Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 1, 199-207.

On Maximal Primitive Fixing Systems

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Abstract. Let $\rho_{\max}(M)$ denote the maximal number of points forming a primitive fixing system for a convex body $M \subset \mathbf{R}^n$. Sharpening results of B. Bollobás with respect to the quantity $\rho_{\max}(M)$ for $n \geq 3$, we construct counterexamples to the conjecture $\rho_{\max}(M) \leq 2(2^n - 1)$ of L. Danzer. These counterexamples M satisfy $\rho_{\max}(M) = \infty$, and they can belong to the following classes of convex bodies: cap bodies, zonoids, and zonotopes.

MSC 1991: 52A20

Keywords: convex body, illumination, primitive fixing system, cap body, zonoid, zonotope, Danzer conjecture.

1. Introduction

L. Fejes Tóth [7] introduced the notion of a *primitive fixing system* for a compact, convex body in $\mathbf{R}^n$. We recall the definition and some related results.

Let $M \subset \mathbf{R}^n$ be a compact, convex body and $F \subset \text{bd } M$. A direction l, defined by a nonzero vector $e \in \mathbf{R}^n$, is said to be an *outer moving direction* with respect to M and F if for every real number $\lambda > 0$ the intersection of F and $(-\lambda e + \text{ int } M)$ is empty. If there is no outer moving direction with respect to F, then F is a *fixing system* of the body M (against translations). Visually, it is impossible to translate M in any direction if we assume to have "fixing nails" at the points of F. A fixing system F of the body M is *primitive* if no proper subsystem $F' \subset F$ is a fixing system of M.

Denote by $\rho_{\min}(M)$ the *smallest* one of the integers s such that there exists a primitive fixing system of M consisting of s points. Analogously, $\rho_{\max}(M)$ is the *largest* one of the integers s such that there exists a primitive fixing system of M consisting of s points.

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[1]This research was partially supported by International Science Foundation, Grant no. M3D300, and by Deutsche Forschungsgemeinschaft

L. Fejes Tóth [7] showed that fixing systems for smooth convex bodies are strongly connected with the classification of *primitive polytopes*, i.e., of compact, convex polytopes with the following property of the facet system: if any facet is omitted, then the intersection of the supporting half-spaces of the remaining facets is no longer bounded; see also [13] for extensions and related notions.

B. Grünbaum [10] showed that for every compact, convex body $M \subset \mathbf{R}^n$ the inequality $\rho_{\min}(M) \leq 2n$ holds, and he posed the task to show that only the parallelotopes can attain this bound. (This is covered by Theorem 2 below.) Furthermore, P. Mani [12] established the following result: there is a convex n-polytope $P \subset \mathbf{R}^n$, $n \geq 3$, with $2n+2$ vertices which is fixed by its vertex set, but no polytope with less than $2n+2$ vertices can be fixed by its vertex set. In view of containment problems for convex bodies (which are, in a sense, related to the fixing system problem), a reformulation of Mani's result was given in [11]. S. Fudali [9] investigated interesting relations between fixing systems and homothetical coverings of convex bodies.

L. Fejes Tóth [7] also mentioned that $\rho_{\max}(M) \leq 6$ for any compact, convex body $M \subset \mathbf{R}^2$, i.e., the *maximal* number of points in a primitive fixing system in the plane is 6, attained by the vertices of a hexagon with pairwise parallel sides. B. Tomor [15]) and S. Fudali [8] showed that these plane figures are the only ones attaining this upper bound.

In [6], L. Danzer formulated the conjecture that the cardinalities of primitive fixing systems are bounded. More detailed, the rhombic dodecahedron $M \subset \mathbf{R}^3$ has a primitive fixing system consisting of 14 points (its vertices), and Danzer assumed that this is the *maximal* value of the cardinality of a primitive fixing system for *any* compact, convex body $M \subset \mathbf{R}^3$, i.e., $\rho_{\max}(M) \leq 14$ for $n = 3$. The analogous zonotope in $\mathbf{R}^n$ (i.e., the vector sum of $n+1$ line segments, no n of which are parallel to a hyperplane) has $2(2^n - 1)$ vertices, and Danzer assumed that this body has its vertex set as the primitive fixing system of maximal cardinality, i.e., $\rho_{\max}(M) \leq 2(2^n - 1)$ for any compact, convex body $M \subset \mathbf{R}^n$.

