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On some Special Fano Manifolds

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Abstract. Some properties of Fano manifolds of product type involving the coindex and the number of factors are discussed and a contribution to the study of special Fano **P**-bundles of low corank is given; in particular standard special Fano **P**-bundles of corank 3 are classified. A question related to a vector bundle adjunction conjecture is also discussed.

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Introduction

In this paper we deal with Fano manifolds of special kind, such as products and **P**-bundles with some extra conditions.

For a Fano manifold of product type all factors are Fano. So we simply call such a manifold a Fano product. Let $X = Y_1 \times \ldots \times Y_s$ be a Fano product. By investigating some inequalities involving the coindex c of X, the coindexes of the factors and their number s, we prove that $s \leq c+1$ and characterize the X for which $c-2 \leq s \leq c+1$. In some sense our results are best possible. The case $s = c-2$ involves at worst Mukai manifolds as factors; but in some cases with $s = c-3$ one of the Y_i belongs to the still unexplored class of Fano manifolds of coindex 4.

Besides products, some **P**-bundles are Fano manifolds. Recently, several papers appeared on the classification of Fano **P**-bundles of low dimensions [11], [12], while other recent papers related to a conjecture of Mukai drew attention to ample vector bundles $\mathcal{E}$ on a projective manifold B such that $K_B + \det \mathcal{E}$ is not nef (or not ample, or trivial) [3], [8], [10], [14]. The obvious link between these subjects is that if $-(K_B + \det \mathcal{E})$ is nef, then $P = \mathbf{P}_B(\mathcal{E})$ is a Fano manifold. In this paper we also investigate some properties of Fano **P**-bundles of the above type $\mathbf{P}_B(\mathcal{E})$ satisfying the condition $\det \mathcal{E} = -K_B$. We

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define these to be special (see section 2). Up to replacing $\mathcal{E}$ with $\mathcal{E} \otimes L$, for a suitable line bundle L on B, they simply coincide with Fano $\mathbf{P}$-bundles of maximal index. If P is a special Fano $\mathbf{P}$-bundle as above, letting $r = \operatorname{rk} \mathcal{E}$ and $b = \dim B$, it follows from [3] that $\operatorname{cork} P \geq 0$, where, by definition, $\operatorname{cork} P = b + 1 - r$, and several results scattered in the literature provide the classification of special Fano $\mathbf{P}$-bundles with $\operatorname{cork} P \leq 2$, up to a still undecided case when $\operatorname{cork} P = 2$. In this paper, after providing several examples in the range $\operatorname{cork} P \geq 3$ we classify a special class of Fano $\mathbf{P}$-bundles of corank 3 on product manifolds, which we call standard. Roughly speaking, this means that $\mathcal{E}$ comes from vector bundles on the factors (for the precise definition see section 3). There are several such $\mathbf{P}$-bundles of corank 3, while they are very few for $\operatorname{cork} P = 2$ and do not appear at all when $\operatorname{cork} P \leq 1$. Although this classification relies on the known results for Fano $\mathbf{P}$-bundles with $\operatorname{cork} P \leq 2$, we would like to stress that our result is effective: actually the undecided case mentioned above in this context is irrelevant.

The paper is organized as follows. In section 1 we study Fano products. In section 2, after discussing general properties of special Fano $\mathbf{P}$-bundles, we review the results which are known for corank ≤ 2 and produce some new examples of these bundles with corank ≥ 3. In section 3 we classify standard special Fano $\mathbf{P}$-bundles of corank 3. Finally in section 4 we relate the above setting to a conjecture on adjunction for vector bundles [9].

1. Fano products

(1.0) Let $X = Y_1 \times \ldots \times Y_s$ $(s \geq 2)$ be a Fano manifold. Put $N = \dim X$, $r = \operatorname{index}(X)$, $c = \operatorname{coindex}(X)$ and $n_i = \dim Y_i$, $i = 1, \ldots, s$. So

(1.0.1)
$$\sum_{i=1}^{s} n_i = N = r + c - 1.$$

We always assume that no Y_i is a product. The aim of this section is to provide an upper bound for s in terms of c and to classify the X as above near the upper bound.

(1.1) Remark. *Let X be as in (1.0) and let p_i denote the projection onto the i-th factor. Then*

(1) *Y_i is a Fano manifold for all i;*
(2) $\operatorname{Pic}(X) \cong \prod_{i=1}^{s} p_i^* \operatorname{Pic}(Y_i)$;
(3) *$r = g.c.d.\{r_i\}$, where $r_i = \operatorname{index}(Y_i)$.*

Proof. 1) Since the normal bundle $N_{Y_i|X}$ is trivial, Y_i being a factor of X, by adjunction we get $K_{Y_i} = (K_X)_{Y_i}$, hence $-K_{Y_i}$ is ample since $-K_X$ is.

2) Note that $\operatorname{Pic}(X) \cong H^2(X, \mathbf{Z})$. Similarly $\operatorname{Pic}(Y_i) \cong H^2(Y_i, \mathbf{Z})$, by (1). So the assertion follows from the Künneth decomposition.

3) Write $-K_X = rH$, $-K_{Y_i} = r_i H_i$, with $H \in \operatorname{Pic}(X)$ and $H_i \in \operatorname{Pic}(Y_i)$ ample, with no proper submultiples. We thus get

$$rH = -K_X = -\sum_{i=1}^{s} p_i^* K_{Y_i} = \sum_{i=1}^{s} r_i p_i^* H_i,$$

which proves the assertion. $\square$

(1.2.0) Obviously $s \le N$ with equality if and only if $n_i = 1$ for all i, which means that $X = (\mathbf{P}^1)^N$ by (1.1.(1)). Note that in this case $r = 2$, i.e., $N = c + 1$, so that $s = c + 1$. Now set

$$\sigma = \frac{N - \sum_{i=1}^{s} c_i}{N - c},$$

where $c_i = n_i + 1 - r_i$ denotes the coindex of Y_i. We can prove the following

(1.2) Lemma. *Let X be as in* (1.0).

