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Two Problems Concerning the Area-perimeter Ratio of Lattice-point-free Regions in the Plane

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Abstract. We give a generalization of Bender's area-perimeter relation for plane lattice-point-free convex regions to simply connected regions, thus we solve a problem posed by M. Silver [10]. Further the result is used for a lattice version of the Dido problem.
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1. Introduction and Results

For a plane lattice-point-free convex region with area A and perimeter P Bender [1] proved the relation

$$\frac{A}{P} < \frac{1}{2}.$$

This result has been generalized in different ways. There are complete solutions for the generalization to higher dimensions [4] and to arbitrary lattices for $d = 2$ [9]. Further there are results for convex bodies containing a certain number of lattice points ([2],[7],[8]).

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Here we replace the condition of convexity by the condition 'simply connected' (throughout this paper a plane region is called simply connected if its boundary is a rectifiable Jordan curve and hence area and perimeter are defined). This problem was first posed by Silver [10] who conjectured that $1/2$ has to be replaced by $1/\pi + 1/4 = 0.568\ldots$. In this paper we shall prove that the correct constant is $0.582822\ldots$.

Theorem 1 *Let M be a lattice-point-free simply connected plane region and let $P(M)$ and $A(M)$ denote its perimeter and its area, respectively. Then*

$$\frac{A(M)}{P(M)} < \gamma, \tag{1}$$

where $\gamma = r(x_0) = 0.582822\ldots$ with

$$r(x) = \frac{1}{4\sin x} - \frac{\cos x}{4x} + \frac{\sin x}{2x} \tag{2}$$

and x_0 is the unique solution of the equation

$$\sin^2 x(x\sin x + 2x\cos x + \cos x - 2\sin x) = x^2\cos x. \tag{3}$$

The inequality is asymptotically tight, i.e. γ cannot be replaced by a smaller number.

A related problem is a lattice version of the well-known Dido problem. Here we add the condition, that there are latticelike arranged trees in the interior of the land and Queen Dido is not allowed to enclose a tree. More exactly, for a given length P we want to find a simply connected region $M \subset H^+ := \{(x,y) \in \mathbf{R}^2 : y \geq 0\}$ with free boundary length $\ell(\partial M \cap \operatorname{int}(H^+)) = P$ (ℓ denotes the length of a curve) and $\operatorname{int}(M) \cap \mathbf{Z}^2 = \emptyset$, which maximizes the enclosed area.

Based on Theorem 1 we prove the following result.

Theorem 2 *The optimal region in the Dido problem with lattice constraints consists of a lattice rectangle with height 1 and its basis on the shoreline, a number of circular caps, each joining two adjacent lattice points of level one and either two congruent caps or two different caps which form a semicircle together, joining the first and the last lattice point with the shoreline. The radii of the circular arcs are identical and this common radius is contained in $[1, \sqrt{5}/2]$. The intersection of the boundary of an optimal set and the shoreline is orthogonal. Further the ratio of the area and the perimeter is bounded by $\delta = t(y_0) = 1.04151\ldots$, where*

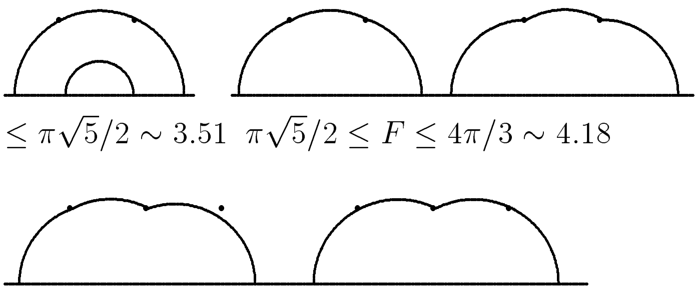
$$t(x) = \frac{1}{4\sin x} - \frac{\cos x}{4x} + \frac{\sin x}{x} \tag{4}$$

and y_0 is the unique solution of the equation

$$\sin^2 x(x\sin x + 4x\cos x + \cos x - 4\sin x) = x^2\cos x. \tag{5}$$

This bound is also asymptotically tight.

The optimal set is sketched for small values of P in Figure 1.



$F \leq \pi\sqrt{5}/2 \sim 3.51$ $\pi\sqrt{5}/2 \leq F \leq 4\pi/3 \sim 4.18$

$4\pi/3 \leq F \leq (\pi + 2\arctan(1/2))\sqrt{5}/2 \sim 4.54$

Figure 1

If the length P is enlarged the lattice rectangle corresponding to the optimal region will be stepwise increased by one unit. These limit lengths can be determined numerically.

