Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 1, 31-40.

Complete Lifts of Harmonic Maps and Morphisms between Euclidean Spaces

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Abstract. In this paper, the concept of the complete lifts of maps between (real and complex) Euclidean spaces is introduced. As applications, we are able to use this concept to characterize holomorphic maps $\phi : \mathbb{C}^m \supset U \longrightarrow \mathbb{C}^n$ and to construct many new examples of harmonic morphisms. Finally we show that the complete lift of the quaternion product followed by the complex product is a simple and explicit example of a harmonic morphism which does not arise from any Kähler structure in the sense of Baird and Wood [Internat. J. Math. 6, 161-192 (1995)].
MSC 1991: 53C40, 58E20
Keywords: complete lifts, harmonic morphisms, holomorphic maps

1. Introduction

Let(M^m, g) and (N^n, h) be two Riemannian manifolds. A map $\phi : (M^m, g) \longrightarrow (N^n, h)$ is called a harmonic map if the divergence of its differential vanishes. Such maps are the extrema of the energy functional $\frac{1}{2}\int_D |d\phi|^2$ over compact domain $D \subset M$. For a detailed account on harmonic maps we refer to [7],[8],[9] and the references therein. Harmonic morphisms are a special subclass of harmonic maps which preserve solutions of Laplace's equation in the sense that for any harmonic function $f : U \longrightarrow \mathbb{R}$, defined on an open subset U of N with $\phi^{-1}(U)$ non-empty, $f \circ \phi : \phi^{-1}(U) \longrightarrow \mathbb{R}$ is a harmonic function. In other words, ϕ pulls back germs of harmonic functions on N to germs of harmonic functions on M. In the theory of stochastic processes, harmonic morphisms $\phi : (M, g) \longrightarrow (N, h)$ are found to be Brownian path preserving mappings meaning that they map Brownian motions on M to Brownian

*The author was supported by the Chinese Education Commission as "Visiting scholar program (94A04)".

motions on N([6],[16]). The following nice characterization of a harmonic morphism is due to Fuglede and Ishihara independently.

Theorem 1.1. *([10],[15])A map ϕ is a harmonic morphism if and only if it is both a harmonic and a horizontal weakly conformal map.*

For a map $\phi : \mathbb{R}^m \supset U \longrightarrow \mathbb{R}^n$ between Euclidean spaces, with $\phi(x) = (\phi^1(x), ..., \phi^n(x))$, the harmonicity and weakly conformality are just equivalent to the following conditions respectively:

$$\sum_{i=1}^{m} \frac{\partial^2 \phi^k}{\partial x_i^2} = 0 \tag{1}$$

$$\sum_{i=1}^{m} \frac{\partial \phi^k}{\partial x_i}\frac{\partial \phi^l}{\partial x_i} = \lambda^2(x)\delta_{kl} \tag{2}$$

For $k, l = 1, 2, ..., n$.

It is well-known that (i) if dim $M <$ dim N then any harmonic morphism is a constant map, (ii) if dim $M =$ dim $N = 2$, the harmonic morphisms are precisely the weakly conformal maps, (iii) if $\underline{\text{dim } N = 1}$ they are precisely the harmonic maps, (iv) if dim $M =$ dim $N \geq 3$ they are precisely the homothetic maps([10], [15]). Though the composition of harmonic maps does not generally turn out to be a harmonic map , we have

Lemma 1.2. *If $\phi_1 : (M, g) \longrightarrow (\overline{M}, \overline{g})$ and $\phi_2 : (\overline{M}, \overline{g}) \longrightarrow (N, h)$ are harmonic morphisms, then so is their composition $\phi_2 \circ \phi_1 : (M, g) \longrightarrow (N, h)$.*

Lemma 1.3. *([13]) Let $\pi : (\overline{M}, \overline{g}) \longrightarrow (M, g)$ be a non-constant harmonic morphism, $\phi : (M, g) \longrightarrow (N, h)$ be a map, and $\Phi = \phi \circ \pi : (\overline{M}, \overline{g}) \longrightarrow (N, h)$ their composition, then Φ is a harmonic morphism if and only if ϕ is a harmonic morphism.*

Since the equations determining a harmonic morphism form an overdetermined system of partial differential equations one should not expect to find too many examples of harmonic morphisms, however, one can easily check that all the maps in the following example are in this class.

