Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 37 (1996), No. 1, 161-198.

A Unified Treatment of the Theories of Matroids with Coefficients and of Δ-Matroids with Coefficients

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Abstract. The representation theory of matroids and that of even Δ-matroids have much in common. In this paper, a unified representation theory is developed which will also encompass the theory of matroids with coefficients and that of Δ-matroids with coefficients. This is established by identifying set systems with special cosets in Coxeter groups as was done by I. M. Gelfand and V. V. Serganova in their work concerning (W, P)-*matroids*; however, in order to extend the theory of matroids with coefficients it was necessary to study a slightly different concept, named a *Combinatorial* $(W; P; U)$-*geometry* which is defined for a Coxeter group W, a subset P of its generating involutions, and some subgroup U of W containing P.

Keywords and phrases: Representation theory of matroids, matroids with coefficients, Grassmann-Plücker relations, Δ-matroids, skew-symmetric matrices, Pfaffian forms, the base graph of a matroid, antipodal graphs, convexity in graphs, Coxeter groups.

MSC 1991: 05B25, 05B35, 20B25, 51D20 (Primary)
15A03, 15A15, 51D10, 51F15, 52A30, 68E10 (Secondary)

1. Introduction

In 1986, A. Dress introduced the concept of a matroid with coefficients in a fuzzy ring K in order to establish a framework for a unified theory of representable, orientable, valuated, etc. matroids ([6]). In [8], it was shown that matroids with coefficients can be defined in terms of Grassmann-Plücker maps, and in [9] we proved many representation theorems which demonstrate why representable and orientable matroids show very similar aspects.

* This work was done while the author stayed at the University of Augsburg.

In 1987, A. Bouchet introduced the concept of a Δ-matroid obtained by weakening the structure of a matroid ([3]). Representability problems for Δ-matroids are studied in [4] and [5], and it turned out (cf. [4, §4]) that for every field K a matroid M is representable over K if and only if it is representable over K as a Δ-matroid by some skew-symmetric matrix. This result suggested the existence of an identity concerning skew-symmetric matrices which should be quite similar to the Grassmann-Plücker identity. Indeed, by studying the Pfaffians of skew-symmetric matrices it turned out (cf. [22, Theorem 2.2]) that for any $n \in \mathbf{N}$, $E := \{1, ..., n\}$, and for any field K there is a canonical one-to-one correspondence between the skew-symmetric $n \times n$-matrices $A = -A^T$ with coefficients in K and maps $p : \mathcal{P}(E) \to K$ satisfying $p(\emptyset) = 1$ and the following two axioms:

(P1) For any $I \subseteq E$ with odd cardinality $\#I$ one has $p(I) = 0$.

(P2) For $I_1, I_2 \subseteq E$ and $I_1 \Delta I_2 := (I_1 \cup I_2) \setminus (I_1 \cap I_2) = \{i_1, ... i_\ell\}$ with $i_j < i_{j+1}$ for $1 \leq j \leq \ell - 1$ one has

$$\sum_{j=1}^{\ell} (-1)^j \cdot p\left(I_1 \Delta \{i_j\}\right) \cdot p\left(I_2 \Delta \{i_j\}\right) = 0.$$

Such maps are called *Pfaffian maps.* For a skew-symmetric matrix $A = (a_{ij})_{i,j \in E}$ the associated map p is defined in terms of its Pfaffian Pf; that is, $p(I) = Pf\left((a_{ij})_{i,j \in I}\right)$. This result proved to be essentially a consequence of the fact that the map p induced by some skew-symmetric matrix A via its Pfaffian form satisfies (P2) – an identity unknown before except in the particular case $\#I_1 = 1$ or $\#I_2 = 1$. The significance of this identity lies in the fact that it has allowed to establish a theory of Δ-matroids with coefficients in any fuzzy ring K (cf. [23]) such that in the special case of K being a field the Δ-matroids with coefficients in K are the Δ-matroids representable by a skew-symmetric matrix.

The conspicuous similarity between the well-known Grassmann-Plücker identities and the relations (P2) suggests the existence of a unified representation theory which should encompass the theory of matroids with coefficients and that of Δ-matroids with coefficients; the present paper is devoted to this subject. As in the work of I. M. Gelfand and V. V. Serganova concerning (W, P)-*matroids* which unify – among other combinatorial structures – matroids, Δ-matroids and a large class of greedoids (cf. [14], [15]), set systems are identified with special cosets in Coxeter groups. In order to extend the theory of matroids with coefficients and that of representable Δ-matroids, it was necessary to introduce the slightly different concept of a *combinatorial* $(W; P; U)$-*geometry*; however, as it turned out in [25, §4], for many important classes of W and P we consider the same structures. While I. M. Gelfand and V. V. Serganova reformulated the exchange condition for bases in matroids into some "minimality condition" which is related to word problems in Coxeter groups and to the Greedy Algorithm, I have tried to generalize the strong exchange condition for bases in matroids. This approach is built up mainly graph theoretically; more precisely, the strong exchange axiom is reformulated in the framework of antipodal graphs. While the combinatorial framework is already established in [25], the present paper concerns the representation theory of combinatorial $(W; P; U)$-geometries.