2. The Existence of Optimal Regions

We will prove the existence of an optimal region for the first problem (for the Dido problem the arguments work analogously). Therefore it is convenient to consider *open* lattice-point-free simply connected regions bounded by a curve since the limit of closed lattice-point-free sets may have lattice points in its interior.

It is sufficient to consider curves for which the pieces between two lattice points are convex curves, since one can give for an arbitrary curve a curve of this form with at most the original length and at least the original area by taking the convex hull and eliminating the lattice points in the interior in a suitable way.

Now let $\{\gamma_n\}$ be a series of such curves $\gamma_n : [0,1] \to \mathbf{R}^2$ with $\ell(\gamma_n) = \text{ const } = P$ and let the interior of the enclosed sets M_n contain no lattice points. Further let

$$A(M_n) \to \sup_{\ell(\gamma)=P} A(M(\gamma)) =: S.$$

We may assume that the sequences of those lattice points passed through are identical.

The series of convex caps enclosed by the convex pieces of γ_n and the line segment joining the two lattice points have converging subsequences by the selection theorem of Blaschke (see e.g. [4]) (this subsequence can be chosen such that all caps converge).

So we obtain a limit set M as the union of the lattice polygon and all the open convex limit caps with length $P(M) = \lim \ell(\gamma_n) = P$ and area $A(M) = \lim A(M_n) = S$. Since M contains no lattice points it is optimal.

Note that two adjacent caps may have a straight line segment in common. In this case the length is counted twice. Further it follows that the boundary consists of circular arcs with equal curvature except the straight line segments, because if that were not the case the set could be improved locally by the usual isoperimetric property.

3. Proof of Theorem 1

Let $P = P(M)$ be fixed. For $P \leq \pi\sqrt{2} = 4.4428\ldots$ the optimal region is a circle with $A/P = r/2 \leq \sqrt{2}/4 < 1/2$.

For $P > \pi\sqrt{2}$ an optimal region exists and it consists of a lattice polygon Π and a union of convex caps $D_i, i = 1, \ldots, k$, whose boundaries join two adjacent points of $\partial\Pi \cap \mathbf{Z}^2$.

Let p_i be the length of $\partial D_i \backslash \partial\Pi$ and let l_i be the distance of the two lattice points at the end.

Further let $a_i = A(D_i)$ be the enclosed area.

By a generalization of Pick's formula [6] by Hadwiger and Wills [3] for the number of lattice points L of Π we have

$$L(\Pi) = A(\Pi) + \frac{k}{2} + \chi(\Pi),$$

where k is the number of edges of Π (counting the edges which are not contained in the closure of the interior of Π twice) and χ denotes the Euler characteristic. It follows $A(\Pi) + 1 \leq \frac{k}{2}$ and hence

$$A(M) + 1 = A(\Pi) + 1 + \sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} (a_i + 1/2)$$

In the following it suffices to prove

$$a_i + 1/2 \leq \gamma p_i, \quad i = 1, \ldots, k. \tag{6}$$

because then it follows

$$A(M) + 1 \leq \gamma \sum_{i=1}^{k} p_i = \gamma P(M). \tag{7}$$

First we shall see that it suffices to consider lattice-point-free circular caps $D_i = C$. Assume that we have shown

$$\frac{A(C) + 1/2}{P(C)} \leq \gamma, \tag{8}$$

for lattice-point-free circular caps C, then we can prove it for general caps D by induction on $\lceil P \rceil$, i.e. the least integer greater than the length $P = P(D)$. For $\lceil P \rceil = 1$ D is necessarily a line segment of length one and $A(D) = 0$. For general P one may assume that D is an optimal convex cap with length P. If it does not contain a further lattice point in its boundary it is a circular cap by local arguments. If D contains a further lattice point then we can divide D into three parts two of them being caps D', D'' of this type with $\lceil P(D') \rceil, \lceil P(D'') \rceil < \lceil P(D) \rceil$ and one lattice triangle with an area of $1/2$. By the induction hypothesis we obtain inequalities for D' and D'' and it follows

$$\frac{1}{2} + A(D) = A(D') + A(D'') + 1 \leq \gamma P(D') + \gamma P(D'') = \gamma P(D).$$

In the following we will prove (8) depending on $l = l_i$.

