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The Divisor of the Resultant

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Abstract. Let A be an integrally closed noetherian domain, $A[T_0, \ldots, T_n]$ a graded polynomial algebra with indeterminates having arbitrary positive (integral) weights and $F_0, \ldots, F_n \in A[T] := A[T_0, \ldots, T_n]$ homogeneous polynomials of positive degrees, which form a regular sequence. The properties of the subvariety $\operatorname{Proj} C$, $C := A[T]/(F_0, \ldots, F_n)$, of the weighted projective space $\operatorname{Proj} A[T]$ and its canonical image in $\operatorname{Spec} A$ are best described in terms of the *resultant ideal* $\mathfrak{R}$ which is a principal ideal and can be defined either by a suitable Fitting ideal or a cancelling rule for determinants. By means of duality available for regular sequences the following is proved: The divisor of $\mathfrak{R}$ equals the divisor defined by the algebra of global sections of $\operatorname{Proj} C$ (which is an A–torsion–module). Some applications on the degree of elimination and on the fibres of the elimination mapping are given.

Introduction

In this paper we continue the study [8] of resultants and their use in elimination theory.

The situation will always be the following: The base ring A is an integrally closed noetherian domain. $A[T] = A[T_0, \ldots, T_n]$ is a graded polynomial algebra where the indeterminates T_i have arbitrary positive weights $\gamma_i \in \mathbf{N}^*$. We consider a regular sequence $F_0, \ldots, F_n$ of homogeneous polynomials in $A[T]$ of positive degrees $\delta_0, \ldots, \delta_n$ and the closed subvariety $\operatorname{Proj} C$, $C := A[T]/(F_0, \ldots, F_n)$, they define in the (weighted) projective space $\operatorname{Proj} A[T]$. The (homogeneous) ideal $\mathfrak{T}$ of the Trägheitsformen in C describes the same variety, i.e. $\operatorname{Proj} C = \operatorname{Proj} \bar{C}$ with $\bar{C} := C/\mathfrak{T}$, and the zero–set of the elimination ideal $\mathfrak{T}_0 = \mathfrak{T} \cap A \subseteq A$ is the image of $\operatorname{Proj} C$ in $\operatorname{Spec} A$.

Far better than the elimination ideal is the resultant ideal $\mathfrak{R} \subseteq A$ which is not only a principal ideal with the same zero–set as $\mathfrak{T}_0$ but also has suitable functorial properties.

Furthermore, its divisor $\operatorname{div}\mathfrak{R} = \chi(\bar{C}_\sigma)$, $\sigma := (\delta_0 + \cdots + \delta_n) - (\gamma_0 + \cdots + \gamma_n)$, describes precisely the degree of $\operatorname{Proj} C$, which means

$$\operatorname{div}\mathfrak{R} = \chi(\Gamma(\operatorname{Proj} C,\ \mathcal{O}_{\operatorname{Proj} C}))$$

cf. Theorem 2.1. [1]) For the proof we use Lemma 3.1 from duality theory which, in addition, provides means to compute $\operatorname{div}\mathfrak{R}$ in a way that generalizes long–known techniques used by Sylvester, Hurwitz, Macaulay and others, cf. Theorem 2.2.

In Section 1 we introduce the resultant ideal $\mathfrak{R}$ and describe its relations to the elimination ideal $\mathfrak{T}_0$. The main results and some applications concerning the degree of elimination are given in Section 2. Section 3 supplies the necessary tools from duality theory.

1. The resultant ideal

Let A be an integrally closed noetherian domain. We consider a regular sequence $F_0, \ldots, F_n$ of homogeneous polynomials of positive degrees $\delta_0, \ldots, \delta_n$ in the graded polynomial algebra

$$A[T] = A[T_0, \ldots, T_n]\,, \quad \deg T_i = \gamma_i \in \mathbf{N}^*,\ i = 0, \ldots, n\,,$$

and the graded A–algebra

$$C = A[T]/(F_0, \ldots, F_n)\,.$$

The zero–ideal in C has the decomposition

$$0 = \mathfrak{Q} \cap \mathfrak{T}$$

where $\mathfrak{Q}$ is the irrelevant primary component of 0 and $\mathfrak{T}$ the ideal of the Trägheitsformen. $\mathfrak{Q}$ is nothing else but the A–torsion of C, furthermore $\mathfrak{T} = 0 : \mathfrak{Q}$ and $\mathfrak{Q} = 0 : \mathfrak{T}$. The graded residue class algebra

$$\bar{C} := C/\mathfrak{T}$$

has the same projective spectrum as C. The homogeneous parts of degree m in $A[T]$, C, $\mathfrak{Q}$, $\mathfrak{T}$, $\bar{C}$, ... will be indicated by attaching m as an index. Of special importance is the degree

$$\sigma := (\delta_0 + \cdots + \delta_n) - (\gamma_0 + \cdots + \gamma_n)\,.$$

The so–called *elimination ideal*

$$\mathfrak{T}_0 = \mathfrak{T}_0(F_0, \ldots, F_n) \subseteq A$$

of $F_0, \ldots, F_n$ describes the (closed) image of $\operatorname{Proj} C = \operatorname{Proj} \bar{C}$ in $\operatorname{Spec} A$.