(a) *If $r = 1$ then $s \le c - 1$, with equality if and only if $X = (\mathbf{P}^1)^{N-2} \times S$, where S is either $\mathbf{P}^2$ or any Del Pezzo surface of index* 1;

(b) *if $r \ge 2$, then $s \le \sigma$, with equality if and only if $r = r_1 = \ldots = r_s$.*

Proof. (a) If $r = 1$ it cannot be $s = N$ by what we said before; hence $s \le N - 1 = c - 1$ and of course $s = c - 1$ if and only if $(s-1)$ factors are $\mathbf{P}^1$ and the remaining one has dimension 2 and index $\ne 2$. So it can only be either $\mathbf{P}^2$ or a Del Pezzo surface of index 1.
(b) Let $r \ge 2$; by (1.1.(3)) we have $r_i = rh_i$ for some positive integer h_i, $i = 1, \ldots, s$. So

$$n_i = r_i + c_i - 1 = rh_i + c_i - 1. \tag{1.2.1}$$

By summing up and comparing with (1.0.1) we thus get

$$s = r\sum_{i=1}^{s} h_i + \sum_{i=1}^{s} c_i - N. \tag{1.2.2}$$

Since $\sum_{i=1}^{s} h_i \ge s$, (1.2.2) gives

$$s(r-1) \le N - \sum_{i=1}^{s} c_i, \tag{1.2.3}$$

where the equality holds if and only if $h_i = 1$ for all i, i.e., if and only if $r = r_i$ for all i; recalling (1.0.1) this proves the assertion. $\square$

As a first consequence of (1.2) we get

(1.3) Corollary. *Let X be as in* (1.0). *Then $c \ge \sum_{i=1}^{s} c_i$, equality holding if and only if $r = r_1 = \ldots = r_s = 1$.*

Proof. If $r > 1$ we have that $\sigma \ge s > 1$ by (1.2). Recalling the definition of σ this gives $c > \sum_{i=1}^{s} c_i$. So if

$$c = \sum_{i=1}^{s} c_i, \tag{1.3.1}$$

then $r = 1$. But this is equivalent to $c = N$, so (1.3.1) gives

$$N = \sum_{i=1}^{s}(n_i + 1 - r_i) = \sum_{i=1}^{s} n_i + s - \sum_{i=1}^{s} r_i,$$

which means that $s = \sum_{i=1}^{s} r_i$ and so $r_i = 1$ for all i. To see the converse, let $r = r_1 = \ldots = r_s = 1$; then

$$c = N = \sum_{i=1}^{s} n_i = \sum_{i=1}^{s} c_i. \qquad \square$$

(1.4) Remark. *Let $r > 1$. Then*

$$\sigma \leq 1 + \frac{c}{r-1},$$

equality occurring if and only if $Y_i = \mathbf{P}^{n_i}$ for all i. In particular $\sigma \leq c + 1$, equality implying $r = 2$ and $Y_i = \mathbf{P}^{n_i}$ for all i.

Proof. Actually by definition of σ,

$$\sigma \leq \frac{N}{N-c} = 1 + \frac{c}{N-c} = 1 + \frac{c}{r-1} \leq 1 + c,$$

and the equality on the left holds if and only if $c_i = 0$ for all i, while the equality on the right occurs if and only if $r = 2$. $\square$

The material above gives the following

(1.5) Theorem. *Let X be as in* (1.0). *Then $s \leq c + 1$ with equality if and only if $X = (\mathbf{P}^1)^N$.*

Proof. (1.2) combined with (1.4) gives $s \leq \sigma \leq c + 1$ and the equalities imply both $r = r_1 = \ldots = r_s$ and $Y_i = \mathbf{P}^{n_i}$ with $r = 2$, hence $r_i = 2$, i.e., $Y_i = \mathbf{P}^1$ for all i. $\square$

The next step is to characterize those X as in (1.0) in case $s = c$.

(1.6) Theorem. *Let X be as in* (1.0). *Then $s = c$ if and only if $X = \mathbf{P}^2 \times \mathbf{P}^2$.*

Proof. (1.2) combined with (1.4) gives

$$2 \leq c = s \leq \sigma \leq 1 + \frac{c}{r-1} \leq c + 1;$$

then either $s = \sigma = c$ or $s = c$ and $\sigma = s + 1 = c + 1$. Let $s = \sigma = c$. By (1.2) we have $r = r_1 = \ldots = r_s$.

Claim. $r = r_1 = \ldots = r_s > 2$.
Proof. As $s = c$ we get $r = r_1 = \ldots = r_s > 1$ from (1.2). Suppose that $r = 2$; then

$$\sum_{i=1}^{s} n_i - \sum_{i=1}^{s} c_i = \sum_{i=1}^{s} r_i - s = 2s - s = s = N - 1,$$

hence $\sum_{i=1}^{s} c_i = 1$. Up to reordering factors we can thus assume that $c_1 = \ldots = c_{N-2} = 0$ and $c_{N-1} = 1$; therefore $n_1 = \ldots = n_{N-2} = 1$ and $n_{N-1} = 2$, so $X = (\mathbf{P}^1)^{N-2} \times (\mathbf{P}^1 \times \mathbf{P}^1) = (\mathbf{P}^1)^N$. But recalling (1.2.0) this leads to a contradiction, since $s = N - 1$. This proves the claim.
From (1.4) we get

$$c = \sigma \leq 1 + \frac{c}{r-1},$$

hence

$$c \leq (r-1)/(r-2).$$

By the claim this implies that either $r = 3$ with $c = 1, 2$ or $r > 3$ with $c = 1$. Recalling that $s = c \geq 2$ we see that only case $c = 2$, $r = 3$ can occur, i.e., X is a Del Pezzo 4-fold. Then Fujita's classification of Del Pezzo manifolds [2, Ch. I, §8] gives $X = \mathbf{P}^2 \times \mathbf{P}^2$.

Now let $s = c$ and $\sigma = s + 1 = c + 1$. We prove that this case cannot occur. From (1.4) we get $r = 2$ and $Y_i = \mathbf{P}^{n_i}$ for all i, hence $s = c = N - 1$. It thus follows that $s - 1$ factors have dimension 1, hence they are $\mathbf{P}^1$, and the last one has dimension 2, but since $r = 2$, it has to be $\mathbf{P}^1 \times \mathbf{P}^1$. Thus $X = (\mathbf{P}^1)^N$, which once again leads to a contradiction since $s = N - 1$. □

Case $s = c - 1$ gives a far richer situation.

