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Redundance of Vertices of the Cube relatively to its Minimal Ellipsoid

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Abstract. Each convex body K has a unique ellipsoid of minimal volume containing K (*Minimal ellipsoid, Loewner ellipsoid*). In this paper we look for subsets of the vertex set of the cube, which have the same minimal ellipsoid as the cube itself. All subsets with this property are determined in dimensions 2,3 and 4. For some higher dimensions (for example 4 to 423) we present the minimal cardinal number of such subsets.

MSC 1991: 52B12, 52A20, 90C34

1. Introduction

Each convex body in $\mathbb{R}^n$ has a uniquely determined minimal ellipsoid, cf. [3],[6]. Let K be a polytope given as the convex hull of its vertices. According to [9],[10],[11] there is a number r, $n+1 \leq r \leq \frac{n(n+3)}{2}$, of vertices of K whose convex hull has the same minimal ellipsoid as K. Therefore, one can remove some vertices of "most" polytopes without changing their minimal ellipsoid. In this paper we investigate the cube in $\mathbb{R}^n$ intending to find sets of vertices that have the same minimal ellipsoid as the cube. All such sets are presented in dimensions 2,3 and 4. Finally, we ask for the minimal number of vertices necessary to obtain this property. This question can be answered for a large class of dimensions because it is closely connected with the existence of Hadamard matrices. In theorem 1 we prove the following: If an Hadamard matrix exists for $n = 4(m+1)$, then one can figure out the minimal number for dimensions $n = 4m$ to $4m+3$. This includes the solution of the problem up to dimension 423.

2. General results

In the following let $E(X^{0^{-1}}, x^0) := \{x \in \mathbb{R}^n | (x - x^0)^T X^{0^{-1}} (x - x^0) \leq 1\}$ be an ellipsoid with centre x^0 where X^0 is a positive definite symmetric matrix. $B_n = \{x \in \mathbb{R}^n | \; \|x\| \leq 1\}$ denotes the unit ball, ∂B_n stands for the boundary of B_n and I_n is the n-dimensional unit matrix. Let $Q_n = \text{conv}\{q_1, ..., q_{2^n}\}$ be the n-dimensional cube with minimal ellipsoid B_n. In this connection $q_i^T = (q_{i1}, ..., q_{in})$, $i = 1, ..., 2^n$, are vectors with $q_{ik} \in \{1/\sqrt{n}, -1/\sqrt{n}\}$ for $k = 1, ..., n$. The following result, see [10],[11], will prove to be useful: An ellipsoid $E(X^{0^{-1}}, x^0)$ is the minimal ellipsoid of a convex body K if and only if $E(X^{0^{-1}}, x^0) \supseteq K$ and the conditions (1) to (5) are fulfilled.

There are vectors

$$u_1, ..., u_r \in \partial B_n \;\text{ with }\; n + 1 \leq r \leq \frac{n(n+3)}{2} \tag{1}$$

and scalars

$$\mu_0 > 0, \;\mu_1 > 0, ..., \mu_r > 0 \tag{2}$$

with

$$u_j^T x^0 + \sqrt{u_j{}^T X^0 u_j} \;=\; k_K(u_j) \;\text{ for }\; j = 1, ..., r \tag{3}$$

$$\sum_{j=1}^{r} \mu_j u_j \;=\; 0 \tag{4}$$

$$(2\mu_0 det X^0) I_n \;=\; \sum_{j=1}^{r} \frac{\mu_j X^0 u_j u_j{}^T}{\sqrt{u_j{}^T X^0 u_j}}, \tag{5}$$

where k_K is the Minkowski support function of K , cf. [4]. If K is the cube Q_n, then $E(X^{0^{-1}}, x^0) = B_n$ holds. This implies $X^0 = I_n$ and $x^0 = 0$. Then (3) and (5) change into

$$\sqrt{u_j{}^T u_j} = 1 = k_{Q_n}(u_j) \tag{3'}$$

and

$$2\mu_0 I_n = \sum_{j=1}^{r} \mu_j u_j u_j{}^T. \tag{5'}$$

Due to $(3')$ only the vertices of Q_n are possible candidates for $u_1, ..., u_r$. Each subset $\{u_1, ..., u_r\}$, $n + 1 \leq r \leq n(n + 3)/2$, of the vertex set $\{q_1, ..., q_{2^n}\}$, which fulfils (4) and $(5')$ with suitable $\mu_i > 0$, $i = 0, ..., r$, has the same minimal ellipsoid (B_n) as the cube Q_n, cf. [10], [11]. Which different (in particular minimal) subsets of this type exist in a given dimension? To answer this question we investigate (similar to [2]) the system