In every degree m, there is a canonical isomorphism

$$\mathfrak{T}_m \xrightarrow{\sim} C^*_{\sigma-m} = \operatorname{Hom}_A(C_{\sigma-m},\ A)$$

[1]) $\chi(M)$ denotes the divisor associated with a finite A–torsion–module M.

given by $t_m \mapsto (c_{\sigma-m} \mapsto t_m c_{\sigma-m}/\Delta)$ where $t_m \in \mathfrak{T}_m$, $c_{\sigma-m} \in C_{\sigma-m}$ and $\Delta \in C_\sigma$ denotes the residue class in C of the determinant $\det(A_{ij})$, the polynomials $A_{ij} \in A[T]$ given by any representations $F_j = \sum_{i=0}^n A_{ij}T_i$, $j = 0, \ldots, n$. The element Δ is an A–base of $\mathfrak{T}_\sigma$, and $\bar{C}_\sigma = C_\sigma/\mathfrak{T}_\sigma$ is a finite A–torsion–module. $\mathfrak{T}_0 \cong C_\sigma^*$ is a reflexive A–module of rank 1 and therefore a divisorial ideal in A, moreover $\mathfrak{T}_0 = \operatorname{Ann}_A \bar{C}_\sigma$.

For details of these constructions and for proofs we refer to [8] or [9]. Furthermore we shall need the following result, cf. Lemma 4.1 in [8] or Lemma 8.2 in [9]:

1.1 $\operatorname{div} \mathfrak{f}_0(\bar{C}_\sigma) = \chi(\bar{C}_\sigma) = \chi(\mathfrak{Q}_\sigma) + \chi(A/\mathfrak{T}_0) = \operatorname{div} \mathfrak{f}_1(C_\sigma) + \operatorname{div} \mathfrak{T}_0.$

In these formulas $\mathfrak{f}_i(M)$ denotes the i–th Fitting ideal of a finite A–module M, $\operatorname{div} \mathfrak{a} = \operatorname{div} \mathfrak{a}^{**}$ the divisor defined by an ideal $\mathfrak{a} \neq 0$ in A and

$$\chi(M) = \operatorname{div} \mathfrak{f}_0(M) = \sum_{\operatorname{codim} \mathfrak{p}=1} (\operatorname{length} M_\mathfrak{p}) \cdot \mathfrak{p}$$

the divisor associated with a finite A–torsion–module M. Obviously, $\chi(A/\mathfrak{a}) = \operatorname{div} \mathfrak{a}$ for every ideal $\mathfrak{a} \neq 0$.

Because $F_0, \ldots, F_n$ is a regular sequence, C_σ and therefore $\bar{C}_\sigma$, too, has a finite free A–resolution (e.g. given by the Koszul complex). The mere existence of such a resolution implies that $\chi(\bar{C}_\sigma)$ is a principal divisor, cf. [1], §4, sect.7. The principal ideal in A corresponding to this principal divisor is by definition the *resultant ideal*

$$\mathfrak{R} = \mathfrak{R}(F_0, \ldots, F_n) = \mathfrak{R}_A(F_0, \ldots, F_n)$$

of the polynomials $F_0, \ldots, F_n \in A[T]$. Because of $\mathfrak{T}_0 = \operatorname{Ann}_A \bar{C}_\sigma$ and $\operatorname{div} \mathfrak{R} = \operatorname{div} \mathfrak{f}_0(\bar{C}_\sigma)$, the ideals $\mathfrak{T}_0$ and $\mathfrak{R}$ describe the same variety in $\operatorname{Spec} A$. By 1.1 the divisorial ideals $\mathfrak{T}_0$ and $\mathfrak{R}$ coincide if and only if the divisor $\chi(\mathfrak{Q}_\sigma)$ of the torsion submodule $\mathfrak{Q}_\sigma$ of C_σ vanishes, i.e. the support of $\mathfrak{Q}_\sigma$ has codimension ≥ 2 in $\operatorname{Spec} A$.

The resultant ideal $\mathfrak{R}$ has better functorial properties than the elimination ideal $\mathfrak{T}_0$; in addition, there is a canonical generator for $\mathfrak{R}$ (uniquely determined up to sign), the so–called *resultant*

$$R = \mathrm{R}(F_0, \ldots, F_n) = \mathrm{R}_A(F_0, \ldots, F_n)$$

of $F_0, \ldots, F_n$, which is compatible with base change $A \to A'$ (as long as A' is an integrally closed domain and $F_0, \ldots, F_n$ form a regular sequence in $A'[T]$). The divisorial ideal $\mathfrak{T}_0$, which is completely determined by its divisor

$$\operatorname{div} \mathfrak{T}_0 = \operatorname{div} \mathfrak{R} - \operatorname{div} \mathfrak{f}_1(C_\sigma),$$

does not commute with base changes, even though $\mathfrak{R}$ (and $\operatorname{div} \mathfrak{R}$) and $\mathfrak{f}_1(C_\sigma)$ do commute. In addition, $\mathfrak{T}_0$ need not be a principal ideal.