Technically, this paper is organized as follows:

In Section 2, we recall the fundamental concepts of matroids with coefficients, of Δ-matroids with coefficients and of antipodal graphs.

In Section 3, we provide the technical tools to establish the theory of $(W;P;U)$-matroids with coefficients for any Coxeter group W, any subset P of its generating involutions and specified subgroups U of W containing P. The axioms are chosen so that, on the one hand, at least the matroids with coefficients and the Δ-matroids with coefficients are included (cf. **5.**), and that, on the other hand, many results concerning matroids with coefficients remain true in this more general framework.

Section 4 is devoted to the study of equivalent representations of a given combinatorial $(W;P;U)$-geometry, that is, those which induce the same $(W;P;U)$-matroid with coefficients. To this end, we shall make use of the extension of Maurer's homotopy theorem for ordinary matroids as developed in [19] to combinatorial $(W;P;U)$-geometries as established in [25, §5].

Finally, in Section 5, we study examples. In particular, it is shown that matroids with coefficients as well as Δ-matroids with coefficients fit into the theory developed in the present paper.

In a forthcoming paper (cf. [27]), I shall study the geometric algebra of a combinatorial $(W;P;U)$-geometry by generalizing the concept of the Tutte group of an ordinary matroid as introduced in [11]. A second, equivalent definition of $(W;P;U)$-matroids with coefficients will then be derived.

Acknowledgement: I would like to thank Andreas Dress for kindly suggesting to me the idea of connecting the theories of matroids with coefficients, of Δ-matroids and of (W,P)-matroids.

2. Preliminaries

In this section, we recall some basic definitions and results concerning matroids with coefficients, Δ-matroids with coefficients and antipodal graphs. We first state the definition of fuzzy rings which serve as coefficient domains for matroids and Δ-matroids.

Definition 2.1. *A fuzzy ring $K = (K;+;\cdot;\varepsilon;K_0)$ consists of a set K together with two compositions:*

$$\begin{aligned} + &: \quad K \times K \to K \,:\, (\kappa,\lambda) \mapsto \kappa+\lambda \quad \textit{and} \\ \cdot &: \quad K \times K \to K \,:\, (\kappa,\lambda) \mapsto \kappa\cdot\lambda, \end{aligned}$$

a specified element $\varepsilon \in K$ and a specified subset $K_0 \subseteq K$ such that the following axioms hold:

(FR0) *$(K,+)$ and $(K,\cdot)$ are abelian semigroups with neutral elements 0 and 1, respectively;*

(FR1) *$0\cdot\kappa = 0$ for all $\kappa \in K$;*

(FR2) *$\alpha\cdot(\kappa_1+\kappa_2) = \alpha\cdot\kappa_1 + \alpha\cdot\kappa_2$ for all $\kappa_1,\kappa_2 \in K$ and $\alpha \in K^* := \{\beta \in K \mid 1 \in \beta\cdot K\}$, the group of units in K ;*

(FR3) $\varepsilon^2 = 1$;

(FR4) $K_0 + K_0 \subseteq K_0$, $K \cdot K_0 \subseteq K_0$; $0 \in K_0$; $1 \notin K_0$;

(FR5) *for* $\alpha \in K^*$ *one has* $1 + \alpha \in K_0$ *if and only if* $\alpha = \varepsilon$;

(FR6) $\kappa_1, \kappa_2, \lambda_1, \lambda_2 \in K$ *and* $\kappa_1 + \lambda_1, \kappa_2 + \lambda_2 \in K_0$ *implies* $\kappa_1 \cdot \kappa_2 + \varepsilon \cdot \lambda_1 \cdot \lambda_2 \in K_0$;

(FR7) $\kappa, \lambda, \kappa_1, \kappa_2 \in K$ *and* $\kappa + \lambda \cdot (\kappa_1 + \kappa_2) \in K_0$ *implies* $\kappa + \lambda \cdot \kappa_1 + \lambda \cdot \kappa_2 \in K_0$.