Example 1.4. (i) $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}, \phi(z, w) = zw$ and $\phi(z, w) = z\overline{w}$;
(ii) $\phi : \mathbb{C}^2 \longrightarrow \mathbb{R} \times \mathbb{C}$, with $\phi(z, w) = (|z|^2 - |w|^2, 2zw)$;
(iii) The quaternion product
$q : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2, q(z_1, z_2, z_3, z_4) = (z_1 z_3 - z_2 \overline{z}_4, z_1 z_4 + z_2 \overline{z}_3)$.
(iv) The hyperbolic analogue of stereographic projection
$\phi : \mathbb{R}^3 \setminus \{(0, 0, x_3) \mid x_3 \geq 0\} \longrightarrow \mathbb{R}^2$ with $\phi(x_1, x_2, x_3) = \left(\frac{x_1}{r - x_3}, \frac{x_2}{r - x_3}\right)$, where $r^2 = x_1^2 + x_2^2 + x_3^2$.
(v) The orthogonal projection $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ with $\phi(x_1, \ldots, x_m) = (x_1, \ldots, x_n)$, and the radial projection $\phi : \mathbb{R}^m \setminus \{0\} \longrightarrow S^{m-1}(m = 1, 2, \ldots)$ with $\phi(x) = x/|x|$.
(vi) The natural projection $\pi : (TM, \overline{g}) \longrightarrow (M, g)$, where $\overline{g}$ is the Sasaki metric on the tangent bundle of the Riemannian manifold (M, g).
(vii) the Hopf maps $S^3 \longrightarrow S^2$ and $S^{2n+1} \longrightarrow \mathbb{C}P^n(n = 1, 2, \ldots)$.
(viii) Any holomorphic maps from a Kähler manifold to a Riemann surface ([1]), in particular, any holomorphic function $\phi : U \longrightarrow \mathbb{C}$ on an open subset U of $\mathbb{C}^m$ is a harmonic morphism.

In recent years, much work has been done in classifying and constructing harmonic morphisms from certain model spaces to other model spaces (see e.g. [2], [3], [4] ,[5], [11], [13] , [12], [14], [18], [19]). More recently Baird and Wood have found (see [19] ,[5]) some strong links between Hermitian structures and harmonic morphisms from open subsets of $\mathbb{R}^{2n}$ to $\mathbb{C}$ or a Riemann surface. They have constructed many interesting locally and globally defined harmonic morphisms. Some of them are holomorphic with respect to non- Kähler structures and some are not holomorphic with respect to any Kähler structure. In this work we use the complete lift of the quaternion product and the composition with the complex product to give a harmonic morphism $\Phi : \mathbb{R}^{16} \supset U \longrightarrow \mathbb{C}$ which is not holomorphic with respe! ct to any Kähler structure.

2. Complete Lifts and Their Properties

Definition 2.1. *Let $\phi : \mathbb{R}^m \supset U \longrightarrow \mathbb{R}^n, \phi(x) = (\phi^1(x), ..., \phi^n(x))$, be a map from an open connected subset of $\mathbb{R}^m$ into $\mathbb{R}^n$. The (real) complete lift of ϕ is a map $\Phi : \mathbb{R}^{2m} \supset U \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$, given by*

$$\Phi(x_1, \dots, x_m; y_1, \dots, y_m) = \left(\frac{\partial \phi^i}{\partial x_j}(x)\right) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \tag{3}$$

where $\left(\frac{\partial \phi^i}{\partial x_j}(x)\right)$ denotes the Jacobian matrix of ϕ at x.