$$\begin{aligned} \lambda_i \;&\geq\; 0 \;\text{ for }\; i = 1, ..., 2^n \\ \sum_{i=1}^{2^n} \lambda_i p_i \;&=\; 0 \\ \sum_{i=1}^{2^n} \lambda_i p_i p_i^T \;&=\; \rho I_n, \end{aligned} \tag{6}$$

where $p_i := \sqrt{n} q_i = (p_{i1}, ..., p_{in})^T$. Therefore p_{ik} is either 1 or -1 for $k = 1, ..., n$ and $i = 1, ..., 2^n$. Let $\Lambda := (\lambda_1, ..., \lambda_{2^n})^T$ be a solution of this system and

$$J(\Lambda) := \{i \mid \lambda_i > 0\} = \{i_1, ..., i_k\} \tag{7}$$

the index set of its positive components. Each solution $\Lambda \neq 0$ with $k = |J(\Lambda)| \leq \frac{n(n+3)}{2}$ corresponds to a solution of (1),(2),(3′),(4),(5′). Just choose $r := k$, $(u_1, .., u_r) := (\frac{1}{\sqrt{n}} p_{i_1}, .., \frac{1}{\sqrt{n}} p_{i_k})$, $(\mu_1, .., \mu_r) := (\lambda_{i_1}, .., \lambda_{i_k})$ and $\mu_0 = \rho/2n$ ($k \geq n+1$ holds automatically, cf. [10],[11]). This vertex set $\{q_{i_1}, ..., q_{i_k}\}$ of Q_n has the same minimal ellipsoid B_n as the initial cube Q_n. Let $L_j = p_j p_j^T$ be the matrix with components $l^j_{ik} := p_{ji} p_{jk}$. It will turn out to be helpful to define $L^j := (l^j_{21}, l^j_{31}, l^j_{32}, l^j_{41}, ..., l^j_{n1}, l^j_{n2}, ..., l^j_{nn-1}) = (L^j_1, L^j_2, ..., L^j_N) \in \mathbb{R}^N$, where $N := (n^2 - n)/2$. The matrices $p_j {p_j}^T$ are symmetric and the elements of the diagonal all equal 1, implying $\rho = \sum_{i=1}^{2^n} \lambda_i$. Thus it is sufficient to consider the following homogeneous linear system instead of (6):

$$\begin{pmatrix} p_1 & p_2 & \dots & p_{2^n} \\ L^1 & L^2 & \dots & L^{2^n} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_{2^n} \end{pmatrix} = 0, \quad \lambda_j \geq 0. \tag{8}$$

For a given solution $\Lambda \neq 0$ the set $M_n(\Lambda) := \{p_j \mid j \in J(\Lambda)\} = \{p_{i_1}, ..., p_{i_k}\}$ is called a *corresponding solution set* and $\lambda := (\lambda_{i_1}, ..., \lambda_{i_k})^T$ is said to be the *corresponding solution vector*. The matrix whose columns correspond to the elements of the solution set is called a *solution matrix.*

For $n = 2$ this leads to the system

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \lambda_j \geq 0, \quad \lambda \neq 0,$$

with the unique solution $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 > 0$. Thus it is impossible to remove one vertex of the cube without getting a smaller minimal ellipsoid.

For $n = 3$ one obtains

$$\begin{pmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_8 \end{pmatrix} = 0, \quad \lambda_j \geq 0, \quad \lambda \neq 0.$$

The vectors $(0, 1, 1, 0, 1, 0, 0, 1)$ and $(1, 0, 0, 1, 0, 1, 1, 0)$ form a basis of the solution space of the previous system. Hence the ball turns out to be the minimal ellipsoid of the two vertex disjoint regular simplices $\text{conv}((1, 1, -1)^T, (1, -1, 1)^T, (-1, 1, 1)^T, (-1, -1, -1)^T)$, cf. Fig. 1, and $\text{conv}((1, 1, 1)^T, (1, -1, -1)^T, (-1, 1, -1)^T, (-1, -1, 1)^T)$. These inscribed simplices are known since the days of Euclid, see [7]. Furthermore, each solution set in $\mathbb{R}^3$ contains the vertices of one of these simplices.

