0}{s})(X,X,X)$. So $\gamma_3 \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M$ if and only if $\overline{\nabla}\,\overset{0}{s} \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M$. We proceed by induction on $h = 3, 4, \ldots$

If h is odd, then $\gamma_h \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M$ if and only if (12) is true for $1 \le i \le h$.

By (11) it follows that

$$\gamma_{h+1} = (k_1k_2\ldots k_h)^{-1}(a_{h+1,2}\,\overset{0}{s}(X,X) + a_{h+1,4}(\overline{\nabla}^2 \overset{0}{s})(X^4) + \ldots + (\overline{\nabla}^{h-1} \overset{0}{s})(X^{h+1})\,,$$

so $(\overline{\nabla}^{h-1} \overset{0}{s})(X^{h+1}) \in \bigoplus_{j=0}^{q} \overset{2j+1}{\mathrm{N}} M$ if and only if γ_{h+1} satisfies the condition (10).
For h even the proof is the same.

Now we can prove the statement of the theorem if we show the equivalence of the conditions (12) with $\overline{\nabla}^i \overset{i-1}{S} = 0$ for all $1 \le i \le h$.

For $i = 1$, since $(\overline{\nabla}\,\overset{0}{s})(X^3) = (\overset{1}{\nabla}\,\overset{0}{s})(X^3) + \overset{1}{S}(X^3)$, we have

$$(\overline{\nabla}\,\overset{0}{s})(X^3) \in \bigoplus_{j=0}^{p} \overset{2j}{\mathrm{N}} M \Longleftrightarrow (\overset{1}{\nabla}\,\overset{0}{s})(X^3) = 0\,.$$

For $i = 2$

$$(\overline{\nabla}^2 \overset{0}{s})(X^4) = -\overset{2}{\mathrm{A}}(X, \overset{1}{S}(X^3)) + \overset{2}{S}(X^4) + \overset{2}{\nabla}\,\overset{1}{S}(X^4)$$

and $(\overline{\nabla}^2 \overset{0}{s})(X^4) \in \bigoplus_{j=0}^{q} \overset{2j+1}{\mathrm{N}} M$ if and only if $(\overset{2}{\nabla}\,\overset{1}{S})(X^4) = 0$. If we calculate the other derivatives, since $\overset{h+1}{\nabla}\,\overset{h}{S} = 0$ is equivalent to $\overset{h-1}{\nabla}\,\overset{h}{\mathrm{A}} = 0$ for all h, [1], the component of the i-th order derivative in the normal space $\overset{i}{\mathrm{N}} M$ is given by $\overset{i}{\nabla}\,\overset{i-1}{S}(X^{i+2})$. So we have the equivalence of the conditions (12) with the $(\overline{\nabla}^i \overset{i-1}{S})(X^{i+2}) = 0$. Since $\overline{\nabla}^i \overset{i-1}{S}$ is symmetric by Codazzi's generalized equations [8], it follows that $\overline{\nabla}^i \overset{i-1}{S} = 0$. By theorem 3.2 we have the statement. $\square$

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Received June 30, 1995