To give *examples*, we start from the generic polynomials

$$\mathsf{F}_0 := XT_0^3 + YT_1^2\,, \quad \mathsf{F}_1 := UT_0^6 + VT_0^3T_1^2 + WT_1^4$$

over $\mathsf{A} := \mathbf{Z}[X, Y, U, V, W]$, where $\deg T_0 := 2$, $\deg T_1 := 3$. Then $\sigma = (6+12)-(2+3) = 13$ and

$$\mathfrak{f}_0(\bar{\mathsf{C}}_\sigma) = (X(XW - YV) + Y^2U)\,, \quad \mathfrak{f}_1(\mathsf{C}_\sigma) = (X, Y)\,.$$

Therefore $\mathsf{R} = \mathrm{R}\,(\mathsf{F}_0, \mathsf{F}_1) = X(XW - YV) + Y^2U$ and $\mathfrak{R}(\mathsf{F}_0, \mathsf{F}_1) = \mathfrak{T}_0(\mathsf{F}_0, \mathsf{F}_1)$. Note that R is a prime element and $\operatorname{div} \mathfrak{f}_1(\mathsf{C}_\sigma) = 0$.

Now we specialize. First we take $A := k[x, y, z] = k[X, Y, Z]/(Y^2 - XZ)$, k any field, with $X \mapsto x$, $Y \mapsto y$, $U \mapsto 1$, $V \mapsto y$, $W \mapsto z$. Then the polynomials

$$F_0 = xT_0^3 + yT_1^2, \quad F_1 = T_0^6 + yT_0^3T_1^2 + zT_1^4$$

form a regular sequence in $A[T]$ and

$$\mathfrak{f}_0(\bar{C}_\sigma) = (y^2), \quad \mathfrak{f}_1(C_\sigma) = (x, y)\,.$$

Therefore $R = \mathrm{R}\,(F_0, F_1) = y^2$ and $\mathfrak{R} = \mathfrak{R}(F_0, F_1) = (y^2)$, $\mathfrak{T}_0 = \mathfrak{T}_0(F_0, F_1) = \mathfrak{p} \cap (z) = \mathfrak{p}z = (xz,\, yz)$ because of $\operatorname{div} \mathfrak{T}_0 = \operatorname{div} \mathfrak{R} - \operatorname{div} \mathfrak{f}_1(C_\sigma) = 2\mathfrak{p} + 2\mathfrak{q} - \mathfrak{p} = \mathfrak{p} + 2\mathfrak{q} = \mathfrak{p} + \operatorname{div} z$ with $\mathfrak{p} := (x, y)$, $\mathfrak{q} := (z, y)$. $\operatorname{div} \mathfrak{T}_0$ is not a principal divisor because the ideal $\mathfrak{p}$ is not principal.

Secondly we specialize to $A := \mathbf{Z}[\sqrt{-5}]$, $X \mapsto 2$, $Y \mapsto 1 + \sqrt{-5}$, $U \mapsto 1$, $V \mapsto 1 - \sqrt{-5}$, $W \mapsto 3$. Then again the polynomials

$$F_0 = 2T_0^3 + (1 + \sqrt{-5})T_1^2, \quad F_1 = T_0^6 + (1 - \sqrt{-5})T_0^3T_1^2 + 3T_1^4$$

form a regular sequence in $A[T]$ and

$$\mathfrak{f}_0(\bar{C}_\sigma) = (2(-2 + \sqrt{-5})), \quad \mathfrak{f}_1(C_\sigma) = (2, 1 + \sqrt{-5}\,)\,.$$

Therefore $R = 2(-2 + \sqrt{-5})$, $\mathfrak{R} = (2(-2 + \sqrt{-5}))$, $\mathfrak{T}_0 = \mathfrak{p} \cdot (-2 + \sqrt{-5})$ because of $\operatorname{div} \mathfrak{T}_0 = \operatorname{div} \mathfrak{R} - \operatorname{div} \mathfrak{f}_1(C_\sigma) = 2\mathfrak{p} + 2\mathfrak{q} - \mathfrak{p} = \mathfrak{p} + 2\mathfrak{q} = \mathfrak{p} + \operatorname{div}(-2 + \sqrt{-5})$ with $\mathfrak{p} := (2, 1 + \sqrt{-5}\,)$, $\mathfrak{q} := (3, 1 + \sqrt{-5}\,)$. Again $\operatorname{div} \mathfrak{T}_0$ is not a principal divisor because the ideal $\mathfrak{p}$ is not principal.

Finally, we specialize to $A := k[x, y, v, w] = k[X, Y, V, W]/(XW - YV)$, k any field (or any integrally closed domain), with $X \mapsto x$, $Y \mapsto y$, $U \mapsto 1$, $V \mapsto v$, $W \mapsto w$. In this case $\operatorname{div} \mathfrak{R} = 2 \operatorname{div} y = 2\mathfrak{p} + 2\mathfrak{q}$, $\operatorname{div} \mathfrak{f}_1(C_\sigma) = \mathfrak{p}$ and $\operatorname{div} \mathfrak{T}_0 = \mathfrak{p} + 2\mathfrak{q}$, i.e. $\mathfrak{T}_0 = \mathfrak{q}y$, where $\mathfrak{p} := (x, y)$, $\mathfrak{q} := (w, y)$. $\mathfrak{q}$ and therefore $\mathfrak{T}_0$ is not a principal ideal. Its divisor class has infinite order in the class group $\mathrm{Cl}(A)$.

In the next section we will put forward some properties of the resultant ideal $\mathfrak{R}(F_0, \ldots, F_n)$ which are related to the projective variety $\operatorname{Proj} A[T]/(F_0, \ldots, F_n)$ described by the polynomials $F_0, \ldots, F_n$.