Remarks:

i) (FR4), (FR5) and (FR7) imply $\kappa + \varepsilon \cdot \kappa \in K_0$ for all $\kappa \in K$.

ii) (FR4) yields directly $K^* \cap K_0 = \emptyset$.

iii) (FR2), (FR4) and (FR5) imply

(FR5') For $\alpha, \beta \in K^*$ one has $\alpha + \beta \in K_0$ if and only if $\beta = \varepsilon \cdot \alpha$.

iv) (FR7) implies: If $\lambda_1, ..., \lambda_n, \kappa_{11}, ..., \kappa_{1m}, \kappa_{21}, ..., \kappa_{2m}, ..., \kappa_{n1}, ..., \kappa_{nm} \in K$ and

$$\sum_{i=1}^{n} \lambda_i \cdot \Big(\sum_{j=1}^{m} \kappa_{ij}\Big) \in K_0, \qquad \text{then}$$

(FR7')
$$\sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_i \cdot \kappa_{ij} \in K_0.$$

Examples:

i) The commutative rings $R = (R; + ; \cdot\,)$ with $1 \in R$ are (in a canonical correspondence to) exactly those fuzzy rings $(K; + ; \cdot\,; \varepsilon; K_0)$ for which $K_0 = \{0\}$. In this case we have necessarily $\varepsilon = -1$.

ii) Let $\mathbf{R}^+$ and $\mathbf{R}^-$ denote the positive and negative units in $\mathbf{R}$, respectively, and put $\mathbf{R}//\mathbf{R}^+ := \{\{0\}, \mathbf{R}^+, \mathbf{R}^-, \mathbf{R}\}$. By adding and multiplying the elements of $\mathbf{R}//\mathbf{R}^+$ as *complexes* one gets the following tables:

$+$	0	$\mathbf{R}^+$	$\mathbf{R}^-$	$\mathbf{R}$
0	0	$\mathbf{R}^+$	$\mathbf{R}^-$	$\mathbf{R}$
$\mathbf{R}^+$	$\mathbf{R}^+$	$\mathbf{R}^+$	$\mathbf{R}$	$\mathbf{R}$
$\mathbf{R}^-$	$\mathbf{R}^-$	$\mathbf{R}$	$\mathbf{R}^-$	$\mathbf{R}$
$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$

$\cdot$	0	$\mathbf{R}^+$	$\mathbf{R}^-$	$\mathbf{R}$
0	0	0	0	0
$\mathbf{R}^+$	0	$\mathbf{R}^+$	$\mathbf{R}^-$	$\mathbf{R}$
$\mathbf{R}^-$	0	$\mathbf{R}^-$	$\mathbf{R}^+$	$\mathbf{R}$
$\mathbf{R}$	0	$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$

Then $K := \mathbf{R}//\mathbf{R}^+$ is a fuzzy ring with $K_0 = \{\{0\}, \mathbf{R}\}$, $K^* = \{\mathbf{R}^+, \mathbf{R}^-\}$ and $\varepsilon = \mathbf{R}^-$. In [8, §6] it is shown that $\mathbf{R}//\mathbf{R}^+$ is an appropriate domain of coefficients for oriented matroid theory.

These and some further examples of fuzzy rings are considered in more detail in [6, (1.3)] and [23, §1].

Now we want to define matroids with coefficients of finite rank in terms of their Grassmann-Plücker maps (cf. [8]).

Definition 2.2. *Assume E is some – possibly infinite – set and $K = (K; + ; \cdot\,; \varepsilon; K_0)$ is a fuzzy ring. For $m \in \mathbf{N}_0$ a map $b : E^m \to K^* \dot\cup \{0\}$ is called a Grassmann-Plücker map of degree m, if the following axioms hold:*

(GP0) *There exist* $e_1, ..., e_m \in E$ *with* $b(e_1, ..., e_m) \neq 0$.

(GP1) b *ist* ε*-alternating; this means, for* $e_1, ..., e_m \in E$ *and every odd permutation* $\tau \in S_m$ *one has*

$$b\left(e_{\tau(1)}, ..., e_{\tau(m)}\right) = \varepsilon \cdot b(e_1, ..., e_m) ,$$

and in case $\#\{e_1, ..., e_m\} < m$ *one has* $b(e_1, ..., e_m) = 0$.

(GP2) *For all* $e_0, ..., e_m, f_2, ..., f_m \in E$ *one has*[1]

$$\sum_{i=0}^{m} \varepsilon^i \cdot b(e_0, ..., \hat{e}_i, ..., e_m) \cdot b(e_i, f_2, ..., f_m) \in K_0. \tag{1}$$

The relations (1) *are called the Grassmann-Plücker relations.*

Two Grassmann-Plücker maps $b_1, b_2 : E^m \to K^* \dot{\cup} \{0\}$ are *equivalent* if there exists some $\kappa \in K^*$ such that for all $e_1, ..., e_m \in E$ one has

$$b_1(e_1, ..., e_m) = \kappa \cdot b_2(e_1, ..., e_m) .$$

In case $m = 0$, a Grassmann-Plücker map is defined on a singleton set taking its value in K^*.