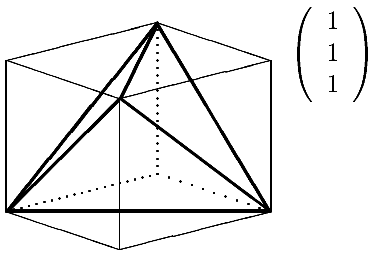

Fig. 1

One easily gets

Lemma 1: *Let $A = \{p_{i_1}, ..., p_{i_k}\}$ be a solution set in $\mathbb{R}^n$. One obtains a solution set in $\mathbb{R}^{n-1}$ by removing the l-th component, $l \in \{1, ..., n\}$, of the vectors p_{i_j}, $j = 1, ..., k$. This set consists of at most k elements.*

Proof: Let $(\lambda_{i_1}, ..., \lambda_{i_k})^T$ be the corresponding solution vector of A. Then we have $\sum_{j=1}^{k} \lambda_{i_j} p_{i_j} = 0$ and $\sum_{j=1}^{k} \lambda_{i_j} p_{i_j} p_{i_j}^T = \rho I_n$ holds. Removing the l-th component , $l \in \{1, ..., n\}$, of the vectors p_{i_j}, we obtain vectors $\overline{p}_{i_j}$, which satisfy $\sum_{j=1}^{k} \lambda_{i_j} \overline{p}_{i_j} = 0$ and $\sum_{j=1}^{k} \lambda_{i_j} \overline{p}_{i_j} \overline{p}_{i_j}^{\,T} = \rho I_{n-1}$. Note that two vectors $\overline{p}_{i_j}$ and $\overline{p}_{i_r}$ with $j \neq r$ may be equal and recall that the matrix $\overline{p}_{i_j} \overline{p}_{i_j}^{\,T}$ results from $p_{i_j} p_{i_j}^T$ by removing the l-th row and the l-th column. In this way we obtain a solution set in $\mathbb{R}^{n-1}$ with at most k elements.

Lemma 2 : *A subset of the vertex set of the $4m$-dimensional cube, $m \geq 1$, which has the same minimal ellipsoid as the cube contains at least $4(m+1)$ elements.*

Proof: Let $A = (p_{i_1} ... p_{i_s})$ be a solution matrix in $\mathbb{R}^{4m}$. We can assume $\sum_{j=1}^{s} \lambda_{i_j} = 4(m+1)$ due to the homogeneity of the solution vector corresponding to A (cf. (8)). Associate an index set S_1, S_2, S_3, S_4, to each constellation $\binom{1}{1}, \binom{1}{-1}, \binom{-1}{1}$ and $\binom{-1}{-1}$ so that $i_j \in S_1$ iff $p_{i_j} = (1, 1, ...)^T$, $i_j \in S_2$ iff $p_{i_j} = (1, -1, ...)^T$, etc. Actually all these constellations do occur in the first two components of the vectors p_{i_j}. Otherwise it would be possible to construct a solution set in $\mathbb{R}^2$ which consists of less than four vertices applying the previous lemma $n-2$ times. By the same argument and the uniqueness (except scalar multiples) of the solution vector in $\mathbb{R}^2$ we get $\sum_{j \in S_1} \lambda_{i_j} = \sum_{j \in S_2} \lambda_{i_j} = \sum_{j \in S_3} \lambda_{i_j} = \sum_{j \in S_4} \lambda_{i_j} = m+1$. Suppose s is smaller than $4(m+1)$. Consequently, at least one constellation of $\binom{1}{1}, \binom{1}{-1}, \binom{-1}{1}$ and $\binom{-1}{-1}$ occurs at most m times in the first two components of the vectors p_{i_j}, or in other words at least one index set of S_1, S_2, S_3, S_4 has a cardinal number less than $m+1$. Let this be S_1. Thus there exists a λ_{i^*}, $i^* \in S_1$, with $\lambda_{i^*} \geq (m+1)/m$; without loss of generality we can assume $\lambda_{i^*} = \lambda_{i_1}$ with $p_{i_1} = (1, 1, ..., 1)^T$. Now look at $p_{i_j} p_{i_j}^T$: Let l be the number of components of p_{i_j} which equal $+1$ and k the number of components of p_{i_j} which equal -1. The matrix $p_{i_j} p_{i_j}^T$ consists of the elements $+1$ (exactly $l^2 + k^2$ times) and -1 (exactly $2lk$ times). Therefore, the sum of all n^2 elements of $p_{i_j} p_{i_j}^T$ turns out to be $l^2 + k^2 - 2lk = (l-k)^2$. Moreover we have