Remark 2.2. (i) The complete lift of a map is a partial linear map in the sense that it depends linearly on half of its variables.
(ii) Let $\phi : \mathbb{C}^m \supset U \longrightarrow \mathbb{C}^n$ be a C^∞ from an open connected subset of $\mathbb{C}^m$ into $\mathbb{C}^n$,then the (complex) complete lift of ϕ can be defined similarly by using the complex Jacobian matrix.

It is well-known that $\mathbb{C}^m$ can be identified with $\mathbb{R}^{2m}$ by identifying $(z_1, \dots, z_m)$ with $(x_1, y_1, \dots, x_m, y_m)$, where $z_k = x_k + iy_k$. Thus any map $\phi : \mathbb{C}^m \supset U \longrightarrow \mathbb{C}^n$ can be identified with a map (called *the real identification of* ϕ) $\phi_r : \mathbb{R}^{2m} \supset W \longrightarrow \mathbb{R}^{2n}$, given by

$$\phi_r(x_1, y_1, \dots, x_m, y_m) = (u^1(x, y), v^1(x, y), \dots, u^n(x, y), v^n(x, y))$$

where $u^k(x, y)$and $v^k(x, y)$ are the real and imaginary parts of the kth component function of ϕ , i.e $\phi^k(x, y) = u^k(x, y) + iv^k(x, y)$.

It should be noted that the operation of the complete lift and the above identification does not commute in general. In fact, the following is easily established:

Proposition 2.3. *Let $\phi : U \longrightarrow \mathbb{C}^n$ be a C^∞ map from an open connected subset U of $\mathbb{C}^m$, then ϕ is holomorphic if and only if the (complex) complete lift Φ of ϕ is identical with the (real) complete lift Φ_r of its real identification ϕ_r, viewed as a map between complex Euclidean spaces.*

Example 2.4. The quaternion product $q : \mathbb{C}^2 \times \mathbb{C}^2 \longrightarrow \mathbb{C}^2, q(z_1, z_2, z_3, z_4) = (z_1 z_3 - z_2 \overline{z}_4, z_1 z_4 + z_2 \overline{z}_3)$ is not holomorphic. The complex complete lift of Q is given by

$$Q(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4) = (z_3 w_1 - \overline{z}_4 w_2 + z_1 w_3, z_4 w_1 + \overline{z}_3 w_2 + z_1 w_4)$$

whilst the real complete lift of q_r, viewed as a complex map, is given by

$$\begin{aligned}&Q_r(z_1,z_2,z_3,z_4,w_1,w_2,w_3,w_4)=\\&\quad(z_3w_1-\overline{z}_4w_2+z_1w_3-z_2\overline{w}_4,z_4w_1+\overline{z}_3w_2+z_2\overline{w}_3+z_1w_4)\end{aligned}$$

which is obviously different from Q.

Theorem 2.5. *The complete lift of any holomorphic map $\phi:\mathbb{C}^m\supset U\longrightarrow\mathbb{C}^n$ is holomorphic.*

Proof: It suffices to check that the component function $\Phi^l(z,w)=\sum_{k=1}^m\frac{\partial\phi^l}{\partial z_k}w_k(l=1,2,\dots,n)$ is holomorphic with respect to variables $z_1,\dots,z_m,w_1,\dots,w_m$. This amounts to check that the following equations

$$\frac{\partial}{\partial\overline{z}_k}(\sum_{j=1}^m\frac{\partial\phi^l}{\partial z_j}w_j)=0$$
$$\frac{\partial}{\partial\overline{w}_k}(\sum_{j=1}^m\frac{\partial\phi^l}{\partial z_j}w_j)=0$$

hold for $l=1,2,\dots,n,\ k=1,2,\dots,m$, which is trivial and is omitted.