Example. Assume K is a field and $E \subseteq K^m$ is a spanning subset of the vector space K^m. View the elements of E as columns. Then $b : E^m \to K$ defined by

$$b(e_1, ..., e_m) = \det(e_1, ..., e_m)$$

is a Grassmann-Plücker map; the relations (1) state nothing but the well-known identity of Grassmann.

Now, we come to the following basic

Definition 2.3. *Assume* E *is some set,* $K = (K; +; \cdot; \varepsilon; K_0)$ *is a fuzzy ring, and suppose* $m \in \mathbf{N}_0$. *A matroid* M *of rank* m *defined on* E *and with coefficients in* K *consists of an equivalence class of Grassmann-Plücker maps* $b : E^m \to K^* \dot{\cup} \{0\}$. *We write* $M = M_b$ *for any Grassmann-Plücker map* b *defining* M.

A subset $B = \{e_1, ..., e_m\}$ *of* E *is called a base of the matroid* M, *if* $b(e_1, ..., e_m) \neq 0$ *for one and thus for all Grassmann-Plücker maps* b *with* $M_b = M$.

Remarks:

i) By definition, for two Grassmann-Plücker maps b, b' we have $M_b = M_{b'}$ if and only if b and b' are equivalent.

ii) If M is a matroid with coefficients defined on E with $\mathcal{B}$ as its set of bases, then (GP2) yields directly that the pair $\underline{M} := (E, \mathcal{B})$ defines a matroid in the classical sense. $\underline{M}$ is called the underlying combinatorial geometry of M.

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[1] As usual, $(e_o, ..., \hat{e}_i, ..., e_m)$ denotes the m-tuple $(e_0, ..., e_{i-1}, e_{i+1}, ..., e_m)$.

Examples:

i) If K is a field, then the matroids of rank m defined on E and with coefficients in K are in a canonical one-to-one correspondence to the equivalence classes of vector representations $\varphi : E \to K^m$ of the underlying combinatorial geometries, where two such representations $\varphi_1, \varphi_2 : E \to K^m$ are called equivalent if and only if there exists some regular $m \times m$-matrix A with coefficients in K such that for all $e \in E$ one has $\varphi_2(e) = A \cdot \varphi_1(e)$. More precisely, if $\varphi : E \to K^m$ is a vector representation of some combinatorial geometry over the field K, then $b : E^m \to K$ defined by

$$b(e_1, ..., e_m) := \det(\varphi(e_1), ..., \varphi(e_m))$$

is a corresponding Grassmann-Plücker map.

ii) Put $K_c := \mathbf{R}/\mathbf{R}^* := \{\{0\}, \mathbf{R}^*, \mathbf{R}\}$. Complex addition and multiplication yield the following tables:

$+$	0	$\mathbf{R}^*$	$\mathbf{R}$
0	0	$\mathbf{R}^*$	$\mathbf{R}$
$\mathbf{R}^*$	$\mathbf{R}^*$	$\mathbf{R}$	$\mathbf{R}$
$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$	$\mathbf{R}$

$\cdot$	0	$\mathbf{R}^*$	$\mathbf{R}$
0	0	0	0
$\mathbf{R}^*$	0	$\mathbf{R}^*$	$\mathbf{R}$
$\mathbf{R}$	0	$\mathbf{R}$	$\mathbf{R}$

$K = K_c$ is a fuzzy ring with $K_0 = \{\{0\}, \mathbf{R}\}, K^* = \{\mathbf{R}^*\}$ and $1 = \varepsilon = \mathbf{R}^*$. In particular, any Grassmann-Plücker map $b : E^m \to K^* \dot{\cup} \{0\}$ is of the form

$$b(e_1, ..., e_m) = \begin{cases} 1, & \text{if } \{e_1, ..., e_m\} \text{ is a base of } M_b \\ 0 & \text{otherwise.} \end{cases}$$

Since $1 + ... + 1 \in K_0$ whenever the sum contains at least two summands, (GP2) states nothing but the strong exchange condition for bases in matroids:

If B_1, B_2 are bases of some matroid M and $e \in B_1 \setminus B_2$, then there exists some $f \in B_2 \setminus B_1$ such that $(B_1 \setminus \{e\}) \cup \{f\}$ and $(B_2 \setminus \{f\}) \cup \{e\}$ are bases of M.