2. Geometric interpretation

As in Section 1 let A denote an integrally closed noetherian domain and $F_0, \ldots, F_n$ a regular sequence of homogeneous polynomials in the graded polynomial algebra $A[T] =$

$A[T_0, \dots, T_n]$ with weighted indeterminates $T_0, \dots, T_n$. Without further notice we take over all the corresponding notations and constructions from Section 1.

The global sections of the projective spectrum of $C = A[T]/(F_0, \dots, F_n)$ form a finite A–algebra

$$B := \Gamma(\operatorname{Proj} C \,, \mathcal{O}_{\operatorname{Proj} C})$$

which is annihilated by the elimination ideal $\mathfrak{T}_0 = \mathfrak{T}_0(F_0, \dots, F_n)$ and thus even an algebra over

$$\bar{A} := A/\mathfrak{T}_0 = \bar{C}_0 \,.$$

The main result is the following theorem:

2.1 Theorem *The resultant $\mathfrak{R} = \mathfrak{R}(F_0, \dots, F_n)$ and the A–algebra B of global sections of $\mathcal{O}_{\operatorname{Proj} C}$ define the same divisor in* $\operatorname{Div} A$:

$$\chi(B) = \operatorname{div} \mathfrak{R} \,.$$

Explicitly: *For every prime ideal $\mathfrak{p}$ of codimension 1 in A*

$$\operatorname{length} B_{\mathfrak{p}} = \operatorname{length} A_{\mathfrak{p}}/\mathfrak{R}_{\mathfrak{p}} = \operatorname{length} (\,\bar{C}_\sigma)_{\mathfrak{p}} \,.$$

Proof. Let $\mathfrak{p}$ be a prime ideal of codimension 1 in A. After localizing by $\mathfrak{p}$ we may assume that A is a discrete valuation ring with maximal ideal $\mathfrak{p}$. In the case $\bar{C} = 0$, i.e. $\mathfrak{T} = C$, we have $\operatorname{Proj} C = \emptyset$ and $\mathfrak{R} = A$.

Now we assume $\bar{C} \neq 0$. Then $\bar{C}$ is (like C) a graded Macaulay ring of dimension 1 and B is the homogeneous part of degree zero of the total graded quotient ring of $\bar{C} = C/\mathfrak{T}$. By N' we denote the non–zero submonoid of $\mathbf{N}$ of all r such that $\bar{C}_r$ contains a non–zero–divisor of $\bar{C}$.

Trivially, $\operatorname{length} B \geq \operatorname{length} \bar{C}_r$ for every $r \in N'$ because of $\bar{C}_r/x_r \subseteq B$ for every non–zero–divisor $x_r \in \bar{C}_r$. The set of all $s \in N'$ such that $\operatorname{length} B = \operatorname{length} \bar{C}_r$ is a non–empty subset N'' of N' with $N' + N'' \subseteq N''$. Therefore there is a $t \in N'$ with $r \in N''$ for all $r \in N'$, $r \geq t$. Certainly, there is some $r \in N''$ with $r > \sigma$. But for all $r \in N'$ with $r > \sigma$

$$\operatorname{length} \bar{C}_r = \operatorname{length} \bar{C}_\sigma = \operatorname{length} A/\mathfrak{R} \,,$$

which we will prove in the next section, cf. Corollary 3.2. □

The last argument in the proof yields the equality $\operatorname{length} B = \operatorname{length} \bar{C}_r$ for all $r \in N'$, $r > \sigma$. As is shown in Corollary 3.2, this equality holds even for all $r > \sigma$ with $C_r \not\subseteq \mathfrak{P}$ for all $\mathfrak{P} \in \operatorname{Ass}_C \bar{C}$.

In general we set

$$N_C := \{ r \in \mathbf{N} : C_r \not\subseteq \mathfrak{P} \;\text{ for all }\; \mathfrak{P} \in \operatorname{Ass}_C \bar{C} \;\text{ with }\; \operatorname{codim}_A \mathfrak{P}_0 = 1 \} \,.$$

Note that $C_r = \bar{C}_r = \mathfrak{Q}_r$ for $r > \sigma$ because of $\mathfrak{T}_r \cong C^*_{\sigma - r} = 0$ for $r > \sigma$. Thus:

2.2 Theorem *For all $r \in N_C$ with $r > \sigma$*

$$\operatorname{div} \mathfrak{R} = \chi(C_r) = \operatorname{div} \mathfrak{f}_0(C_r) \,.$$

We note that in spite of the (defining) equation $\operatorname{div}\mathfrak{R} = \chi(\bar{C}_\sigma)$ the degree σ does not need to be an element of N_C. In the simple example $F_0 = T_0^2$, $F_1 = \pi T_1 \in A[T_0, T_1]$ over the discrete valuation ring A with maximal ideal $A\pi$, $\deg T_0 := 2$, $\deg T_1 := 3$, one has $\sigma = 2$, $N_C = 3\mathbf{N}$. In the classical case $\gamma_0 = \cdots = \gamma_n = 1$ the monoid N_C always coincides with $\mathbf{N}$.

The formula in 2.2 generalizes the well known old computations of resultants in the line of the so–called Macaulay determinants, a very special case ($n = 1$, $\gamma_0 = \gamma_1 = 1$, $r = \sigma + 1$) being covered by the Sylvester determinant.