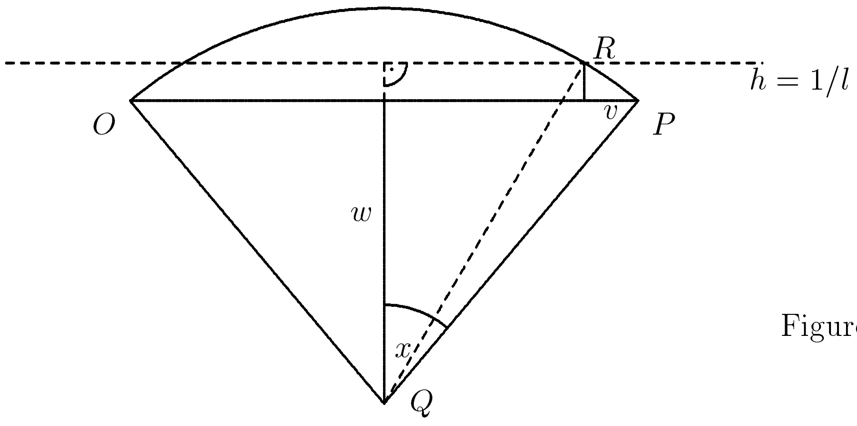


Figure 2

For this we have to show that the angle x (see Figure 2) corresponding to $C = C_i$ cannot be too large for large l. Let Q be the center corresponding to C and let O and P be the two lattice points at the end of ∂C. There is a lattice point R with distance $h = 1/l$ to $\overline{OP}$ whose projection is contained in OP.

Let v be the smaller one of the distances of the projection to P and O, without restriction to P. Since $\|P - R\| \geq 1$ it follows $v^2 + (1/l)^2 \geq 1$ and hence

$$v \geq \sqrt{1 - 1/l^2} =: v_0 \tag{9}$$

Let w be the radius minus the height of the cap, then $(l/2)^2 + w^2 = r^2 = (w + 1/l)^2 + (l/2 - v)^2$ and hence $y = lv/2(l - v) - 1/(2l)$. Since $v(l - v)$ is increasing for $0 \leq v \leq l/2$ it follows with (9)

$$w \geq l/2v_0(l - v_0) - 1/(2l) = l/2(\sqrt{l^2 - 1} - 1).$$

Hence

$$\tan x = l/(2w) \leq \frac{1}{\sqrt{l^2 - 1} - 1} \tag{10}$$

For the area and the perimeter of C we have with $r = l/(2 \sin x)$

$$p = P(C) = 2xr = lx/(\sin x) \tag{11}$$

and

$$a = A(C) = xr^2 - r^2 \sin x \cos x = l^2/4 \frac{x - \sin x \cos x}{\sin^2 x}. \tag{12}$$

It is easy to see that

$$\frac{x - \sin x \cos x}{\sin^2 x} \leq x \leq \tan x, \ 0 \leq x \leq \pi/2. \tag{13}$$

For $l \geq \sqrt{10}$ we have with (10) and (13)

$$\frac{a + 1/2}{p} \leq a/p + 1/(2l) \leq \frac{l}{4(\sqrt{l^2 - 1} - 1)} + 1/(2l).$$

Since the function $f(l) = \frac{l}{4(\sqrt{l^2-1}-1)} + 1/(2l)$ is decreasing it follows

$$\frac{a+1/2}{p} \leq f(\sqrt{10}) = 0.5533\ldots < \gamma.$$

Since OP contains no further lattice point, the cases $l = 1, \sqrt{2}, \sqrt{5}$ are still to be considered.

For $l = \sqrt{5}$ it follows from (10) $\tan x \leq 1$ and

$$\frac{a+1/2}{p} = \sqrt{5}/20 \left(\frac{5}{\sin x} - \frac{5\cos x}{x} + \frac{2\sin x}{x} \right)$$

This function is increasing in $[0, \pi/4]$ and so the ratio is at most its value in $x = \pi/4$ which is $0.488\ldots < \gamma$.

For $l = \sqrt{2}$ we have necessarily $x \leq \pi/2$ and

$$\frac{a+1/2}{p} = \sqrt{2}/4 \left(\frac{1}{\sin x} - \frac{\cos x}{x} + \frac{\sin x}{x} \right)$$

This function is increasing in $[0, \pi/2]$ and so the ratio is at most its value in $x = \pi/2$ which is $0.5786\ldots < \gamma$.

For $l = 1$ we have necessarily $x \leq 3\pi/4$ and

$$\frac{a+1/2}{p} = \frac{1}{4\sin x} - \frac{\cos x}{4x} + \frac{\sin x}{2x} := r(x)$$

By elementary discussion of $r(x)$ one can see that $r(x) \leq 0.5828 < \gamma$ for $\pi/2 \leq x \leq 3/4\pi$. Further, one can see that $r(x)$ is strict concave in $[0, \pi/2]$ and it takes its unique maximum value in the only solution $x_0 = 1.03\ldots$ of $r'(x) = 0$ which is equivalent to

$$\sin^2 x(x\sin x + 2x\cos x + \cos x - 2\sin x) = x^2 \cos x.$$

So we have proved that in any case

$$\frac{a+1/2}{p} \leq r(x_0) = \gamma.$$

Obviously the ratio can be arbitrary close to γ, for instance a long 'worm' with all angles $x = x_0$.