(1.7) Theorem. *Let X be as in* (1.0). *Then $s = c - 1$ if and only if, up to reordering factors, X is one of the following products:*

(1) $(\mathbf{P}^1)^{N-2} \times S$, *where S is either $\mathbf{P}^2$ or a Del Pezzo surface of index* 1;
(2) $(\mathbf{P}^1)^{N-3} \times Y$, *where Y is either $\mathbf{P}^3$ or a Del Pezzo 3-fold of index* 2;
(3) $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{P}^2$;
(4) $\mathbf{P}^2 \times \mathbf{Q}^3$;
(5) $\mathbf{P}^3 \times \mathbf{P}^3$.

Proof. If $r = 1$ then X is as in (1), by (1.2.(a)). So let $r \geq 2$. Keep the same notation as in the proof of (1.2) and recall that

$$\sum_{i=1}^{s} h_i \geq s. \tag{1.7.1}$$

First assume that (1.7.1) is an equality. Then $r = r_1 = \ldots = r_s$ and by (1.2.2) $s = \sigma = c-1$. Let $r = 2$; then $N = c + 1$, hence $\sum_{i=1}^{s} c_i = 2$. Therefore either $c_1 = \ldots = c_{N-3} = 0$, $c_{N-2} = 2$, or $c_1 = \ldots = c_{N-4} = 0$, $c_{N-3} = c_{N-2} = 1$. In the former case we have $n_1 = \ldots = n_{N-3} = 1$, $n_{N-2} = 3$, hence X is as in (2) with Y a Del Pezzo 3-fold of index 2. In the latter case we have $n_1 = \ldots = n_{N-4} = 1$, $n_{N-3} = n_{N-2} = 2$, hence $X = (\mathbf{P}^1)^{N-4} \times (\mathbf{P}^1 \times \mathbf{P}^1) \times (\mathbf{P}^1 \times \mathbf{P}^1) = (\mathbf{P}^1)^N$, but this is impossible, since $s = N-2$. So this case does not occur. Now let $r \geq 3$. Recalling (1.4) we get

$$c \leq \sigma + 1 \leq 2 + \frac{c}{r-1}, \quad \text{hence}$$

$$c \leq \frac{2r-2}{r-2}. \tag{1.7.2}$$

For $r = 3$, (1.7.2) gives $c \leq 4$ and $\sigma = s \leq 3$, so that either $c = 4$, with $\sigma = s = 3$, or $c = 3$, with $\sigma = s = 2$. In the former case we get $N = 6$ and $c_1 + c_2 + c_3 = 0$, hence $c_i = 0$ and $n_i = 2$ for $i = 1, 2, 3$. This exactly means that X is as in (3). In the latter case we get $N = 5$ and $c_1 + c_2 = 1$, hence we can assume $c_1 = 0$, $c_2 = 1$ and then $n_1 = 2$, $n_2 = 3$. So X is as in (4). For $r = 4$ (1.7.2) gives $c \leq 3$; therefore $c = 3$ and $\sigma = s = 2$. Then $N = 6$ and $c_1 + c_2 = 0$; so $c_i = 0$, $n_i = 3$ for $i = 1, 2$, which leads to case (5). Note that it cannot be $r \geq 5$; otherwise we would get $c \leq 2$ from (1.7.2), which is impossible since $s \geq 2$.

Now assume that (1.7.1) is a strict inequality. Then from (1.2.2) we get the inequality

$$s \leq \sigma - \frac{r}{r-1} < \sigma - 1.$$

Combining this with (1.4) and the assumption we obtain

$$c - 1 = s \leq \sigma - 2 \leq c + 1 - 2 = c - 1,$$

which gives $\sigma = c + 1$. By (1.4) this is equivalent to $r = 2$ and $Y_i = \mathbf{P}^{n_i}$ with n_i odd, for all i. Furthermore from (1.0.1) we get $n_1 + \ldots + n_s + 1 - 2 = c = s + 1$, hence

$$n_1 + \ldots + n_s = s + 2. \tag{1.7.3}$$

By what we said there exists an index i such that $n_i \geq 3$. So, up to reordering the factors of X, we can assume $n_i \geq 3$ for $i = s-t, \ldots, s-1, s$. Then by (1.7.3)

$$s - t - 1 \leq n_1 + \ldots n_{s-(t+1)} \leq s + 2 - 3(t+1) = s - 3t - 1,$$

which shows that $t = 0$, $n_1 = \ldots = n_{s-1} = 1$, $n_s = 3$. So X is as in case (2) with $Y = \mathbf{P}^3$. This concludes the proof. □

Along the same line we can classify Fano products with $s = c - 2$. The proof is long but similar to the one of (1.7), and we omit it. The result is the following

(1.8) Theorem. *Let X be as in* (1.0)*. Then $s = c - 2$ if and only if, up to reordering factors, X is one of the following products:*

(1) $(\mathbf{P}^1)^{N-3} \times Y$, *with Y either a quadric or a Fano 3-fold of index* 1*;*
(2) $(\mathbf{P}^1)^{N-4} \times Y \times Y'$, *where Y is either $\mathbf{P}^2$ or a Del Pezzo surface of index* 1 *and Y' is a Del Pezzo surface of index* 1*;*

(3) $(\mathbf{P}^1)^{N-4} \times Y$, *where* Y *is either a quadric or a Fano 4-fold of index* 2*;*
(4) $\mathbf{P}^2 \times Y$, *with* Y *a Del Pezzo 4-fold;*
(5) $\mathbf{Q}^3 \times \mathbf{Q}^3$*;*
(6) $\mathbf{P}^2 \times \mathbf{P}^2 \times \mathbf{Q}^3$*;*
(7) $(\mathbf{P}^2)^4$*;*
(8) $\mathbf{P}^3 \times \mathbf{Q}^4$*;*
(9) $\mathbf{P}^4 \times \mathbf{P}^4$.

For 4-folds occurring in case (3), which fit into the class of Mukai manifolds, see [4].