Theorem 2.6. *The complete lift of any harmonic map $\phi:\mathbb{R}^m\supset U\longrightarrow\mathbb{R}^n$ is harmonic.*

Proof: Let $\phi:\mathbb{R}^m\supset U\longrightarrow\mathbb{R}^n,\phi(x)=(\phi^1(x),...,\phi^n(x))$, be a harmonic map, then we have $\sum_{j=1}^m\frac{\partial^2\phi^k}{\partial x_j^2}=0$ for $k=1,2,\dots,n$. We must check that the component function of the complete lift $\Phi^k(x,y)=\sum_{i=1}^m\frac{\partial\phi^k}{\partial x_i}y_i,\ k=1,2,\dots,n$, is a harmonic function of $x_1,\dots,x_m,y_1,\dots,y_m$. But
$\sum_{j=1}^m\frac{\partial^2}{\partial x_j^2}(\sum_{i=1}^m\frac{\partial\phi^k}{\partial x_i}y_i)+\sum_{j=1}^m\frac{\partial^2}{\partial y_j^2}(\sum_{i=1}^m\frac{\partial\phi^k}{\partial x_i}y_i)=\sum_{i=1}^m y_i\frac{\partial}{\partial x_i}(\sum_{j=1}^m\frac{\partial^2\phi^k}{\partial x_j^2})=0$. This ends the proof.

For the complete lift of a horizontal conformal map we have

Theorem 2.7. *Let $\phi:\mathbb{R}^m\supset U\longrightarrow\mathbb{R}^n$ be a horizontal weakly conformal map, then the complete lift Φ of ϕ is horizantal weakly conformal if and only if the following conditions hold for $\alpha,\beta=1,2,\dots,n.$:*

$$(\text{hess }\phi^\alpha)^2=(\text{hess }\phi^\beta)^2\tag{4}$$
$$(\text{hess }\phi^\alpha)(\text{hess }\phi^\beta)=-(\text{hess }\phi^\beta)(\text{hess }\phi^\alpha)\tag{5}$$

where $(\text{hess }\phi^\alpha)$ *denotes the Hessian matrix of the component function* ϕ^α*.*

Proof: At each point (x,y), the Jacobian matrix of Φ is given by

$$J(\Phi)=\begin{pmatrix}\partial_1\nabla\phi^1\bullet y&\dots&\partial_m\nabla\phi^1\bullet y&\frac{\partial\phi^1}{\partial x_1}&\dots&\frac{\partial\phi^1}{\partial x_m}\\\partial_1\nabla\phi^2\bullet y&\dots&\partial_m\nabla\phi^2\bullet y&\frac{\partial\phi^2}{\partial x_1}&\dots&\frac{\partial\phi^2}{\partial x_m}\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\\partial_1\nabla\phi^n\bullet y&\dots&\partial_m\nabla\phi^n\bullet y&\frac{\partial\phi^n}{\partial x_1}&\dots&\frac{\partial\phi^n}{\partial x_m}\end{pmatrix}$$

Where $\bullet$ denotes the inner product in Euclidean space.
From this together with the horizontal weak conformality of ϕ it follows that Φ is horizontally weakly conformal if and only if the following equations hold

$$\sum_{i=1}^{m}(\partial_i\nabla\phi^\alpha \bullet y)^2 \equiv \sum_{i=1}^{m}(\partial_i\nabla\phi^\beta \bullet y)^2 \tag{6}$$

$$\sum_{i=1}^{m}(\partial_i\nabla\phi^\alpha \bullet y)(\partial_i\nabla\phi^\beta \bullet y) \equiv 0 \tag{7}$$