This means that $K = K_c$ is an appropriate domain for classical matroid theory (see also [10, (4.7)]).

iii) As already remarked above, $\mathbf{R}//\mathbf{R}^+$ is an appropriate domain for oriented matroids; indeed, it is trivially verified that matroids with coefficients in $\mathbf{R}//\mathbf{R}^+$ are in canonical one-to-one correspondence to chirotopes as considered in [7], see also [2], [17] and [18].

Next we recall the concept of a Δ-matroid.

Definition 2.4. (cf. [3], [4]) *Assume E is some finite set and $\emptyset \neq \mathcal{F} \subseteq \mathcal{P}(E)$.*

i) *The pair $(E, \mathcal{F})$ is a Δ-matroid if $\mathcal{F}$ satisfies the following symmetric exchange axiom:*

(SEA) *For $F_1, F_2 \in \mathcal{F}$ and $e \in F_1 \Delta F_2 := (F_1 \cup F_2) \setminus (F_1 \cap F_2)$ there exists some $f \in F_1 \Delta F_2$ such that $F_1 \Delta \{e, f\} \in \mathcal{F}$.*

ii) *A Δ-matroid $(E, \mathcal{F})$ is even if for all $F_1, F_2 \in \mathcal{F}$ one has $\#F_1 \equiv \#F_2 \bmod 2$.*

Note that the ordinary matroids $(E, \mathcal{F})$ with base set $\mathcal{F}$ are exactly the Δ-matroids defined on E with equicardinal feasible sets. For $\mathcal{F} \subseteq \mathcal{P}(E)$ and $I \subseteq E$ put

$$\mathcal{F}\Delta I := \{F\Delta I \mid F \in \mathcal{F}\}. \tag{2}$$

Trivially, $(E, \mathcal{F})$ is a Δ-matroid if and only if $(E, \mathcal{F}\Delta I)$ is a Δ-matroid.

Definition 2.5. (cf. [4, §4]) *Assume K is a field and $(E, \mathcal{F})$ is a Δ-matroid. $(E, \mathcal{F})$ is representable over K by some skew-symmetric matrix $A = (a_{ij})_{i,j \in E}$ if there exists some $I \subseteq E$ such that*

$$\mathcal{F}\Delta I = \mathcal{F}(A) := \left\{F \subseteq E \mid A' := (a_{ij})_{i,j \in F} \text{ is nonsingular}\right\}, \tag{3}$$

where $(a_{ij})_{i,j \in \emptyset}$ is considered to be nonsingular. The matrix A is then called a representation of $(E, \mathcal{F})$ relative to I.

Now, we are also able to recall the concept of a Δ-matroid with coefficients as introduced in [23].

Definition 2.6. *Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring and $E = \{1, ..., n\}, n \in \mathbf{N}$.*

i) *A map $p : \mathcal{P}(E) \to K^* \dot{\cup} \{0\}$ is a twisted Pfaffian map if the following axioms hold:*
   (TP0) *There exists some $I_0 \subseteq E$ with $p(I_0) \neq 0$.*
   (TP1) *For all $I_1, I_2 \subseteq E$ with $p(I_1) \neq 0$ and $p(I_2) \neq 0$ one has $\#I_1 \equiv \#I_2 \bmod 2$.*
   (TP2) *If $I_1, I_2 \subseteq E$ and $I_1 \Delta I_2 = \{i_1, ..., i_\ell\}$ with $i_j < i_{j+1}$ for $1 \leq j \leq \ell - 1$, then one has*
   $$\sum_{j=1}^{\ell} \varepsilon^j \cdot p(I_1 \Delta \{i_j\}) \cdot p(I_2 \Delta \{i_j\}) \in K_0.$$

   Two twisted Pfaffian maps $p_1, p_2 : \mathcal{P}(E) \to K^* \dot{\cup} \{0\}$ are called equivalent if there exists some $\kappa \in K^*$ such that for all $I \subseteq E$ one has $p_1(I) = \kappa \cdot p_2(I)$.

ii) *A Δ-matroid M defined on E and with coefficients in K consists of an equivalence class of twisted Pfaffian maps $p : \mathcal{P}(E) \to K^* \dot{\cup} \{0\}$. We write also $M = M_p$ for any twisted Pfaffian map p defining M.*
   A subset $F \subseteq E$ is called free or feasible in M if $p(F) \neq 0$ for one and thus for all twisted Pfaffian maps p with $M_p = M$.