2. Special Fano P-bundles

Let $P = \mathbf{P}_B(\mathcal{E})$ be the projective bundle associated to a vector bundle $\mathcal{E}$ of rank r on a projective manifold B. Let $\pi : P \to B$ be the bundle projection and let ξ denote the tautological bundle $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$. By the canonical bundle formula we have

$$K_P = -r\xi + \pi^*(K_B + \det \mathcal{E}). \tag{2.0.1}$$

If P is Fano we call it a Fano **P**-bundle. It follows from (2.0.1) that a Fano **P**-bundle has index $\leq r$. In particular if

$$\mathcal{E} \text{ is ample and } K_B + \det \mathcal{E} = \mathcal{O}_B, \tag{2.0.2}$$

we immediately get from (2.0.1) that $K_P = -r\xi$, where ξ is an ample line bundle with no proper submultiples in $\mathrm{Pic}(P)$. So P is a Fano manifold of index r.

We will call P a *special* Fano **P**-bundle if it corresponds to a vector bundle $\mathcal{E}$ satisfying (2.0.2).

Of course not every Fano **P**-bundle is special. E.g., for any integer $t \geq 1$ let $B = \mathbf{P}^{3t-1}$ and $\mathcal{E} = \mathcal{O}(1)^{2t}$. Then for $P = \mathbf{P}_B(\mathcal{E})$ we have $K_P = -t(2\xi + \pi^*\mathcal{O}(1))$. As $2\xi + \pi^*\mathcal{O}(1)$ is ample with no proper submultiples in $\mathrm{Pic}(P)$ we thus see that P is a Fano **P**-bundle of index $t < \mathrm{rk}\,\mathcal{E}$, hence not special. For other examples in low dimensions, see [11], [12]. In fact we have

(2.1) Proposition. *Let* $P = \mathbf{P}_B(\mathcal{E})$ *be a Fano* **P***-bundle;* P *is special if and only if it has maximal index* r*.*

Proof. We prove the "only if" part, the converse having been already shown. Let $-K_P = rh$, with $h \in \mathrm{Pic}(P)$ ample. By (2.0.1) we thus see that $K_B + c_1(\mathcal{E}) \in r\,\mathrm{Pic}(B)$. Writing $K_B + c_1(\mathcal{E}) = rL$ for a suitable line bundle L on B, (2.0.1) gives $-K_P = r(\xi - \pi^*L)$: so $h = \xi - \pi^*L$ is the tautological bundle of $\mathcal{E}' := \mathcal{E} \otimes L^{-1}$, which is ample, since h is. Therefore $\mathcal{E}'$ satisfies (2.0.2). On the other hand $P = \mathbf{P}_B(\mathcal{E}) = \mathbf{P}_B(\mathcal{E}')$. □

(2.2) Recall that for an ample vector bundle $\mathcal{E}$ of rank r on a projective n-fold B the adjoint line bundle $K_B + \det \mathcal{E}$ is always ample if $r \geq n+1$, unless $(B, \mathcal{E}) = (\mathbf{P}^n, \mathcal{O}(1)^{(n+1)})$ [3]. This suggest to define the *corank* of P as

$$\mathrm{cork}\, P = \dim B + 1 - \mathrm{rk}\,\mathcal{E} \tag{2.2.1}$$

and the above result can be restated by saying that $\operatorname{cork} P \geq 0$, with equality if and only if

$$(B, \mathcal{E}) = (\mathbf{P}^n, \mathcal{O}(1)^{(n+1)}). \tag{2.2.2}$$

Rephrasing known results of several authors [3], [8] and [10] in terms of corank we get the following lists of special Fano $\mathbf{P}$-bundles with corank 1 and 2.

(2.3) Theorem. *Let $P = \mathbf{P}_B(\mathcal{E})$ be a special Fano $\mathbf{P}$-bundle. Then* $\operatorname{cork} P = 1$ *if and only if $(B, \mathcal{E})$ is one of the following pairs:*

(1) $(\mathbf{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(1)^{(n-1)})$,
(2) $(\mathbf{P}^n, T_{\mathbf{P}^n})$, *where $T_{\mathbf{P}^n}$ is the tangent bundle,*
(3) $(\mathbf{Q}^n, \mathcal{O}(1)^n)$.

(2.4) Theorem. *Let $P = \mathbf{P}_B(\mathcal{E})$ be a special Fano $\mathbf{P}$-bundle. Then* $\operatorname{cork} P = 2$ *if and only if $(B, \mathcal{E})$ is one of the following pairs:*

(1) $(\mathbf{P}^3, N(2))$, *where N is the null correlation bundle,*
(2) $(\mathbf{P}^2 \times \mathbf{P}^1, p^* T_{\mathbf{P}^2} \otimes \mathcal{O}(0,1))$, *where p stands for the first projection,*
(3) $(\mathbf{P}^2 \times \mathbf{P}^1, \mathcal{O}(2,1) \oplus \mathcal{O}(1,1))$,
(4) $(\mathbf{Q}^3, S(2))$, *where S is the spinor bundle,*
(5) $(\mathbf{Q}^4, S(2) \oplus \mathcal{O}(1))$, *where S is one of the two spinor bundles,*
(6) $(Y, H^{(n-1)})$, *where Y is any Del Pezzo manifold and $-K_Y = (n-1)H$,*
(7) $(\mathbf{P}^n, \mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{(n-3)})$,
(8) $(\mathbf{P}^n, \mathcal{O}(3) \oplus \mathcal{O}(1)^{(n-2)})$,
(9) $(\mathbf{Q}^n, \mathcal{O}(2) \oplus \mathcal{O}(1)^{(n-2)})$,

or, possibly, P has two $\mathbf{P}^2$-bundle structures on Fano 4-folds with Picard number 1, index 1 and pseudoindex ≥ 3.

As in section 1, Del Pezzo manifolds are intended in the sense of Fujita [2]. Note that case (6) includes the 6 types of Del Pezzo manifolds with $\operatorname{Pic}(B) \cong \mathbf{Z}$ and the following 4 pairs

$$(\mathbf{P}^2 \times \mathbf{P}^2, \mathcal{O}(1,1)),\ (\mathbf{P}(T_{\mathbf{P}^2}), \xi),\ ((\mathbf{P}^1)^3, \mathcal{O}(1,1,1)),\ (B_p(\mathbf{P}^3), \sigma^*\mathcal{O}(2) - E),$$

where $\sigma : B_p(\mathbf{P}^3) \to \mathbf{P}^3$ is the blowing-up at the point p and E is the exceptional divisor [2, p. 72]. For the definitions and the main properties of the bundles N and S appearing in (2.4) we refer to [5, p. 76] and to [6].