Note that Equations (6) and (7) express the identity of quadratics in $y_i^{\prime}s$, they are equivalent to the following equations

$$\partial_i\nabla\phi^\alpha \bullet \partial_i\nabla\phi^\alpha = \partial_i\nabla\phi^\beta \bullet \partial_i\nabla\phi^\beta \tag{8}$$

$$\partial_i\nabla\phi^\alpha \bullet \partial_j\nabla\phi^\alpha = \partial_i\nabla\phi^\beta \bullet \partial_j\nabla\phi^\beta \tag{9}$$

$$\partial_i\nabla\phi^\alpha \bullet \partial_i\nabla\phi^\beta = 0 \tag{10}$$

$$\partial_i\nabla\phi^\alpha \bullet \partial_j\nabla\phi^\beta = -\partial_j\nabla\phi^\alpha \bullet \partial_i\nabla\phi^\beta \tag{11}$$

Now equations (8) and (9) are equivalent to the fact that the square of the Hessian matrix of ϕ^α is the same for all $\alpha = 1, 2, \ldots, n$, while equations (10) and (11) just say that the product matrix $(\mathrm{hess}\phi^\alpha)(\mathrm{hess}\phi^\beta)$ is skew symmetric. On the other hand, since $(\mathrm{hess}\phi^\alpha)$ and $(\mathrm{hess}\phi^\beta)$ are symmetric it is easily seen that (10) and (11) are, in fact, equivalent to (5) which ends the proof.

3. Examples and further results

In this section we give examples of harmonic morphisms whose complete lifts are harmonic morphisms. We prove that the complete lift of a quadratic harmonic morphism is always a quadratic harmonic morphism. This gives us a large class of harmonic morphisms between Euclidean spaces. Two counter examples are also given, and finally we use the complete lift of the quaternion product and the composition with the complex product to construct an example of harmonic morphism $\phi : \mathbb{R}^{16} \longrightarrow \mathbb{C}$ which does not arise from any Kähler structure.

Example 3.1. (i) $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}$, $\phi(z, w) = z\overline{w}$ is a non-holomorphic harmonic morphism. The complex and the real complete lifts are given respectively by
$\Phi(z_1, z_2, w_1, w_2) = \overline{z_2}w_2$, and $\Phi_r(x_1, \ldots, x_4, y_1, \ldots, y_4) = (x_3y_1 + x_4y_2 + x_1y_3 + x_2y_4, -x_4y_1 + x_3y_2 + x_2y_3 - x_1y_4)$. It is easily checked that they are both harmonic morphisms.
(ii) $\phi : \mathbb{C}^2 \longrightarrow \mathbb{R}\times\mathbb{C}, \phi(z, w) = (|z|^2 - |w|^2, 2zw)$, viewed as a map between real Euclidean spaces can be written as
$\phi(x_1, \ldots, x_4) = (x_1^2 + x_2^2 - x_3^2 - x_4^2, 2x_1x_3 - 2x_2x_4, 2x_1x_4 - 2x_2x_3)$.
Then ϕ is a harmonic morphism (see [6]) and its complete lift is given by
$\Phi = (2x_1y_1 + 2x_2y_2 - 2x_3y_3 - 2x_4y_4, 2x_3y_1 - 2x_4y_2 + 2x_1y_3 - 2x_2y_4, 2x_4y_1 + 2x_3y_2 + 2x_2y_3 + 2x_1y_4)$.
One can easily check that Φ is also a harmonic morphism.
(iii) The real complete lift of the quaternion product is again a harmonic morphism while the complex complete lift of the quaternion product is no longer a harmonic morphism. In

fact, the real complete lift of q is given by

$$Q_r(x,y) = \begin{pmatrix} x_5 & -x_6 & -x_7 & -x_8 & x_1 & -x_2 & -x_3 & -x_4 \\ x_6 & x_5 & x_8 & -x_7 & x_2 & x_1 & -x_4 & x_3 \\ x_7 & -x_8 & x_5 & x_6 & x_3 & x_4 & x_1 & -x_2 \\ x_8 & x_7 & -x_6 & x_5 & x_4 & -x_3 & x_2 & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_8 \end{pmatrix}$$