Remarks:

i) If $\mathcal{F}$ is the set of feasible subsets of a Δ-matroid $M = M_p$ with coefficients in some fuzzy ring K, then (TP2) yields at once that $\mathcal{F}$ satisfies the following strong symmetric exchange condition:
   (SSEA) For $F_1, F_2 \in \mathcal{F}$ and $e \in F_1 \Delta F_2$ there exists some $f \in (F_1 \Delta F_2) \setminus \{e\}$ with $F_1 \Delta \{e, f\} \in \mathcal{F}$ and $F_2 \Delta \{e, f\} \in \mathcal{F}$,

which obviously strengthens (SEA). We call $\underline{M} := \left(E, p^{-1}(K^*)\right)$ the underlying Δ-matroid of $M = M_p$.

ii) For $K = K_c$ as in Example ii) following Definition 2.3, axiom (TP2) is clearly equivalent to the strong symmetric exchange condition. In [24], it is shown that a Δ-matroid is even if and only if it satisfies (SSEA). Therefore, K_c is an appropriate coefficient domain for even Δ-matroids; that is, every even Δ-matroid may be interpreted as a Δ-matroid with coefficients in K_c.

iii) Assume K is a fuzzy ring, $\alpha \in K^*$, $I_0 \subseteq E = \{1, ..., n\}$, and $p, p' : \mathcal{P}(E) \to K^* \dot\cup \{0\}$ are two maps related by the formula

$$p'(I) = \alpha \cdot p(I \Delta I_0) \text{ for all } I \subseteq E.$$

Then p is a twisted Pfaffian map if and only if p' is a twisted Pfaffian map. Note that for $I_1, I_2 \subseteq E$ one has $(I_1 \Delta I_0) \Delta (I_2 \Delta I_0) = I_1 \Delta I_2$.

iv) If K is a field, then by [23, Theorem 2.9] a Δ-matroid $(E, \mathcal{F})$ is representable over K by some skew-symmetric matrix if and only if $(E, \mathcal{F})$ is the underlying Δ-matroid of M_p for some twisted Pfaffian map $p : \mathcal{P}(E) \to K$. The correspondence is as follows: If $\mathcal{F} \Delta I_0 = \mathcal{F}(A)$ for some $I \subseteq E$ and some skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le n}$, then – according to the last remark – we may assume $I_0 = \emptyset$, and a corresponding twisted Pfaffian map $p = p_A$ is then given in terms of the Pfaffian form Pf; namely $p(I) := Pf\left((a_{ij})_{i,j \in I}\right)$ for $I \subseteq E$.

Vice versa, if $p : \mathcal{P}(E) \to K$ is a twisted Pfaffian map with $p(\emptyset) = 1$, then it is easily verified that the matrix $A_p = A = (a_{ij})_{1 \le i,j \le n}$ given by

$$a_{ii} := 0 \text{ for } 1 \le i \le n,$$
$$a_{ij} = -a_{ji} = p(\{i, j\}) \text{ for } 1 \le i < j \le n$$

is a representation of $\underline{M_p}$.

For more details see [22, §2] and [12], where it is also shown that for any skew-symmetric matrix $A = (a_{ij})_{1 \le i,j \le n}$ the corresponding twisted Pfaffian map $p = p_A$ satisfies (TP2).

v) In [23, Proposition 2.15], it is verified that for a given even Δ-matroid $(E, \mathcal{F})$ and a given fuzzy ring K the existence of some twisted Pfaffian map $p : \mathcal{P}(E) \to K^* \dot\cup \{0\}$ with $p^{-1}(K^*) = \mathcal{F}$ does not depend on the natural order on the set $E = \{1, ..., n\}$.

In the rest of this section, we are concerned with the concept of an antipodal graph as established in [1], [16], [20]. First, we state the following

Definition 2.7. *Assume $\Gamma = (V, \mathcal{K})$ is a graph.*

i) *The map $d_\Gamma : V \times V \to \mathbf{N}_0 \cup \{\infty\}$, induced by Γ, is given by*

$$d_\Gamma(v, v') := \begin{cases} \infty \text{ if } v \text{ and } v' \text{ lie in distinct connected components of } \Gamma \\ \min\{\ell \in \mathbf{N}_0 \mid \text{there exist } v_0, ..., v_\ell \in V \text{ with } v_0 = v, v_\ell = v' \\ \qquad \text{and } \{v_{i-1}, v_i\} \in \mathcal{K} \text{ for } 1 \le i \le \ell\} \text{ otherwise.} \end{cases} \tag{4}$$

ii) *For $v, v' \in V$ with $d_\Gamma(v, v') < \infty$ the hull $\langle v, v' \rangle_\Gamma$ is defined by*