We will finish this section with some applications of Theorem 2.1.

Assume the elimination ideal $\mathfrak{T}_0$ to be prime. Then $\operatorname{div}\mathfrak{R}$ is a multiple of $\operatorname{div}\mathfrak{T}_0$ (note that this implies that the divisor class of $\mathfrak{T}_0$ in $\operatorname{Cl} A$ is necessarily of finite order). Since $\mathfrak{T}_0\,\bar{C}_\sigma = 0$ the length of $(\bar{C}_\sigma)_{\mathfrak{T}_0}$ is simply the rank of $\bar{C}_\sigma$ over the integral domain $\bar{A} = A/\mathfrak{T}_0$. For the same reasons the length of $B_{\mathfrak{T}_0}$ is the rank $[B : \bar{A}]$ of B over $\bar{A}$ which is called the *degree* of $\operatorname{Proj} C$ over $\operatorname{Spec}\bar{A}$ or the *degree of elimination.* By 2.2 we have:

2.3 Theorem *If the elimination ideal $\mathfrak{T}_0$ is prime, then*

$$\operatorname{div}\mathfrak{R} = (\operatorname{rank}_{\bar{A}}\, \bar{C}_\sigma)\cdot\mathfrak{T}_0 = [B : \bar{A}\,]\cdot\mathfrak{T}_0\,,$$

i.e. the degree of elimination is the multiplicity of the resultant divisor $\operatorname{div}\mathfrak{R}$ *with respect to the elimination divisor* $\operatorname{div}\mathfrak{T}_0$.

Let us now consider the case that even the ideal $\mathfrak{T}$ of inertia forms is prime, that means that the projective variety defined by the polynomials $F_0, \ldots, F_n$ is irreducible. Then B is an integral domain and a finite extension of $\bar{A} \subseteq B$.

2.4 Theorem *If $\mathfrak{T}$ is a prime ideal the following conditions are equivalent*:
(1) $\mathfrak{R} = \mathfrak{T}_0$, *i.e.* $\operatorname{rank}_{\bar{A}}\bar{C}_\sigma = 1$.
(2) $\chi(\mathfrak{Q}_\sigma) = 0$.
(3) $\mathfrak{Q}_\sigma = 0$, *i.e. C_σ is torsionfree over A.*
(4) *$\bar{A}$ and B have the same quotient field, i.e. the projection* $\operatorname{Proj} C = \operatorname{Proj}\bar{C} \to \operatorname{Spec}\bar{A}$ *is birational.*

Proof. (1) and (2) are equivalent by 1.1, and (1) and (4) are equivalent by 2.3. Only the implication (2) $\Rightarrow$ (3) has yet to be proved. $\bar{C}_\sigma = C_\sigma/\mathfrak{T}_\sigma$ as an $\bar{A}$–submodule of the integral domain $\bar{C} = C/\mathfrak{T}$ is torsionfree over $\bar{A}$. The canonical inclusion $\mathfrak{Q}_\sigma \to \bar{C}_\sigma$ shows that $\mathfrak{Q}_\sigma$ is also torsionfree as an $\bar{A}$–module. Therefore $\chi(\mathfrak{Q}_\sigma) = 0$, which means $\operatorname{rank}_{\bar{A}}\mathfrak{Q}_\sigma = 0$, implies $\mathfrak{Q}_\sigma = 0$. □

Next we consider a generic situation. For $j = 0, \ldots, n$ let N_j be a subset of $N_{\delta_j}(\gamma) := \{\nu \in \mathbf{N}^{n+1} : \gamma_0\nu_0 + \cdots + \gamma_n\nu_n = \delta_j\}$ and $F_j = \sum_{\nu\in N_j} U_{j\nu}T^\nu$ the generic polynomial with indeterminate coefficients $U_{j\nu}$, $\nu \in N_j$. We call $F_0, \ldots, F_n$ strictly admissible if they form a regular sequence in $A[T]$, $A := \mathbf{Z}[U_{j\nu}]_{j,\nu}$, and if, in addition, their ideal $\mathfrak{T}$ of the Trägheitsformen is prime. There is a combinatorial algorithm to decide whether $F_0, \ldots, F_n$

are strictly admissible, cf. [8], Lemma 2.6, or the full account in [9], Section 4. In this case the degree of elimination is

$$d := [B : \bar{A}\,] = [\Gamma : (\Gamma_0 + \cdots + \Gamma_n)]\,,$$

where Γ is the group of syzygies of the weights $\gamma_0, \ldots, \gamma_n$ of the indeterminantes $T_0, \ldots, T_n$ and Γ_j for $j = 0, \ldots, n$ its subgroup generated by the differences $\nu - \mu$, $\nu, \mu \in N_j$, cf. [8], Theorem 5.2. By 2.3 we have:

2.5 Theorem *For strictly admissible generic polynomials* $F_j = \sum_{\nu \in N_j} U_{j\nu} T^\nu, j = 0, \ldots, n,$

$$\mathfrak{R}(F_0, \ldots, F_n) = (\mathfrak{T}_0(F_0, \ldots, F_n))^d, \quad d = [\Gamma : (\Gamma_0 + \cdots + \Gamma_n)]\,.$$

As a last application we generalize a result of Chardin about the degree of the fibres of $\operatorname{Proj} C$ over $\operatorname{Spec} A$, cf. [3], Proposition 3.