It follows from Theorem 2.6 that $Q_r(x,y)$ is harmonic. $Q_r(x,y)$ is also horizontal weakly conformal as one can see this at a glance of its Jacobian matrix

$$J(Q_r(x,y)) = \left(\; J(q_r(y)) \mid J(q_r(x)) \; \right)_{(4\times 16)}.$$

On the other hand, the complex complete lift Q of q is given by

$$Q(z,w) = (z_3w_1 - \overline{z}_4w_2 + z_1w_3, z_4w_1 + \overline{z}_3w_2 + z_1w_4).$$

When viewed as a map between Euclidean spaces its Jacobian matrix can be calculated as

$$J(Q(x,y)) =$$
$$\begin{pmatrix} y_5 & -y_6 & 0 & 0 & y_1 & -y_2 & -y_3 & -y_4 & x_5 & -x_6 & -x_7 & -x_8 & x_1 & -x_2 & 0 & 0 \\ y_6 & y_5 & 0 & 0 & y_2 & y_1 & -y_4 & y_3 & x_6 & x_5 & x_8 & -x_7 & x_2 & x_1 & 0 & 0 \\ y_7 & -y_8 & 0 & 0 & y_3 & y_4 & y_1 & -y_2 & x_7 & -x_8 & x_5 & x_6 & 0 & 0 & x_1 & -x_2 \\ y_8 & y_7 & 0 & 0 & y_4 & -y_3 & y_2 & y_1 & x_8 & x_7 & -x_6 & x_5 & 0 & 0 & x_2 & x_1 \end{pmatrix}$$

Therefore, the complex complete lift of q is not horizontal weakly conformal though it is a harmonic map.
(iv) The complete lift of the hyperbolic analogue of stereographic projection in Example 1.4(iv) is not a harmonic morphism.

Note that the maps in $(i) - (iii)$ of Example 3.1 are among a large class of harmonic morphisms whose complete lifts are always harmonic morphisms:

Definition 3.2. *A map $\Phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is called a quadratic map if all of its components are homogeneous polynomials of degree* 2.

Theorem 3.3. *The complete lift of any quadratic harmonic morphism is again a quadratic harmonic morphism.*

Proof: Let $\phi : \mathbb{R}^m \longrightarrow \mathbb{R}^n$ be a quadratic harmonic morphism, then by definition we can write

$$\phi(X) = (X^tA_1X, \ldots, X^tA_nX) \tag{12}$$

where X denotes the column vector in $\mathbb{R}^m$, X^t the transpose of X and $A_i (i = 1, ..., n)$ is a symmetric matrix of $m \times m$. We can write the Jacobian matrix of $\phi(X)$ as

$$J(\phi(X)) = \begin{pmatrix} 2X^tA_1 \\ \vdots \\ 2X^tA_n \end{pmatrix}_{n\times m}$$

By definition, the complete lift of ϕ can be written as

$$\begin{aligned}\Phi(X,Y) &= \begin{pmatrix} 2X^tA_1 \\ \vdots \\ 2X^tA_n \end{pmatrix} Y \\ &= (2X^tA_1Y, \dots, 2X^tA_nY)\end{aligned}$$

Now the harmonicity of $\Phi(X,Y)$ follows from that of ϕ by Theorem 2.6. On the other hand, a routine calculation gives

$$\begin{aligned}J(\Phi(X,Y)) &= \left(\begin{array}{c|c} 2Y^tA_1 & 2X^tA_1 \\ \vdots & \vdots \\ 2Y^tA_n & 2X^tA_n \end{array}\right)_{n\times 2m} \\ &= (J(\phi(Y))|J(\phi(X)))_{n\times 2m}\end{aligned} \tag{13}$$

where the second equality holds because of the fact that $A_i (i = 1, \dots, n)$ is symmetric. From equation (13) one can easily see that $\Phi(X,Y)$ is horizontal weakly conformal if and only if ϕ possesses that property. Thus we have proved the Theorem.