$$\sum_{\nu=1}^{N} L_{\nu}^{i_j} = \frac{(l-k)^2}{2} - \frac{n}{2} \geq -\frac{n}{2} = -2m, \tag{9}$$

where equality holds iff $l = k$.

In the case $j = 1$ this turns to $\sum_{\nu=1}^{N} L_\nu^{i_1} = N = 2m(4m-1) = 8m^2 - 2m$. Thus we obtain

$$\sum_{j=2}^{s} \lambda_{i_j} = 4(m+1) - \lambda_{i_1} \leq 4m + 4 - \frac{m+1}{m} = \frac{4m^2+3m-1}{m}.$$

Since $\sum_{j=1}^{s} \lambda_{i_j} L^{i_j}$ equals 0, we also have $\sum_{j=1}^{s} \lambda_{i_j} \sum_{\nu=1}^{N} L_\nu^{i_j} = 0$. Using this one can show that

$$\begin{aligned} 0 = \sum_{j=1}^{s} \lambda_{i_j} \sum_{\nu=1}^{N} L_\nu^{i_j} &\geq \left(\frac{m+1}{m}\right) N + \sum_{j=2}^{s} \lambda_{i_j} \sum_{\nu=1}^{N} L_\nu^{i_j} \\ &\geq 8m^2 + 6m - 2 + \sum_{j=2}^{s} \lambda_{i_j}(-2m) \\ &\geq 8m^2 + 6m - 2 - 2m\frac{(4m^2+3m-1)}{m} = 0. \end{aligned}$$

Hence the equality sign holds in all these inequalities, which requires $\sum_{\nu=1}^{N} L_\nu^{i_j} = -2m$ for $j = 2, ..., s$. According to (9) this is only possible if the vectors p_{i_j}, $j = 2, .., s$, contain +1 and -1 both exactly $2m$ times. From this it follows that $\sum_{k=1}^{n} \lambda_{i_j} p_{i_j k} = 0$ holds for $j = 2, ..., s$. As a result we obtain $\sum_{j=1}^{s} \lambda_{i_j} \sum_{k=1}^{n} p_{i_j k} = n\lambda_{i_1}$, in contradiction to $\sum_{j=1}^{s} \lambda_{i_j} p_{i_j} = 0$. Thus $s \geq 4(m+1)$ is proved, q.e.d.

Our problem is closely connected with the existence of Hadamard matrices. (An Hadamard matrix H is an $n \times n$-matrix (h_{ik}) with $h_{ik} \in \{1, -1\}$ and $H^T H = nI_n$.) It is known that an Hadamard matrix can only exist for $n = 1, 2$ and $n \equiv 0 \bmod 4$. The problem whether such a matrix exists for all $n \equiv 0 \bmod 4$, first considered by Sylvester [14], remains unsolved to this day. Summaries about the results concerning this topic are given in [1],[12],[13]. 428 is the smallest number for which no Hadamard matrix has been found until now, cf. [12].

The proof of Theorem 1 makes use of the following lemma, cf. for example [8]:

Lemma 3: *An $(n-1)$-dimensional regular simplex can be inscribed in an $(n-1)$-dimensional cube such that each vertex of the simplex coincides with a vertex of the cube if and only if an Hadamard matrix of order n exists.*

Summarizing lemma 1 and lemma 3 we obtain the following statement. If an Hadamard matrix of order n exists, then we can construct solution sets of at most n elements for all dimensions smaller than n. On the other hand, such sets consist of at least n elements in dimension $n-4$, $n \geq 8$ ($n \equiv 0 \bmod 4$), as proved in lemma 2. This yields

Theorem 1: *If an Hadamard matrix of order n, $n \geq 8$, exists, then the minimal number of vertices of the cube which have the same minimal ellipsoid as the cube itself amounts to n in the dimensions $n-4$, $n-3$, $n-2$ and $n-1$.*

Thus we get the following table:

dim	1	2	3	4	5	6	7	8	...	11	12	...	15	.....	420	...	423
min	2	4	4	8	8	8	8	12	...	12	16	...	16	.....	424	...	424

,

where dim denotes the dimension of the cube and min stands for the minimal number of vertices whose minimal ellipsoid is the ball, too.