In the range $\operatorname{cork} P \geq 3$ no classification result is known. Let us produce some examples.

(2.5) Examples. Obvious special Fano $\mathbf{P}$-bundles with $\operatorname{cork} P = 3$ are given by the following pairs:

(1) $B = \mathbf{P}^n$, with $\mathcal{E} = \mathcal{O}(4) \oplus \mathcal{O}(1)^{n-3}$, $\mathcal{O}(3) \oplus \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-4}$, or $\mathcal{O}(2)^3 \oplus \mathcal{O}(1)^{n-5}$;
(2) $B = \mathbf{Q}^n$, with $\mathcal{E} = \mathcal{O}(3) \oplus \mathcal{O}(1)^{n-3}$, or $\mathcal{O}(2)^2 \oplus \mathcal{O}(1)^{n-4}$.

(3) A less obvious example is given by the pair $(\mathbf{Q}^6, S(2))$, where S is one of the two spinor bundles on $\mathbf{Q}^6$. Actually $\operatorname{rk} S = 4$ and $S(2) = S(1) \otimes \mathcal{O}(1)$ is ample, since $S(1)$ is spanned, as a quotient of $\mathcal{O}^8$ [6, (2.8), (ii)]. Moreover $c_1(S(2)) = c_1(S) + 8 = 6$. Therefore $\mathcal{E} := S(2)$ satisfies (2.0.2).

In fact it is easy to show that if $(\mathbf{Q}^n, \mathcal{E})$ gives rise to a special Fano $\mathbf{P}$-bundle, $\mathcal{E}$ being or having as a summand a twisted spinor bundle, then only cases (2.4.(4)), (2.4.(5)) or (2.5.(3)) can occur.

(4) Another example related to spinor bundles is the following. Let S^* be the dual of the spinor bundle S on $\mathbf{Q}^5$ and consider the exact sequence [6, (3.1)]

$$0 \to \mathcal{O} \to S^* \to F := S^*/\mathcal{O} \to 0.$$

F has rank 3 and is spanned, as S^* is; actually $S^* \simeq S(1)$, which is a quotient of $\mathcal{O}^8$ [6, (2.8), (i)]. Moreover $c_1(F) = c_1(S^*) = -c_1(S) = 2$ [6, (2.9)]. Therefore $\mathcal{E} := F(1)$ satisfies (2.0.2) and the corresponding P has corank 3.

(2.6) Let B be any Fano manifold of index r and coindex c and let $\mathcal{E} = H^r$, where $H \in \operatorname{Pic}(B)$ is the ample line bundle such that $-K_B = rH$. Then obviously $P = \mathbf{P}_B(\mathcal{E})$ is a special Fano $\mathbf{P}$-bundle with $\operatorname{cork} P = c$. In particular, taking as B any Mukai manifold [4], we get in this way a lot of special Fano $\mathbf{P}$-bundles of corank 3.

To conclude this section let us produce a non-obvious example of special Fano $\mathbf{P}$-bundle of corank 4.

(2.7) Example. Let F be as in (2.5.(4)). By [7, (2.2)] there is an exact sequence

$$0 \to \mathcal{O} \to F^*(1) \to C(1) \to 0$$

defining a rank-2 bundle C on $\mathbf{Q}^5$ called the Cayley bundle. From this sequence we get $c_1(C(1)) = 1$. Moreover $C(2)$ is spanned [7, (3.7)]. It thus follows that the rank-2 bundle $\mathcal{E} := C(3)$ satisfies (2.0.2), hence the corresponding P is a special Fano $\mathbf{P}$-bundle of corank 4.

3. Special Fano P-bundles of corank 3 on products

Let $P = \mathbf{P}_B(\mathcal{E})$ be a special Fano $\mathbf{P}$-bundle over a product manifold $B = B_1 \times B_2$. Let $r = \operatorname{rk} \mathcal{E}$, $b = \dim B$, $b_i = \dim B_i$ and let p_i be the projection of B onto the i-th factor.

We say that P is *standard* to mean that

(3.0.1) $$\mathcal{E} = (p_1^*\mathcal{E}_1 \otimes p_2^*L_2) \oplus (p_2^*\mathcal{E}_2 \otimes p_1^*L_1),$$

where $\mathcal{E}_i$ and L_i are ample vector bundles of rank r_i and ample line bundles on B_i respectively, $i = 1, 2$, one of the two summands being possibly zero. The aim of this section is to determine all standard special Fano $\mathbf{P}$-bundles satisfying the condition

(3.0.2) $$\operatorname{cork} P = 3.$$

In the simpler case cork $P = 2$ the same analysis we are going to do leads to the pairs (2.4.(2)), (2.4.(3)) and the two products fitting in case (2.4.(6)). Our result is the following

(3.1) Theorem. *Up to exchanging the factors of B, the pairs $(B, \mathcal{E})$ corresponding to the standard special Fano $\mathbf{P}$-bundles satisfying (3.0.2) are the following:*

(1) $(\mathbf{P}^3 \times \mathbf{P}^3, \mathcal{O}(1,1)^4)$,
(2) $(\mathbf{P}^1 \times \mathbf{P}^3, \mathcal{O}(1,2)^2)$,
(3) $(\mathbf{P}^2 \times \mathbf{Q}^3, \mathcal{O}(1,1)^3)$,
(4) $(\mathbf{P}^3 \times \mathbf{P}^2, \mathcal{O}(2,1) \oplus \mathcal{O}(1,1)^2)$,
(5) $(\mathbf{P}^3 \times \mathbf{P}^2, p_1^* T_{\mathbf{P}^3} \otimes \mathcal{O}(0,1))$,
(6) $(\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(2,1,1) \oplus \mathcal{O}(1,1,1))$,
(7) $(\mathbf{P}^2 \times \mathbf{P}^1 \times \mathbf{P}^1, p_1^* T_{\mathbf{P}^2} \otimes \mathcal{O}(0,1,1))$,
(8) $(\mathbf{P}^3 \times \mathbf{P}^1, p_1^* N(2) \otimes \mathcal{O}(0,1))$,
(9) $(\mathbf{Q}^3 \times \mathbf{P}^1, p_1^* S(2) \otimes \mathcal{O}(0,1))$,
(10) $(Y \times \mathbf{P}^1, p_1^* H^2 \otimes \mathcal{O}(0,1))$, *$(Y, H)$ being any Del Pezzo threefold,*
(11) $(\mathbf{P}^3 \times \mathbf{P}^1, \mathcal{O}(3,1) \oplus \mathcal{O}(1,1))$,
(12) $(\mathbf{Q}^3 \times \mathbf{P}^1, \mathcal{O}(2,1) \oplus \mathcal{O}(1,1))$,
(13) $(\mathbf{P}^2 \times \mathbf{P}^2, \mathcal{O}(1,2) \oplus \mathcal{O}(2,1))$,
(14) $(\mathbf{P}^2 \times \mathbf{P}^2, \mathcal{O}(1,1) \oplus \mathcal{O}(2,2))$.