As before let A be an integrally closed domain. For the moment, the homogeneous polynomials $F_0, \ldots, F_n \in A[T_0, \ldots, T_n]$ are not supposed to form a regular sequence. Nevertheless the resultant $R = \mathrm{R}_A(F_0, \ldots, F_n)$ is defined as the element one gets by specializing the resultant of the generic homogeneous polynomials of the same degrees. (If these generic polynomials do not form a regular sequence — which can happen when some weights are > 1 — then $R = 0$ by definition.) As explained in Section 1, if $F_0, \ldots, F_n$ form a regular sequence in $A[T]$ this resultant generates the resultant ideal $\mathfrak{R} = \mathfrak{R}(F_0, \ldots, F_n)$.

For every prime ideal $\mathfrak{p} \subseteq A$ of codimension 1 we denote by $\mathrm{v}_\mathfrak{p}(R)$ the multiplicity of $\mathfrak{p}$ in the principal ideal AR and by $X_\mathfrak{p} = \operatorname{Proj}(C_\mathfrak{p}/\mathfrak{p}C_\mathfrak{p})$ the fibre of $X := \operatorname{Proj} C$, $C = A[T]/(F_0, \ldots, F_n)$, over $\mathfrak{p} \in \operatorname{Spec} A$. If this fibre is finite,

$$\deg X_\mathfrak{p} = \dim_{\kappa(\mathfrak{p})} \Gamma(X_\mathfrak{p}, \mathcal{O}_{X_\mathfrak{p}})$$

denotes its degree. As above, $B = \Gamma(X, \mathcal{O}_X)$, which is finite over A in every case. With these notations we have:

2.6 Theorem *If for the prime ideal* $\mathfrak{p} \subseteq A$ *of codimension* 1 *the fibre* $X_\mathfrak{p}$ *is finite then*

$$\deg X_\mathfrak{p} \le \mathrm{v}_\mathfrak{p}(R)\,.$$

Equality holds if and only if $\mathfrak{p}B_\mathfrak{p} = 0$.

Proof. We may assume right away that A is a discrete valuation ring with maximal ideal $\mathfrak{p} = A\pi$. The assumption on the fibre is equivalent with the following condition: $\operatorname{codim}(\pi, F_0, \ldots, F_n) \ge n + 1$.

First we consider the case that $F_0, \ldots, F_n$ is not a regular sequence. Then there is a homogeneous prime ideal $\mathfrak{P}$ of codimension $\le n$ containing $F_0, \ldots, F_n$. Therefore $\pi \notin \mathfrak{P}$ and $F_0, \ldots, F_n$ generate an ideal of codimension $\le n$ in $K[T]$ where K is the quotient field of A. Then $R = 0$, $K \otimes_A B \ne 0$ and $\mathfrak{p}B \ne 0$, which settles the theorem in this case (because $\mathrm{v}_\mathfrak{p}(0) = \infty$).

Now let us assume that $F_0, \ldots, F_n$ is a regular sequence. Then $X \cong \operatorname{Spec} B$ is affine and finite over $\operatorname{Spec} A$ which implies $\Gamma(X_\mathfrak{p}, \mathcal{O}_{X_\mathfrak{p}}) = B/\mathfrak{p}B$. With Theorem 2.1 one gets $\deg X_\mathfrak{p} = \operatorname{length} B/\mathfrak{p}B \le \operatorname{length} B = \mathrm{v}_\mathfrak{p}(R)$, and equality holds if and only if $\mathfrak{p}B = 0$. $\square$

3. Supplements in duality

In this section the base ring A is a Dedekind domain. Again $F_0, \dots, F_n$ is a regular sequence of homogeneous polynomials of positive degrees in $A[T] = A[T_0, \dots, T_n]$ and $C = A[T]/(F_0, \dots, F_n)$. We adopt freely notations and constructions from Section 1. Among other things we prove a result we needed for Theorems 2.1 and 2.2.

3.1 Lemma *Assume A to be a Dedekind domain. Let $\bar{C} = C/\mathfrak{T}$, $\bar{A} = A/\mathfrak{T}_0 = \bar{C}_0$ and $\omega_{\bar{A}} := \mathfrak{T}_0^{-1}/A$. For every homogeneous element $z \in C$ of positive degree r, which is a non–zero–divisor in $\bar{C}$ (which is true if and only if C/zC is a finite A–algebra), there is an isomorphism*

$$\mathfrak{Q}/z\mathfrak{Q} \cong \mathrm{Hom}_{\bar{A}}\,(\bar{C}/z\bar{C},\, \omega_{\bar{A}})(-\sigma - r)$$

of graded C–modules.

$\mathfrak{Q}$ denotes as in Section 1 the irrelevant component of the zero–ideal in C. Recall that for any graded module V over a graded ring the shifted module $V(m)$, $m \in \mathbf{Z}$, is the graded module having as homogeneous part of degree $i \in \mathbf{Z}$ the subgroup $V(m)_i = V_{m+i}$.