4. Proof of Theorem 2

For $P \leq \pi\sqrt{5}/2$ the optimal solution of the Dido problem with lattice constraints is a semicircle. For fixed $P > \pi\sqrt{5}/2$ the optimal solution is the union of a lattice polygon and circular caps of equal curvature, which have lattice points as endpoints. Let a and b be the two endpoints of ∂M at the shoreline and let (without restriction) the most left point of $\partial M \cap \mathbf{Z}^2$ be contained in $x = 1$ and the most right one in $x = k$.

First we will see that ∂M moves from a to a lattice point of level 1, then to the adjacent one and so on and finally it moves from the last lattice point to b without touching any other lattice point.

Let us assume, ∂M moves from a to a lattice point with level ≥ 2. Then the corresponding lattice point q of level 1 has to be contained in ∂M (else it is contained in the interior of M) and the length of ∂M from a to q is at least $l \geq 3$.

By reflecting the set with respect to $y = 0$ and connecting q with its image we obtain with (7) for the enclosed area v $2v + 1 \leq 0.583(2l + 2)$ and hence $v \leq 0.583l + 0.083 < l - 1$, for $l \geq 2.6$.

But this is a contradiction, since we have the better alternative to move from q in a rectangular form to the shoreline and to enclose an area of $l - 1$.

Now let q be a lattice point of level 1 with $x(q) \leq k - 1$, which is contained in ∂M. By the same reasons as above, the adjacent lattice point q' of level 1 is contained in ∂M.

Let l be the length of ∂M from q to q' and let v be the enclosed area. If a further lattice point is contained in that part then $l \geq 2.4$ and as above $2v + 1 \leq 0.583 \cdot 2l$ and so $v \leq 0.583l - 1/2$. So the contribution of that part to $A(M)$ is $v + 1 \leq 0.583l + 1/2 < l$, for $l \geq 1.2$.

Hence, ∂M does not contain another lattice point in this part, for if it did we could we cut out the part $\{(x, y) : x(q) \leq x \leq x(q')\}$ and use the length l to obtain an additional area of l.

Now we consider the first and the last arc.

We have to solve the Dido problem with the condition, that a point of level 1 is contained in the boundary. First we remark that the intersection of the arcs and the shoreline has to be orthogonal, i.e. the center of the arc is contained in the shoreline.

Consider a point on the arc which is close to the shoreline (for example this point such that the length of the arc from the shoreline to this one is $1/2$).

Using a reflection with respect to $y = 0$ this last part of the arc has to be the half of a circular arc with center on the shoreline. It follows that the intersection is orthogonal.

There are three possibilities for the two arcs.

(a) one small arc and one large arc, forming together a semicircle.

(b) two of the small arcs,

(c) two of the large arcs.

Type (c) cannot be optimal, because in that case we can replace the two equal caps above the line $y = 1$ with a corresponding angle x by one larger cap and one smaller cap with angles $x + y$ and $x - y$, respectively and the same common length and it turns out by elementary discussion of the area depending on $y > 0$ that this is an improvement.

The next question is: Which common radii r are possible?

Of course, $r \geq 1$, since the centers of the last arcs are contained in $y = 0$ and they join a lattice point of level 1 with the shoreline. Further, in case (a), obviously $r \leq \sqrt{5}/2$, because otherwise a lattice point is contained. But also in case (b) $r > \sqrt{5}/2$ is not allowed. Assume case (b) with $r > \sqrt{5}/2$ is optimal. Then consider the last arc and the adjacent one (with endpoints p and q).

The total length of the two arcs is between 2 and the length of the circular arc with center $1/2(p + q) - (0, 1)$ and radius $\sqrt{5}/2$. Using a reflection with respect to $y = 0$ it follows that the circular arc with the same length (which does not enclose any lattice point) is better than the two arcs.

Hence, $1 \leq r \leq \sqrt{5}/2$.

Now we estimate the ratio of area and perimeter piecewise as in Theorem 1. For the two last arcs the inequality is obvious. For the other ones the area is $1+(x-\sin x \cos x)/(4\sin^2 x)$ and the perimeter is $x/(\sin x)$. So the ratio is $r(x)+1/2 \sin x/x$ which is a concave function and takes its maximum value $1.04151\ldots$ at $0.5007\ldots$.

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