Note that (10) includes just one case in which B is the product of three or more factors: it corresponds to the pair

(3.1.0) $$((\mathbf{P}^1)^4, \mathcal{O}(1,1,1,1)^2).$$

The proof of Theorem (3.1) takes the remainder of this section. First note that B is Fano, by (2.0.2), hence

(3.1.1) $$\operatorname{Pic}(B) = p_1^* \operatorname{Pic}(B_1) \times p_2^* \operatorname{Pic}(B_2),$$

by (1.1.(2)). Moreover $K_B = p_1^* K_{B_1} + p_2^* K_{B_2}$.

(3.2) First assume that the second summand in (3.0.1) is zero and set $L = L_2$. Then, in view of (3.1.1), the fact that P is special gives

(3.2.1) $$K_{B_1} + \det \mathcal{E}_1 = \mathcal{O}_{B_1},$$

i.e. $\mathbf{P}_{B_1}(\mathcal{E}_1)$ itself is special, and

(3.2.2) $$K_{B_2} + rL = \mathcal{O}_{B_2}.$$

Condition (3.0.2) means that $r = b - 2 = b_1 + (b_2 - 2)$. This combined with (3.2.1) and the results of section 2 shows that $b_2 \leq 3$ and

if $b_2 = 3$, then $r = b_1 + 1$ and $(B_1, \mathcal{E}_1)$ is as in (2.2.2),
if $b_2 = 2$, then $r = b_1$ and $(B_1, \mathcal{E}_1)$ is as in (2.3),

if $b_2 = 1$, then $r = b_1 - 1$ and $(B_1, \mathcal{E}_1)$ is as in (2.4).

Note however that not all pairs in (2.4) can occur; actually we see from (3.2.2) that r divides the index of B_2. Since $r \geq 2$ we thus conclude that

if $b_2 = 1$, then $r = 2$ and $(B_2, L) = (\mathbf{P}^1, \mathcal{O}(1))$;
if $b_2 = 2$, then $r = 3$ or 2 and $(B_2, L) = (\mathbf{P}^2, \mathcal{O}(1))$ or $(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1,1))$ accordingly,
if $b_2 = 3$, then either

$B_2 = \mathbf{P}^3$ with $L = \mathcal{O}(1), \mathcal{O}(2)$ and $r = 4, 2$ respectively,
$(B_2, L) = (\mathbf{Q}^3, \mathcal{O}(1))$ with $r = 3$, or
(B_2, L) is a Del Pezzo threefold.

By combining the possibilities for $(B_1, \mathcal{E}_1)$ with those for (B_2, L) and taking into account the admissible values for r, we thus get the pairs (1)–(12) in (3.1).

(3.3) Now assume that the summands in (3.0.1) are nontrivial. Then, recalling (3.1.1), the fact that P is special gives rise to the following two symmetric conditions:

(3.3.1) $$K_{B_i} + \det \mathcal{E}_i + r_j L_i = \mathcal{O}_{B_i}, \quad (i,j) = (1,2) \text{ or } (2,1).$$

In particular this means that $K_{B_i} + \det \mathcal{E}_i$ is not nef. By [14, Theorem 1] this implies that $r_i \leq b_i + 1$, $i = 1, 2$, which, combined with the condition $r_1 + r_2 = b_1 + b_2 - 2$ coming from (3.0.2), leads, up to exchanging the factors, to the following possibilities:

A) $r_1 = b_1$ and $r_2 = b_2 - 2$,
B) $r_i = b_i - 1$ for $i = 1, 2$.

Moreover, by [14, Theorem 2], we see that in case A)

(3.3.2) $$(B_1, \mathcal{E}_1) = (\mathbf{P}^{b_1}, \mathcal{O}(1)^{b_1}).$$

Let us continue the study of

Case A). From (3.3.1) with $i = 1$ and (3.3.2) it follows that $r_2 = 1$ and $L_1 = \mathcal{O}_{\mathbf{P}}(1)$. Hence $b_2 = 3$. Now (3.3.1) with $i = 2$ becomes

$$K_{B_2} + \det \mathcal{E}_2 + b_1 L_2 = \mathcal{O}_{B_2}.$$

This says that $\mathcal{E}_2 \oplus L_2^{b_1}$ is an ample vector bundle of rank $(1 + b_1)$ on B_2 with trivial adjoint bundle. So recalling what we said in section 2, we get $1 + b_1 \leq b_2 + 1 = 4$, i.e. $b_1 \leq 3$. Moreover recalling (2.2.2), (2.3), (2.4) and noting that $\mathcal{E}_2 \oplus L_2^{b_1}$ is split, we get:

(3.3.3) if $b_1 = 3$ then $B_2 = \mathbf{P}^3$, $\mathcal{E}_2 = L_2 = \mathcal{O}_{\mathbf{P}^3}(1)$;

(3.3.4) if $b_1 = 2$, then either $(B_2, \mathcal{E}_2, L_2) = (\mathbf{P}^3, \mathcal{O}(2), \mathcal{O}(1))$, or $B_2 = \mathbf{Q}^3$ and $\mathcal{E}_2 = L_2 = \mathcal{O}(1)$;