$$\langle v, v' \rangle_\Gamma := \{v'' \in V \mid d_\Gamma(v, v') = d_\Gamma(v, v'') + d_\Gamma(v'', v')\}.$$

iii) *If Γ is connected, then the diameter $diam\Gamma$ of Γ is given by*

$$diam\Gamma := \sup\{d_\Gamma(v, w) \mid v, w \in V\}.$$

Definition 2.8. *Assume $\Gamma = (V, \mathcal{K})$ is a connected graph.*

i) *Γ is even, if for every $v \in V$ there exists a unique $\widehat{v} \in V$ with $d_\Gamma\left(v, \widehat{v}\right) = diam\Gamma$.*

ii) *An even graph Γ is antipodal, if for all $v \in V$ and $\widehat{v}$ as in* i) *one has $\left\langle v, \widehat{v} \right\rangle_\Gamma = V$. In this case, for $v \in V$ the vertex $\widehat{v}$ given in* i) *is called the antipode of v.*

Remarks:

i) By definition, the diameter of an antipodal graph is finite.

ii) If $\Gamma = (V, \mathcal{K})$ is an antipodal graph, then it is easily verified that the map $\alpha : V \to V$ given by $\alpha(v) := \widehat{v}$ is an automorphism of Γ, which satisfies $\alpha^2 = id_V$. However, in [16, Figure 2] there are listed some even graphs for which the map α is not an automorphism.

By unifying the concepts of matroids with coefficients and of Δ-matroids with coefficients, we shall be concerned with antipodal subgraphs of certain graphs of finite diameter.

Definition 2.9. *A connected graph $\Gamma = (V, \mathcal{K})$ is convex-antipodal if for all $v_1, v_2 \in V$ the induced subgraph $\Gamma_{\langle v_1, v_2 \rangle}$ of Γ on the vertex set $\langle v_1, v_2 \rangle := \langle v_1, v_2 \rangle_\Gamma$ is finite and antipodal. In this case the antipode of $v \in \langle v_1, v_2 \rangle$ in $\Gamma_{\langle v_1, v_2 \rangle}$ is denoted by $ap_{\langle v_1, v_2 \rangle}(v)$.*

Remark. Assume the graph $\Gamma = (V, \mathcal{K})$ is convex-antipodal, and suppose $v_1, v_2 \in V$. Then it is easily verified that for $v \in \langle v_1, v_2 \rangle$ one has $\langle v_1, v_2 \rangle = \left\langle v, ap_{\langle v_1, v_2 \rangle}(v) \right\rangle$ and that $\langle v_1, v_2 \rangle$ is a convex set in the metric space (V, d_Γ); that means, for $v, v' \in \langle v_1, v_2 \rangle$ one has $\langle v, v' \rangle \subseteq \langle v_1, v_2 \rangle$. (See also [25, Lemma 3.9] for more details.)

Examples:

i) Assume $m \in \mathbf{N}_0$ and E is some – possibly infinite – set with at least m elements. Let $\Gamma_1 = (V(\Gamma_1), \mathcal{K}(\Gamma_1))$ denote the graph with $V(\Gamma_1) = \binom{E}{m}$ and

$$\mathcal{K}(\Gamma_1) := \left\{ \{A, B\} \subseteq \binom{E}{m} \,\middle|\, \#(A \cap B) = m - 1 \right\}. \tag{5}$$

Then Γ_1 is convex-antipodal. For $A, B \in \binom{E}{m}$ one has

$$\langle A, B \rangle := \langle A, B \rangle_{\Gamma_1} = \left\{ C \in \binom{E}{m} \,\middle|\, A \cap B \subseteq C \subseteq A \cup B \right\},$$

and for $C \in \langle A, B\rangle$ the antipode $ap_{\langle A,B\rangle}(C)$ is given by

$$ap_{\langle A,B\rangle}(C) = (A \cup B) \setminus (C \setminus (A \cap B)) = A \Delta B \Delta C.$$

We shall be concerned with the graph Γ_1 when studying matroids with coefficients defined on E and of rank m.

ii) Assume E is some finite set, and let $\Gamma_2 = (V(\Gamma_2), \mathcal{K}(\Gamma_2))$ denote the bipartite graph with $V(\Gamma_2) = \mathcal{P}(E)$ and

$$\mathcal{K}(\Gamma_2) := \{\{A, B\} \subseteq \mathcal{P}(E) \mid \#(A\Delta B) = 1\}. \tag{6}$$