3. Dimension $n = 4$

The determination of all solution sets in dimension 4 becomes more difficult than in dimensions 2 and 3. In this case ($n = 4$), the following six vectors, $v_1, ..., v_6$, form a basis of the solution space of (8):

$$
\begin{array}{l}
v_1^T = (\ -1,\ \ 1,\ \ 1,\ \ 0,\ \ 1,\ \ 0,\ \ 0,\ 0,\ \ 1,\ \ 0,\ \ 0,\ 0,\ 0,\ 0,\ 0,\ 1) \\
v_2^T = (\ \ 1,\ -1,\ \ 0,\ \ 1,\ \ 0,\ \ 1,\ \ 0,\ 0,\ \ 0,\ \ 1,\ \ 0,\ 0,\ 0,\ 0,\ 1,\ 0) \\
v_3^T = (\ \ 1,\ \ 0,\ -1,\ \ 1,\ \ 0,\ \ 0,\ \ 1,\ 0,\ \ 0,\ \ 0,\ \ 1,\ 0,\ 0,\ 1,\ 0,\ 0) \\
v_4^T = (\ -1,\ \ 1,\ \ 1,\ -1,\ \ 0,\ \ 0,\ \ 0,\ 0,\ \ 1,\ -1,\ -1,\ 1,\ 0,\ 0,\ 0,\ 0) \\
v_5^T = (\ -1,\ \ 1,\ \ 1,\ -1,\ \ 1,\ -1,\ -1,\ 1,\ \ 0,\ \ 0,\ \ 0,\ 0,\ 0,\ 0,\ 0,\ 0) \\
v_6^T = (\ \ 2,\ -1,\ -1,\ \ 1,\ -1,\ \ 1,\ \ 1,\ 0,\ -1,\ \ 1,\ \ 1,\ 0,\ 1,\ 0,\ 0,\ 0)
\end{array}
$$

The order of the vectors $p_1, ..., p_{16}$ is chosen in a canonical way. The vector p_j^T, regarded as a binary number, corresponds to the natural number $j-1$. In this connection the numbers 1 and -1 replace the dual digits 0 and 1, respectively, i.e. $p_1^T = (1, 1, 1, 1)$, $p_2^T = (1, 1, 1, -1)$, $p_3^T = (1, 1, -1, 1), ..., p_{16}^T = (-1, -1, -1, -1)$. We have to consider all linear combinations $\Lambda := \Lambda(\mu) = \sum_{i=1}^{6} \mu_i v_i$, $\mu^T := (\mu_1, ..., \mu_6)$, of these basis vectors to get all possible solution sets in $\mathbb{R}^4$. Additionally, no component of the linear combination Λ is allowed to become negative, cf. (8). This implies $\mu_i \geq 0$, $i = 1, ..., 6$, because each vector v_i, $i = 1, ..., 6$, contains one positive component $(+1)$ at such a place where all other vectors v_j have components equal to 0. Since we actually look for the solution sets, only the index sets $J(\Lambda)$ (cf. (7)) of positive components of Λ are of interest. For example, consider all linear combinations with $\mu_1, \mu_2 > 0$ and $\mu_3 = ... = \mu_6 = 0$. Since $v_{11} = -1$ and $v_{21} = 1$, we have $\mu_2 \geq \mu_1$. On the other hand, $\mu_1 \geq \mu_2$ results from $v_{12} = 1$ and $v_{22} = -1$, which leads to $\mu_1 = \mu_2$. Hence $\Lambda((1, 1, 0, 0, 0, 0)^T) = (0, 0, 1, 1, 1, 1, 0, 0, 1, 1, 0, 0, 0, 0, 1, 1)^T$ holds and $\{p_3, p_4, p_5, p_6, p_9, p_{10}, p_{15}, p_{16}\}$ is a solution set, denoted by S_1.