(3.3.5) if $b_1 = 1$, then we have 5 possibilities: either B_2 is a Del Pezzo threefold with $-K_{B_2} = 2H$ and $\mathcal{E}_2 = L_2 = H$, or $B_2 = \mathbf{P}^3$ and $\mathcal{E}_2 = L_2 = \mathcal{O}(2)$, or, up to exchanging $\mathcal{E}_2$ and L_2, the triple $(B_2, \mathcal{E}_2, L_2)$ is one of the following:

$$(\mathbf{P}^1 \times \mathbf{P}^2, \mathcal{O}(1,2), \mathcal{O}(1,1)), \ (\mathbf{P}^3, \mathcal{O}(3), \mathcal{O}(1)), \ (\mathbf{Q}^3, \mathcal{O}(2), \mathcal{O}(1)).$$

Taking into account these possibilities we thus get for $(B, \mathcal{E})$ the pairs (1)–(4), (6), (10)–(12) in (3.1), all of which already found. Now come to

Case B). It follows from [14, Theorem 3] that both pairs $(B_i, \mathcal{E}_i)$ belong to the following list:

1) ($\mathbf{P}^{(n-1)}$-bundle over a smooth curve, $\mathcal{E}_i$ restricting to any fibre as $\mathcal{O}(1)^{(n-1)}$),
2) $(\mathbf{P}^n, \mathcal{O}(1)^{(n-1)})$,
3) $(\mathbf{P}^n, \mathcal{O}(2) \oplus \mathcal{O}(1)^{(n-2)})$,
4) $(\mathbf{Q}^n, \mathcal{O}(1)^{(n-1)})$.

Let us first discuss case 1).

(3.4) Lemma. *Let* $\operatorname{cork} P = 3$ *and assume that* $(B_1, \mathcal{E}_1)$ *is as in case* 1). *Then* $B_1 = \mathbf{P}^1 \times \mathbf{P}^1$, $\mathcal{E}_1 = L_1 = \mathcal{O}(1,1)$ *and* $b_2 = 2$.

Proof. Set $m = b_1 - 1$. Since B_1 is Fano, its base curve has to be $\mathbf{P}^1$ and so $B_1 = \mathbf{P}_{\mathbf{P}^1}(\mathcal{F})$, where $\mathcal{F}$ is a rank-$(m+1)$ vector bundle on $\mathbf{P}^1$, which we can suppose to be normalized in the form

$$\mathcal{F} = \mathcal{O}_{\mathbf{P}^1} \oplus \mathcal{O}_{\mathbf{P}^1}(a_1) \oplus \ldots \oplus \mathcal{O}_{\mathbf{P}^1}(a_m), \quad \text{with } \ 0 \le a_1 \le \ \ldots \ \le a_m.$$

Let $\delta = \deg \mathcal{F} = a_1 + \ \ldots \ + a_m$; thus $\delta \geq 0$. As

$$-K_{B_1} = (m+1)\xi + (2-\delta)F, \tag{3.4.1}$$

where ξ and F stand for the tautological bundle of $\mathcal{F}$ and a fibre respectively, the ampleness condition for $-K_{B_1}$ gives $2 - \delta > 0$ [1, p. 74, Lemma 3.2.4]. From 1) we have that $(\det \mathcal{E}_1)_F = \mathcal{O}_{\mathbf{P}^m}(m)$; so since ξ and F generate $\operatorname{Pic}(B_1)$ we have

$$\det \mathcal{E}_1 = m\xi + nF, \tag{3.4.2}$$

with $n > 0$ since $\det \mathcal{E}_1$ is ample. Similarly for the ample line bundle L_1 we have

$$L_1 = \alpha\xi + \beta F, \tag{3.4.3}$$

with $\alpha > 0$ and $\beta > 0$. Now recall that $\operatorname{cork} P = 3$. Then putting together (3.4.1), (3.4.2), (3.4.3), condition (3.3.1) with $i = 1$ gives

$$-(m+1)\xi + (\delta - 2)F + m\xi + nF + r_2(\alpha\xi + \beta F) = 0.$$

This shows that $\alpha = r_2 = 1$ and $\delta - 2 + n + \beta = 0$. Therefore $\delta = 0$, $n = \beta = 1$. This proves that $B_1 = \mathbf{P}^1 \times \mathbf{P}^m$, $L_1 = \mathcal{O}(1,1)$ and $b_2 = r_1 + r_2 + 2 - b_1 = m + 1 + 2 - (m+1) = 2$. Now note that $\mathcal{E}_1 \otimes \xi^{-1}$ restricts trivially to every fibre F, by 1). Hence $\mathcal{E}_1 \otimes \xi^{-1} = \rho^*\mathcal{G}$, where $\rho : B_1 \to \mathbf{P}^1$ is the projection and $\mathcal{G}$ is a rank-m vector bundle on $\mathbf{P}^1$. Since

$$\mathcal{G} = \mathcal{O}_{\mathbf{P}^1}(d_1) \oplus \ \ldots \ \oplus \mathcal{O}_{\mathbf{P}^1}(d_m),$$

we thus get

$$\mathcal{E}_1 = \rho^*\mathcal{G} \otimes \xi = \mathcal{O}(d_1, 1) \oplus \dots \oplus \mathcal{O}(d_m, 1) \tag{3.4.4}$$

and the ampleness of $\mathcal{E}_1$ implies that each summand in (3.4.4) is ample. So $d_i \geq 1$ for all i. On the other hand from the expression of $\det \mathcal{E}_1$ we immediately see that $\deg \mathcal{G} = 1$. Hence from

$$m \leq d_1 + \dots + d_m = \deg \mathcal{G} = 1$$

we conclude that $m = 1$. Thus $\mathcal{E}_1 = \det \mathcal{E}_1 = \xi + F = \mathcal{O}(1,1)$. This concludes the proof.□

In other words (3.4) says that case 1) fits into case 4) for $n = 2$. Now come back to (3.3.1) and note that $\mathcal{E}_i \oplus L_i^{r_j}$ is an ample vector bundle of rank $r = r_1 + r_2$ on B_i with trivial adjoint bundle. Therefore, by section 2 we get $b_1 + b_2 - 2 = r \leq b_i + 1$, which gives $b_i \leq 3$ for $i = 1, 2$. On the other hand from $r_i \geq 1$ we also get $b_i \geq 2$, so that $b_i = 2$ or 3 for $i = 1, 2$. Looking again at (3.3.1) and taking into account (3.4) we can now further refine the list of possibilities for $(B_i, \mathcal{E}_i), L_i$ and $b_j = r_j + 1$; they are the following:

(1) $(\mathbf{P}^1 \times \mathbf{P}^1, \mathcal{O}(1,1)), L_i = \mathcal{O}(1,1), b_j = 2,$
(2) $(\mathbf{P}^2, \mathcal{O}(1)), L_i = \mathcal{O}(1), b_j = 3,$
(3) $(\mathbf{P}^2, \mathcal{O}(1)), L_i = \mathcal{O}(2), b_j = 2,$
(4) $(\mathbf{P}^3, \mathcal{O}(1)^2), L_i = \mathcal{O}(1), b_j = 3,$
(5) $(\mathbf{P}^3, \mathcal{O}(1)^2), L_i = \mathcal{O}(2), b_j = 2,$
(6) $(\mathbf{P}^2, \mathcal{O}(2)), L_i = \mathcal{O}(1), b_j = 2,$
(7) $(\mathbf{P}^3, \mathcal{O}(2) \oplus \mathcal{O}(1)), L_i = \mathcal{O}(1), b_j = 2,$
(8) $(\mathbf{Q}^3, \mathcal{O}(1)^2), L_i = \mathcal{O}(1), b_j = 2.$

Let (h,k) denote the pair $(B, \mathcal{E})$ obtained by combining a factor of type (h) with a factor of type (k) in the above list. Then it is immediate to check that up to exchanging h and k, only the following cases can occur: (1,1), (1,3), (1,6), (2,5), (2,7), (2,8), (3,3), (3,6), (4,4), (6,6). Case (1,1) corresponds to the pair (3.1.0), while the remaining ones give rise to the pairs (6), (6), (4), (4), (3), (13), (14), (1), (13) in (3.1), two of which not already found.

4. On the index of the base of a special Fano P-bundle

In this section we assume that

(4.0) $P = \mathbf{P}(\mathcal{E})$ is a special Fano $\mathbf{P}$-bundle over a projective manifold B and $r = \operatorname{rk} \mathcal{E}$.

Then by (2.0.2) B is Fano. This is more generally true if P is simply any Fano $\mathbf{P}$-bundle over B [11, (1.6)]. Let i be the index of B; then $-K_B = iH$, with $H \in \operatorname{Pic}(B)$ ample. A rational curve $l \subset B$ is said to be a line of (B, H) if $lH = 1$. The following fact is well known.

(4.1) Proposition. *Let P be as in* (4.0). *If (B, H) contains a line, then*

$$i \geq r. \tag{4.1.1}$$

Proof. Let l be a line of (B, H). Since l is a rational curve we have $c_1(\mathcal{E}_l) \geq r$; hence from (2.0.2) we get

$$r \leq c_1(\mathcal{E}_l) = -K_B l = iHl = i. \qquad \square$$

If B has a large index then certainly (B, H) contains lines. In fact

(4.2) Proposition. *Let B be a Fano manifold of dimension b and index $i > (b+1)/2$. Then (B, H) contains a line.*

Proof. As B is Fano, by Mori's cone theorem, it contains a rational extremal curve, say C. Since $-K_B C \leq b+1$ we get $b+1 \geq -K_B C = iHC$, i.e., $HC \leq (b+1)/i$. So if $i > (b+1)/2$ we conclude that $HC = 1$. $\square$

In fact the argument shows that all extremal rational curves of B are lines of (B, H). By combining (4.1) with (4.2) we get the following

(4.3) Corollary. *Let P be as in* (4.0). *If $i > (b+1)/2$, then $i \geq r$.*

Special Fano $\mathbf{P}$-bundles with $i < r$ are known, e.g., see (2.4.(2)), (2.4.(3)). However it is conjectured ([10, p. 152], see also [9, (2.12), (3)]) that

(4.3.1) $$\text{if } \operatorname{Pic}(B) \cong \mathbf{Z}, \text{ then } i \geq r.$$

Corollary (4.3) provides a support for this conjecture since B having large index implies that $\operatorname{Pic}(B) \cong \mathbf{Z}$, apart from very special cases [13]. What can be said about the equality $i = r$? Of course for any Fano manifold B of index i the vector bundle $\mathcal{E} = H^i$ gives a special Fano $\mathbf{P}$-bundle satisfying $i = r$ and the known classification results outlined in section 2 show that this obvious way is the only one to produce special Fano $\mathbf{P}$-bundles with $i = r$ when $\operatorname{cork} P \leq 2$. However this is no longer true if $\operatorname{cork} P \geq 3$, as the following example suggested by (3.1.(11)) shows.

(4.4) Example. Let $B = \mathbf{P}^{2n-1} \times \mathbf{P}^1$, $n \geq 3$ odd. Then $i = 2$ with $H = \mathcal{O}(n, 1)$. Let $\mathcal{E} = \mathcal{O}(p, 1) \oplus \mathcal{O}(q, 1)$, with p, q positive, $p \neq q$, $p + q = 2n$. Then $\mathcal{E}$ satisfies (2.0.2), but $\mathcal{E} \neq H^2$. This also shows the irrelevance of the existence of lines on (B, H) for the obvious way being the only one to produce special Fano $\mathbf{P}$-bundles with $i = r$. Similarly we have

(4.5) Example. Let $B = \mathbf{P}^5 \times \mathbf{P}^3$; then $i = 2$ with $H = \mathcal{O}(3, 2)$. Thus $\mathcal{E} = \mathcal{O}(4, 3) \oplus \mathcal{O}(2, 1)$ satisfies (2.0.2) but $\mathcal{E} \neq H^2$. Note that the pair (B, H) does not contain lines. So this shows in particular that the converse to (4.1) does not hold. Note that B has small index in all our examples. We would like to conclude this paper by appending the following question to conjecture (4.3.1).

(4.6) Question. Let things be as above and assume that $i = r$. Is it true that $\mathcal{E} = H^r$

(1) if $\operatorname{Pic}(B) \cong \mathbf{Z}$?

(2) (weaker form) if $i > (b+1)/2$?

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