But in contradiction to this assumption, B. Bollobás [1] gave an unexpected solution to this question for $n \geq 3$. He noticed that for every compact, convex body $M \subset \mathbf{R}^n$ any of its primitive fixing systems contains only a finite number of points. On the other hand, he proved that *for any positive integer $k \geq 6$ there is a compact, convex body $M \subset \mathbf{R}^3$* (Fig. 1) *with a primitive fixing system $F \subset M$ containing k points.*

We remark that the bodies constructed in [1] essentially depend on k, i.e., for different k the bodies are different, and for every body M in [1] the number $\rho_{\max}(M)$ is finite. Moreover, each body constructed in [1] has *infinitely many* nonregular boundary points.

Finally, we indicate some results which are needed in the following. They were established (among other ones) in [4] and [5]. To formulate them, we recall that a boundary point a of a compact, convex body $M \subset \mathbf{R}^n$ is said to be *illuminated* [3] by a direction (defined by a nonzero vector $p \in \mathbf{R}^n$) if $a + \lambda p \in \operatorname{int} M$ for a positive number λ.

Lemma 1: *A point system $F \subset \operatorname{bd} M$ is a fixing system of a compact, convex body $M \subset \mathbf{R}^n$ if and only if each direction illuminates at least one point of F.*

For the following theorem, we recall the definition of the functional md M which was introduced in [2]. Let $H(M)$ denote the system of all outer unit normals of the convex body $M \subset \mathbf{R}^n$ at its regular boundary points. Then md $H(M) =:$ md M is the largest of

Figure 1

the integers k such that there is a system of $k+1$ minimally dependent vectors in $H(M)$. (A system of $k+1$ vectors is *minimally dependent* if it is positively dependent and each of its subsystems of k elements is linearly independent.)

Theorem 2: *For every compact, convex body $M \subset \mathbf{R}^n$ the inequalities*

$$n + \frac{n}{\mathrm{md}M} \leq \rho_{\min}(M) \leq 2n + 1 - \mathrm{md}M$$

hold. Moreover, these estimates are sharp, i.e., for every integer ρ satisfying $n + \frac{n}{m} \leq \rho \leq 2n+1-m$ there exists a compact, convex body $M \subset \mathbf{R}^n$ with md $M = m$ *and* $\rho_{\min}(M) = \rho$.

In particular, the case md $M = 1$ yields the announced characterization of parallelotopes which was conjectured by B. Grünbaum [10].

Lemma 3: *Let M be a compact, convex body in $\mathbf{R}^n$ which is represented as a direct vector sum: $M = M_1 \oplus \cdots \oplus M_k$. Assume that every convex set $M_1, \cdots, M_k$ contains the origin in its relative interior, and let $F_1, \cdots, F_k$ be primitive fixing systems for $M_1, \cdots, M_k$ with respect to their affine hulls, respectively. Then $F_1 \cup \cdots \cup F_k$ is a primitive fixing system for the body M.*

In this note we will construct three new examples. The first one (see section 2 below) is a compact, convex body $M \subset \mathbf{R}^3$ with $\rho_{\max}(M) = \infty$. More detailed, the body M possesses the following property: for any integer $m \geq 3$ there is a primitive fixing system $F \subset \mathrm{bd}M$ consisting of $m+1$ points. An analogous example can be easily constructed in $\mathbf{R}^n$. Thus, in contrast with [1], we give a *unique* body for any cardinality of a primitive fixing system. We remark that the constructed body M is a *cap body*, i.e., it has *only one* nonregular boundary point b, whereas for any *smooth*, compact, convex body $N \subset \mathbf{R}^n$ the number $\rho_{\max}(N)$ is not larger than $2n$, see [10].