If we consider all possible linear combinations this way, then we get the minimal solution sets $S_1, ..., S_{26}$. In this context minimal solution set means that no proper subset of S_i, $i = 1, ..., 26$, is a solution set itself.

	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}	p_{12}	p_{13}	p_{14}	p_{15}	p_{16}
S_1			+	+	+	+			+	+					+	+
S_2		+		+	+		+		+		+			+		+
S_3	+			+		+	+			+	+		+			+
S_4		+	+			+	+		+			+	+			+
S_5		+	+		+			+		+	+		+			+
S_6	+			+		+	+		+			+		+	+	
S_7	+			+	+			+		+	+			+	+	
S_8		+	+		+			+	+			+		+	+	
S_9	+		+			+		+		+		+	+		+	
S_{10}	+	+					+	+			+	+	+	+		
S_{11}		+	+	+	+	+	+		+	+	+		+			+
S_{12}	+			+	+	+	+		+	+	+			+	+	+
S_{13}		+	+	+	+	+	+		+			+		+	+	+
S_{14}		+	+	+	+			+	+	+	+			+	+	+
S_{15}	+		+	+		+	+		+	+		+	+		+	+
S_{16}	+		+	+	+	+		+		+	+		+		+	+
S_{17}	+	+		+		+	+		+		+	+	+	+		+
S_{18}	+	+		+	+		+	+		+	+		+	+		+
S_{19}	+	+	+			+	+	+		+	+	+	+			+
S_{20}	+		+	+	+	+		+	+	+		+		+	+	
S_{21}	+	+		+	+		+	+	+		+	+		+	+	
S_{22}		+	+		+	+		+	+	+		+	+		+	+
S_{23}		+	+		+		+	+	+		+	+	+	+		+
S_{24}	+			+		+	+	+		+	+	+	+	+	+	
S_{25}	+	+	+			+	+	+	+			+	+	+	+	
S_{26}	+	+	+		+			+		+	+	+	+	+	+	

The sets S_1 to S_{10} consist of eight elements whereas the cardinal number of S_{11} to S_{26} equals eleven. These (minimal) sets can be divided into three classes.
(We call two vertices adjacent or neighbours if they are connected by an edge of the cube.)

1. The set contains no two adjacent vertices. This leads to the two vertex disjoint regular cross-polytopes, see [5], corresponding to S_3 and S_8, cf. Fig. 2.

2. Each vertex p_i is related to exactly one other vertex p_j in the sense that p_j and p_i are neighbours ($S_1, S_2, S_4, S_5, S_6, S_7, S_9$ and S_{10}), cf. Fig. 3.

3. Let p be any vertex of the cube. Remove $-p$ and the vertices adjacent to p (S_{11} to S_{26}), cf. Fig. 4.

All other solution sets contain one of these minimal sets $S_1, ..., S_{26}$.

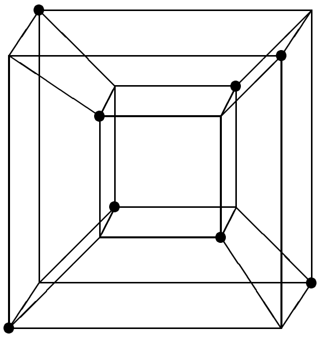

Fig. 2

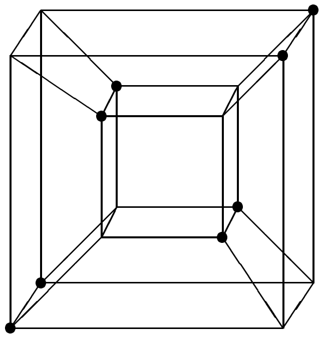

Fig. 3

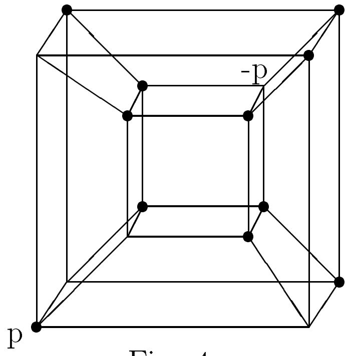

Fig. 4

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