Proof of 3.1. Let $A[Z]$ denote the graded polynomial algebra over A in the single indeterminate Z of degree r. Then C and $\bar{C}$ are finite graded $A[Z]$–algebras by $Z \mapsto z$ and

$$\mathrm{Hom}_{\bar{A}}\,(\bar{C}/z\bar{C}\,, \omega_{\bar{A}}) = \mathrm{Hom}_{\bar{A}[Z]}(\bar{C}/z\bar{C}\,, \omega_{\bar{A}}) = \mathrm{Hom}_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}})\,.$$

Corresponding to the exact sequence $0 \to \omega_{\bar{A}}[Z] \xrightarrow{Z} \omega_{\bar{A}}[Z] \to \omega_{\bar{A}} \to 0$ there is the exact sequence

$$0 \to \mathrm{Hom}_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}}\,[Z]) \xrightarrow{Z} \mathrm{Hom}_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}}\,[Z]) \to \mathrm{Hom}_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}}) \to \mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C})$$

where $\mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C}) := \mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}}[Z]) = 0$, however, which is proved as follows: From the exact sequence $0 \to \bar{C} \xrightarrow{z} \bar{C} \to \bar{C}/z\bar{C} \to 0$ we get an exact sequence

$$\mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C}) \xrightarrow{z} \mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C}) \to \mathrm{Ext}^2_{\bar{A}[Z]}(\bar{C}/z\bar{C}\,, \omega_{\bar{A}}[Z])\,.$$

The Ext^2– group is zero because $\omega_{\bar{A}}[Z]$ is an $\bar{A}[Z]$–module of injective dimension 1. Hence multiplication by z on the finite graded $\bar{A}[Z]$–module $\mathrm{Ext}^1_{\bar{A}[Z]}(\bar{C})$ is surjective, which implies that the module vanishes.

It suffices to show now that

$$\mathrm{Hom}_{\bar{A}[Z]}(\bar{C}\,, \omega_{\bar{A}}[Z]) = \mathrm{Hom}_{A[Z]}(\bar{C}\,, \omega_{\bar{A}}[Z]) \cong \mathfrak{Q}(\sigma + r)$$

as graded C–modules.

First we prove that the homomorphism $C \to \bar{C}$ induces an isomorphism

$$\mathrm{Hom}_{A[Z]}(\bar{C}\,, \omega_{\bar{A}}[Z]) = \mathrm{Hom}_{A[Z]}(C, \omega_{\bar{A}}[Z])$$

by showing that $\mathrm{Hom}_{A[Z]}(\mathfrak{T}, \omega_{\bar{A}}[Z]) = 0$. For this we may assume that A is a discrete valuation ring with maximal ideal $A\pi$. Then we consider the exact sequence

$$\mathrm{Hom}_{A[Z]}(\mathfrak{T}, A[Z]) \to \mathrm{Hom}_{A[Z]}(\mathfrak{T}, \bar{A}[Z]) \to \mathrm{Ext}^1_{A[Z]}(\mathfrak{T}, A[Z]) \xrightarrow{\pi^s} \mathrm{Ext}^1_{A[Z]}(\mathfrak{T}, A[Z])$$

belonging to $0 \to A[Z] \xrightarrow{\pi^s} A[Z] \to \bar{A}[Z] \to 0$, where π^s is a generator of $\mathfrak{T}_0$. For trivial reasons, $\mathrm{Hom}_{A[Z]}(\mathfrak{T}, A[Z]) = 0$. Therefore it remains to show that $\mathrm{Ext}^1_{A[Z]}(\mathfrak{T}, A[Z])$ is a torsionsfree A–module. But this follows from an isomorphism

$$\mathrm{Ext}^1_{A[Z]}(\mathfrak{T}, A[Z]) = \mathrm{Hom}_A(\mathfrak{T}, A)$$

of modules. For the construction of such an isomorphism see [7], Lemma 4.1; one has to use that $\mathfrak{T}$ is a finite projective A–module.

Next we prove that $\mathrm{Hom}_{A[Z]}(C, \omega_{\bar{A}}[Z])$ is isomorphic to the $\mathfrak{T}_0$–torsion of the module $\mathrm{Ext}^1_{A[Z]}(C, A[Z])$. For this we consider the exact sequence

$$\mathrm{Hom}_{A[Z]}(C, \mathfrak{T}_0^{-1}[Z]) \to \mathrm{Hom}_{A[Z]}(C, \omega_{\bar{A}}[Z]) \to \mathrm{Ext}^1_{A[Z]}(C, A[Z]) \to \mathrm{Ext}^1_{A[Z]}(C, \mathfrak{T}_0^{-1}[Z])$$

belonging to $0 \to A[Z] \to \mathfrak{T}_0^{-1}[Z] \to \omega_{\bar{A}}[Z] \to 0$. Again $\mathrm{Hom}_{A[Z]}(C, \mathfrak{T}_0^{-1}[Z]) = 0$ for trivial reasons. Thus $\mathrm{Hom}_{A[Z]}(C, \omega_{\bar{A}}[Z])$ is the kernel of

$$\mathrm{Ext}^1_{A[Z]}(C, A[Z]) \to \mathrm{Ext}^1_{A[Z]}(C, \mathfrak{T}_0^{-1}[Z])\,,$$

which is the $\mathfrak{T}_0$–torsion of $\mathrm{Ext}^1_{A[Z]}(C, A[Z])$ as can be verified locally with respect to A.