The second example (see section 3) is a *zonoid* $M \subset \mathbf{R}^3$ which has *only two* nonregular boundary points and satisfies $\rho_{\max}(M) = \infty$. (By central symmetry, no zonoid can have only one nonregular boundary point.) In $\mathbf{R}^n$, an example can be constructed analogously. Finally, in section 4 below we give our third example. Namely, for any integer $m \geq 9$ we construct a vector sum (= Minkowski sum) of m line segments, i.e., a *zonotope* $Z_m \subset \mathbf{R}^3$, with a primitive fixing system consisting of $m+1$ points (to pass to $\mathbf{R}^n$, it is sufficient to apply Lemma 3 above). Thus, already in the class of zonotopes we have counterexamples to the conjecture of L. Danzer.

2. A body $M \subset \mathbf{R}^3$ with only one nonregular boundary point and $\rho_{\max}(M) = \infty$

In an orthogonal coordinate system (x_1, x_2, x_3) in $\mathbf{R}^3$, denote by $B \subset \mathbf{R}^3$ the ball of radius 2 centered at the origin and by M the body $\operatorname{conv} B \cup \{b\}$, where b is the point $(0, 0, -2)$. Let C_λ be the circumference $\operatorname{bd} B \cap \Gamma_\lambda$, where Γ_λ is the plane consisting of the points $x = (x_1, x_2, x_3)$ with $x_3 = \lambda$, $-\sqrt{2} < \lambda < \sqrt{2}$. Choose an arbitrary point $q \in C_1$ and denote by l_q the direction defined by the vector $q - b$ (Fig. 2). The set $G_q \subset \operatorname{bd} M$ which is illuminated by the direction l_q can be described in the following way.

Figure 2

Consider the open half-space $\Pi_q = \{x : \langle x, q - b\rangle < 0\}$; then $G_q = (\mathrm{bd}M \cap \Pi_q) \setminus [b, q]$. The relative boundary of the set $G_q \subset \mathrm{bd}M$ is the union of the segment $[b, q]$ and the large circle $Z_q = \mathrm{bd}B \cap \mathrm{bd}\Pi_q$.

The intersection $Z_q \cap C_{1-\varepsilon}$ (where $0 < \varepsilon < 1$) consists of two points p', p'', and we denote by $\varphi(\varepsilon)$ the angle between the vectors $p' - g$, $p'' - g$, where g is the center of the circumference $C_{1-\varepsilon}$. When the parameter ε runs through the segment $[0, 1]$, then the angle $\varphi(\varepsilon)$ varies from 0 to π.

Let now $m \geq 3$ be an integer. We choose and fix a positive number ε such that

$$\frac{2\pi}{m} < \varphi(\varepsilon) < \frac{4\pi}{m}.$$

Furthermore, we choose points $a_1, \cdots, a_m$ which divide the circumference $C_{1-\varepsilon}$ into m equal arcs, and we denote by F the set $\{b, a_1, \cdots, a_m\}$. We will show that $F \subset \mathrm{bd}M$ is a primitive fixing system for the body M.

Indeed, the arc $w_q \subset C_{1-\varepsilon}$ (smaller than a half-circle) with the endpoints p', p'' contains at least one of the points $a_1, \cdots, a_m$ (since $\varphi(\varepsilon) > \frac{2\pi}{m}$). Hence for every $q \in C_{-1}$ the direction of the vector $q - b$ (i.e., every direction forming the angle $\frac{\pi}{4}$ with the positive x_3-axis) illuminates at least one of the points $a_1, \cdots, a_m$. Consequently any direction, forming with the positive x_3-axis an angle *not smaller* than $\frac{\pi}{4}$, also illuminates at least one of the points $a_1, \cdots, a_m$. And if a direction forms with the positive x_3-axis an angle *greater* than $\frac{\pi}{4}$, then this direction illuminates the point b. Thus *any* direction illuminates at least one point of the set F, i.e., F is a fixing system for the set M (see Lemma 1 above).