Remark 3.4. a) A complete classification of quadratic harmonic morphisms between Euclidean spaces will appear in [17].
b) Though the complete lift of a quadratic harmonic morphism is always a quadratic harmonic morphism there exist quadratic harmonic morphisms which are not the complete lift of any map as the following example shows.

Example 3.5. The quaternion product is not the complete lift of any map. In fact, if there were a map

$$\phi(x) = (\phi^1(x), ..., \phi^4(x)) : \mathbb{R}^4 \longrightarrow \mathbb{R}^4$$

whose complete lift were the quaternion product, then $\phi^1(x), ..., \phi^4(x)$ would satisfy the following systems of PDEs :

$$\begin{array}{cccc}
\frac{\partial\phi^1}{\partial x_1} = x_1 & \frac{\partial\phi^1}{\partial x_2} = -x_2 & \frac{\partial\phi^1}{\partial x_3} = -x_3 & \frac{\partial\phi^1}{\partial x_4} = -x_4 \\
\frac{\partial\phi^2}{\partial x_1} = -x_2 & \frac{\partial\phi^2}{\partial x_2} = x_1 & \frac{\partial\phi^2}{\partial x_3} = -x_4 & \frac{\partial\phi^2}{\partial x_4} = x_3 \\
\frac{\partial\phi^3}{\partial x_1} = x_3 & \frac{\partial\phi^3}{\partial x_2} = x_4 & \frac{\partial\phi^3}{\partial x_3} = x_1 & \frac{\partial\phi}{\partial x_4} = -x_2 \\
\frac{\partial\phi^4}{\partial x_1} = x_3 & \frac{\partial\phi^4}{\partial x_2} = x_4 & \frac{\partial\phi^4}{\partial x_3} = x_2 & \frac{\partial\phi^4}{\partial x_4} = x_1
\end{array}$$

But this is impossible because, for instance, $\frac{\partial^2\phi^2}{\partial x_2\partial x_1} = -1 \neq \frac{\partial^2\phi^2}{\partial x_1\partial x_2} = 1$.

Example 3.6. Harmonic morphism $\Phi : \mathbb{R}^{16} \longrightarrow \mathbb{C}$ not arising from any Kähler structure. We have seen that $Q_r(x,y) : \mathbb{R}^{16} \longrightarrow \mathbb{C}^2$ is a harmonic morphism, and it is known that $\phi : \mathbb{C}^2 \longrightarrow \mathbb{C}, \phi(z,w) = zw$ is a harmonic morphism. We claim that $\Phi : \mathbb{R}^{16} \longrightarrow \mathbb{C}$ given by $\Phi = \phi \circ Q_r$ is a harmonic morphism not arising from any Kähler structure. To see this we need the following criterion

Proposition 3.7. *(cf.Baird and Wood [5]) Let $\Phi : \mathbb{R}^{2m} \supset U \longrightarrow \mathbb{C}$ be a submersive harmonic morphism. Then Φ is holomorphic with respect to a Kähler structure if and only if there exists an m-dimensional isotropic subspace $W \subset \mathbb{C}^{2m}$ such that $\nabla\Phi(x) \in W$ for any $x \in U$.*

We will prove that $\Phi = \phi \circ Q_r$ is not holomorphic with respect to any Kähler structure by showing that there exists no such isotropic subspace for Φ. It is enough to check that there exist 8 points $x_1, \dots, x_8 \in \mathbb{R}^{16}$ such that $\{\nabla\Phi(x_1), \dots, \nabla\Phi(x_8)\}$ is a linearly independent set of vectors in $\mathbb{C}^{16}$, and that there exists another point $x_9 \in \mathbb{R}^{16}$ such that $\nabla\Phi(x_9) \notin span\,\{\nabla\Phi(x_1), \dots, \nabla\Phi(x_8)\}$.
In fact,