Because $\mathfrak{Q}$ is the $\mathfrak{T}_0$–torsion of C, it finally suffices to show that there is an isomorphism

$$\mathrm{Ext}^1_{A[Z]}(C, A[Z]) \cong C(\sigma + r)$$

of graded C–modules.

C has the representation

$$C = C[Z]/(Z - z) = A[Z, T]/(F_0, \ldots, F_n,\ Z - z)$$

where $F_0, \ldots, F_n,\ Z - z$ is a regular sequence in $A[Z, T]$. Furthermore, C has homological dimension 1 over $A[Z]$ because C is finite over $A[Z]$ and is a Macaulay ring of dimension 1. Thus there is a canonical isomorphism

$$\mathrm{Ext}^1_{A[Z]}(C, A[Z]) \cong \mathrm{Ext}^{n+2}_{A[Z,T]}(C, A[Z, T])(-\gamma_0 - \cdots - \gamma_n)$$

(γ_i being the degree of T_i) of graded C–modules, cf. again [7], Lemma 4.1. From the Koszul resolution of C over $A[Z, T]$ one gets the isomorphism

$$\mathrm{Ext}^{n+2}_{A[Z,T]}(C, A[Z, T]) \cong C(\delta_0 + \cdots + \delta_n + r)$$

(δ_i being the degree of F_i, r the degree of $Z-z$) of graded C–modules. The composition of both isomorphisms yields an isomorphism of $\mathrm{Ext}^1_{A[Z]}(C, A[Z])$ onto $C(\sigma+r)$ we were looking for, because $-\gamma_0-\cdots-\gamma_n+\delta_0+\cdots+\delta_n+r$ is simply $\sigma+r$. □

Let us recall the definition of the monoid N_C (given before Theorem 2.2) which for a Dedekind domain reads as follows:

$$N_C = \{r \in \mathbf{N} : C_r \not\subseteq \mathfrak{P} \text{ for all } \mathfrak{P} \in \mathrm{Ass}_C \bar{C}\}.$$

3.2 Corollary *Assume A to be a Dedekind domain. For every $r \in N_C$ and $s \in \mathbf{Z}$*

$$\chi(\mathfrak{Q}_{r+s}) - \chi(\mathfrak{Q}_s) = \chi(\bar{C}_{\sigma-s}) - \chi(\bar{C}_{\sigma-s-r}).$$

In particular,

$$\chi(\bar{C}_\sigma) = \chi(\mathfrak{Q}_r) \ (= \chi(C_r) = \chi(\bar{C}_r))$$

for every $r \in N_C$ with $r > \sigma$.

Proof. We may assume that A is a discrete valuation ring and, using a suitable flat extension which preserves lengths, that its residue field is infinite. Then N_C is the monoid of all $r \in \mathbf{N}$ such that C_r contains an element z which is a non–zero–divisor in $\bar{C}$. Now the corollary follows from 3.1 simply by the fact that dualizing by $\omega_{\bar{A}} \cong \bar{A}$ preserves lengths. □

As another simple consequence of 3.1 we get:

3.3 Corollary *Assume A to be a Dedekind domain: Let $z \in C_r$ be a homogeneous element of positive degree r, which is a non–zero–divisor in C. Then $(C/zC)_m = 0$ for every $m \geq \sigma + r$.*

There is a canonical way to formulate the duality decribed in Lemma 3.1 (using the mere existence of such a duality): We do this *by constructing a canonical $\bar{A}$–bilinear homogeneous form*

$$\mathfrak{Q}/z\mathfrak{Q} \times (\bar{C}/z\bar{C})(\sigma+r) \longrightarrow \omega_{\bar{A}}$$

which defines a perfect duality.

The isomorphism $\mathfrak{T}_0 \xrightarrow{\sim} C^*_\sigma = (C_\sigma/\mathfrak{Q}_\sigma)^*$ induces canonical isomorphisms $C_\sigma/\mathfrak{Q}_\sigma = (C_\sigma/\mathfrak{Q}_\sigma)^{**} \xrightarrow{\sim} \mathfrak{T}^*_0 = \mathfrak{T}_0^{-1}$ and $\bar{C}_\sigma/\mathfrak{Q}_\sigma = C_\sigma/(\mathfrak{Q}_\sigma + A\Delta) \xrightarrow{\sim} \mathfrak{T}_0^{-1}/A = \omega_{\bar{A}}$. So we may replace $\omega_{\bar{A}}$ by $\bar{C}_\sigma/\mathfrak{Q}_\sigma$. The value of the bilinear form on $([q],[c])$, $q \in \mathfrak{Q}_m$, $c \in C_{\sigma+r-m}$, is

$$\langle [q], [c] \rangle := \left[\frac{qc}{z}\right] \in \bar{C}_\sigma/\mathfrak{Q}_\sigma.$$

Note that $qc \in C_{\sigma+r}$ can be written in the form $qc = zu$ with $u \in C_\sigma$ by 3.3. It suffices to show that this form is non–degenerate over $\bar{A}$ because the modules involved have finite lengths. But non–degeneracy follows from the fact that there is an isomorphism $\mathfrak{Q}/z\mathfrak{Q} \cong \mathrm{Hom}_{\bar{A}}(\bar{C}/z\bar{C}, \bar{C}_\sigma/\mathfrak{Q}_\sigma)(-\sigma-r)$ of graded C–modules, cf. 3.1.

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