On the other hand, if for an integer $k \in \{1, \cdots, m\}$ the point $q \in C_{-1}$ is chosen in such a way that the arc w_q with the endpoints p', p'' has its midpoint at the point a_k, then the direction of the vector $q - b$ illuminates *only* the point a_k of the set F (since $\varphi(\varepsilon) < \frac{4\pi}{m}$). Moreover, the direction of the x_3-axis illuminates only the point b of the set F. This means that the fixing system $F \subset \mathrm{bd}M$ is primitive. Thus $\rho_{\max}(M) = \infty$.

We remark that the constructed body $M \subset \mathbf{R}^3$ is the union $B \cup Q$, where Q is the rotational cone with the base C_{-1} and the apex b. (In other words, M is a *cap body.*) In a sense, the equality $\rho_{\max}(M) = \infty$ is connected with an observation of V. Soltan [14] who noticed that for Q (or for any other cone with a circle as its base) there is a primitive illuminating system consisting of k directions, $k \geq 4$ being any integer.

3. A zonoid $M \subset \mathbf{R}^3$ with two nonregular boundary points and $\rho_{\max}(M) = \infty$

In this section we want to construct a *zonoid* M satisfying $\rho_{\max}(M) = \infty$. (In the sense of Hausdorff metric, zonoids are limits of vector sums of finitely many line segments.)

Denote by K the winding curve with the parametric representation

$$x_1 = \cos\frac{s}{\sqrt{2}},\ \ x_2 = \sin\frac{s}{\sqrt{2}},\ \ x_3 = \frac{s}{\sqrt{2}};\quad -\pi\sqrt{2} \leq s \leq \pi\sqrt{2}.$$

The parameter s is the arc length along K. By M we denote the zonoid generated by the curve K in the sense of Zalgaller-Reshetnyak [16]. In other words, M is the set of the points

$$z = \int_{-\pi\sqrt{2}}^{\pi\sqrt{2}} \mu(s)\tau(s)\, ds,$$

where

$$\tau(s) = \left(-\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}},\ \frac{1}{\sqrt{2}}\right)$$

is the unit tangential vector of K at the point $x \in K$ with $x_3 = \frac{s}{\sqrt{2}}$, and $\mu(s)$ runs through the set of all measurable functions with $|\mu(s)| \le 1$, $-\pi\sqrt{2} \le s \le \pi\sqrt{2}$. The boundary surface of the zonoid M has *only two* nonregular boundary points $b = (0,0,-\pi)$, $b' = (0,0,\pi)$ (since it is obtained by rotating the curve $x_1 = 4\cos\frac{x_3}{2}$, $x_2 = 0$, $-\pi \le x_3 \le \pi$, around the x_3-axis). We show that $\rho_{\max}(M) = \infty$.

Figure 3

Indeed, let l be a tangential ray of the body M at the point b, and l' be the ray symmetric to l with respect to the origin. Denote by G and G' the sets of those points $x \in \mathrm{bd}M$ which are illuminated by the directions of the rays l and l', respectively. The sets G, G' are open in $\mathrm{bd}M$ and have the common relative boundary $\Delta = \mathrm{cl}G \cap \mathrm{cl}G'$ (Fig. 3). The curve Δ has a spinode at the point b' such that both the branches of Δ have l' as their common tangential ray at b'.

Let C_λ, $0 \le \lambda \le \pi$, be the circumference $\mathrm{bd}M \cap \Gamma_\lambda$, where $\Gamma_\lambda = \{x : x_3 = \lambda\}$. The intersection $\Delta \cap C_\lambda$ consists of two points p', p'', and we denote by $\varphi(\lambda)$ the angle between the vectors $p' - g$, $p'' - g$, where g is the center of C_λ. When the parameter λ runs through the segment $[0, \pi]$, the angle $\varphi(\lambda)$ varies from π to 0.