$$\begin{aligned}&\Phi(z_1, z_2, z_3, z_4, w_1, w_2, w_3, w_4) =\\ &\quad(z_3w_1 - \overline{z}_4w_2 + z_1w_3 - z_2\overline{w}_4)(z_4w_1 + \overline{z}_3w_2 + z_2\overline{w}_3 + z_1w_4)\end{aligned}$$

and ,

$$\begin{aligned}&\nabla\Phi(x) =\\ &(w_3B + w_4A, i(w_3B + w_4A), -\overline{w}_4B + \overline{w}_3A, i(-\overline{w}_4B + \overline{w}_3A),\\ &\ w_1B + w_2A, i(w_1B - w_2A), -w_2B + w_1A, i(w_2B + w_1A),\\ &(z_3B + z_4A, i(z_3B + z_4A), -\overline{z}_4B + \overline{z}_3A, i(-\overline{z}_4B + \overline{z}_3A),\\ &z_1B + z_2A, i(z_1B - z_2A), -z_2B + z_1A, i(z_2B + z_1A))\end{aligned}$$

Where $A = z_3w_1 - \overline{z}_4w_2 + z_1w_3 - z_2\overline{w}_4$, $B = z_4w_1 + \overline{z}_3w_2 + z_2\overline{w}_3 + z_1w_4$. We calculate

$$\begin{aligned}\nabla\Phi(0,0,1,0,1,0,0,1) &= (1,i,0,0,0,0,1,i,0,0,1,i,0,0,0,0)\\ \nabla\Phi(0,0,i,0,1,0,0,1) &= (i,-1,0,0,0,0,i,-1,0,0,1,i,0,0,0,0)\\ \nabla\Phi(1,0,0,0,1,0,1,0) &= (0,0,1,i,0,0,1,i,0,0,0,0,0,0,1,i)\\ \nabla\Phi(i,0,0,0,1,0,1,0) &= (0,0,i,-1,0,0,i,-1,0,0,0,0,0,0,-1,-i)\\ \nabla\Phi(1,0,0,1,1,0,0,0) &= (0,0,0,0,1,i,0,0,0,0,-1,-i,1,i,0,0)\\ \nabla\Phi(1,0,0,1,i,0,0,0) &= (0,0,0,0,-1,-i,0,0,0,0,-i,1,i,-1,0,0)\\ \nabla\Phi(1,0,1,0,1,0,0,0) &= (0,0,0,0,0,0,1,i,0,0,1,i,0,0,1,i)\\ \nabla\Phi(1,0,1,0,i,0,0,0) &= (0,0,0,0,0,0,-1,i,0,0,i,-1,0,0,i,-1)\end{aligned}$$

It is easy to see that these eight vectors are mutually orthogonal in $\mathbb{C}^{16}$ with respect to $\langle z, w\rangle^C = z_1w_1 + \ldots + z_{16}w_{16}$. Hence they are linearly independent. It is easy to see that $\nabla\Phi(0,0,1-i,0,1,1,0,0) = (0,0,0,0,2,-2,-2i,2i,2,2i,2,2i,0,0,0,0)$ does not belong to the subspace spanned by the above eight vectors.

Remark 3.8. It is known that the orthogonal multiplications, as maps between Euclidean spaces, can be used to construct some harmonic maps and morphisms. However the complete lift of an orthogonal multiplication need not be an orthogonal multiplication in general as one can check this for the quaternion product.

Acknowledgments: This work was done while the author was visiting Department of Pure Mathematics, University of Leeds, U.K.. The author is grateful to the department for the hospitality and generosity, and to E. Loubeau and many colleagues there for their help and friendship. The author would like to thank S. Gudmundsson for his generosity in sending the author many useful references and for his comments which led to an improvement of the original manuscript. Last and most importantly, the author is greatly indebted to John C. Wood for making the author's visit to Leeds possible and for guiding the author in the study of harmonic maps and morphisms. During the preparation of this work the author benefitted much from his extremely useful comments and suggestions.

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Received: July 4, 1995; revised version: October 3, 1995