Let now $m \ge 3$ be an integer. We choose and fix a number $\lambda > 0$ such that

$$\frac{2\pi}{m} < \varphi(\lambda) < \frac{4\pi}{m}.$$

Furthermore, we choose some points $a_1, \cdots, a_m$ which divide C_λ into m equal arcs, and we denote by F the set $\{b, a_1, \cdots, a_m\}$. Then $F \subset \mathrm{bd}M$ is a primitive fixing system for the body M (the proof can be conducted as in the previous section).

4. Zonotopes with arbitrarily large cardinalities of primitive fixing systems

Now we want to construct three-dimensional zonotopes with analogous properties, and it is easy to extend this construction to n-dimensional zonotopes with $n \geq 4$.

Let $C \subset \{x : x_3 = 1\}$ be the circumference of radius 1 centered at $(0, 0, 1)$, and $p_1, \cdots, p_m$ be points dividing C into m equal arcs, $m \geq 9$. Denote by Z_m the zonotope $I_1 + \cdots + I_m$, where I_k is the segment $[-p_k, p_k]$, $k = 1, \cdots, m$ (see Fig. 4 for $m = 12$; in Fig. 5 the view of Z_m from above is shown).

Figures 4 and 5

We will show that Z_m has a primitive fixing system containing $m + 1$ points. In particular, for $m = 14$ we obtain a primitive fixing system $F \subset \mathrm{bd}Z_{14}$ consisting of 15 points, i.e., we obtain a counterexample to the conjecture of L. Danzer.

We denote by b and b' the vertices $(0, 0, -m)$ and $(0, 0, m)$ of the zonotope Z_m, respectively. The points

$$b' - 2p_1, \cdots, b' - 2p_m$$

are the vertices of Z_m adjacent to b'; the points

$$b' - 2(p_1 + p_2),\ b' - 2(p_2 + p_3),\ \cdots,\ b' - 2(p_{m-1} + p_m),\ b' - 2(p_m + p_1)$$

form the next "circle" of adjacent vertices. The next "circle" consists of the vertices

$$a_1 = b' - 2(p_m + p_1 + p_2),\ a_2 = b' - 2(p_1 + p_2 + p_3),\ \cdots,$$
$$a_m = b' - 2(p_{m-1} + p_m + p_1).$$

We prove that $F = \{b, a_1, \cdots, a_m\} \subset \mathrm{bd} Z_m$ is a primitive fixing system for Z_m.

Consider the direction l_k defined by the vector p_k, where k is one of the integers $1, \cdots, m$. The segments parallel to the direction l_k and situated in $\mathrm{bd}\, Z_k$ are shown in Figure 5. They form a *belt* of parallel segments (see [4]). The points of $\mathrm{bd}\, Z_k$ which are illuminated by the direction l_k are situated *to the right* from the belt, whereas the points of the belt and the points to the left from the belt are not illuminated by l_k. This shows that the direction l_k illuminates a_k, but any other point from F is not illuminated by this direction.

Thus for each vertex $a_k,\ k = 1, \cdots, m$, there is a direction (namely l_k) which illuminates a_k and does not illuminate any other point of F. For $b \in F$ such a direction exists, too (namely, the direction defined by the vector b' illuminates b and does not illuminate any other point from F). If we can prove that F is a fixing system, this means that F is even *primitive*.

By Lemma 1 above, it remains to prove that any direction illuminates (at least) one of the points $b, a_1, \cdots, a_m$. Indeed, consider a ray emanating from b' and passing through a point c which belongs to a facet adjacent to b'. The direction of this ray is defined by a vector $\lambda p_k + \mu a_{k+1},\ k \in \{1, \cdots, m\}$, where the numbers λ, μ are nonnegative and do not vanish simultaneously, say $\lambda > 0$. Then for any $\beta \geq 0$ the direction of the vector $\lambda p_k + \mu a_{k+1} + \beta b$ illuminates the vertex a_k. And if $\beta < 0$, then the direction $\lambda p_k + \mu a_{k+1} + \beta b$ illuminates the vertex b. Thus any direction illuminates at least one point of the set F.

By using Lemma 3, one can construct analogous zonotopes in $\mathbf{R}^n$ for any $n \geq 4$.

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Received: January 10, 1996