Moreover, for $\nu \in \{1,2\}$ let $G_\nu = (V(G_\nu), \mathcal{K}(G_\nu))$ denote the graph with $V(G_\nu) = \{A \subseteq E \mid \#A \equiv \nu \, mod\, 2\}$ and

$$\mathcal{K}(G_\nu) := \{\{A, B\} \subseteq V(G_\nu) \mid \#(A\Delta B) = 2\}. \tag{7}$$

Then the graphs Γ_2, G_1 and G_2 are convex-antipodal.
For $A, B \in \mathcal{P}(E)$ one has

$$\langle A, B\rangle := \langle A, B\rangle_{\Gamma_2} = \{C \subseteq E \mid A \cap B \subseteq C \subseteq A \cup B\},$$

and for $C \in \langle A, B\rangle$ one has, similarly as in Example i)

$$ap_{\langle A,B\rangle}(C) = (A \cup B) \setminus (C \setminus (A \cap B)) = A \Delta B \Delta C.$$

If, moreover, $A, B \in V(G_\nu)$ for one and the same $\nu \in \{1,2\}$, then one has

$$\langle A, B\rangle_{G_\nu} = \langle A, B\rangle_{\Gamma_2} \cap V(G_\nu)$$

and

$$ap_{\langle A,B\rangle_{G_\nu}}(C) = ap_{\langle A,B\rangle_{\Gamma_2}}(C) \text{ for } C \in \langle A, B\rangle_{G_\nu} .$$

We shall have to study the graphs Γ_2, G_1 and G_2 in the case of Δ-matroids with coefficients defined on E.

Throughout the rest of this paper assume that the graphs Γ_1, Γ_2, G_1 and G_2 are as in the last examples, where m and E are fixed. At the end of this section, we recall the concepts of the base graph of a matroid and of the base graph of an even Δ-matroid.

Definition 2.10:

i) *Assume M is a matroid (with or without coefficients) defined on E and of rank m and with $\mathcal{B}$ as its set of bases. The base graph of M is the graph $\Gamma_M = (\mathcal{B}, \mathcal{K})$ with $\mathcal{B}$ as its set of vertices and*

$$\mathcal{K} := \{\{B_1, B_2\} \mid B_1, B_2 \in \mathcal{B}, \#(B_1 \cap B_2) = m - 1\}$$

as its set of edges; that is, Γ_M *is the induced subgraph of* Γ_1 *on the base set* $\mathcal{B}$.

ii) *Assume* $M = (E, \mathcal{F})$ *is an even* Δ*-matroid. The base graph of* M *is the graph* $\Gamma_M = \left(\mathcal{F}, \widetilde{\mathcal{K}}\right)$ *with* $\mathcal{F}$ *as its set of vertices and*

$$\widetilde{\mathcal{K}} := \{\{F_1, F_2\} \mid F_1, F_2 \in \mathcal{F}, \# (F_1 \Delta F_2) = 2\}$$

as its set of edges.

Note that Definition 2.10 i) is a special case of Definition 2.10 ii) and that the base graph of an even Δ-matroid $(E, \mathcal{F})$ is an induced subgraph either of G_1 or of G_2.

3. The Basic Concept of a $(W; P; U)$-Matroid with Coefficients

In order to unify the concepts of matroids with coefficients and of Δ-matroids with coefficients in a graph theoretical framework, we shall study graphs arising from Coxeter groups.

In the sequel, let W denote a Coxeter group, say

$$W = \left\langle R \mid (rs)^{m(r,s)} = 1,\ r, s, \in R \right\rangle$$

for some $R \subseteq W$, the set of generators of W, and $1 \leq m(r, s) = m(s, r) \leq \infty$ for all $r, s \in R$, where $m(r, s) = 1$ if and only if $r = s$.

For $P \subseteq R$ let

$$W_P := \langle P \rangle \tag{8}$$

denote the parabolic subgroup of W generated by P, and let

$$W^P := W/W_P = \{w \cdot W_P \mid w \in W\} \tag{9}$$

denote the set of left cosets, regarded as subsets of W.

Definition 3.1. *Assume* W, P *are as above and put*

$$\mathcal{K} := \left\{ \{w_1 \cdot W_P, w_2 \cdot W_P\} \in \binom{W^P}{2} \,\middle|\, \begin{array}{l} W_P \cdot w_1^{-1} \cdot w_2 \cdot W_P = \\ \qquad W_P \cdot r \cdot W_P \text{ for some } r \in R \setminus P \end{array} \right\}. \tag{10}$$

i) *The* $(W; P)$*-graph* $\Gamma := \left(W^P, \mathcal{K}\right)$ *has* W^P *as its set of vertices and* $\mathcal{K}$ *as its set of edges.*
For $\emptyset \neq L \subseteq W^P$ *let* Γ_L *denote the induced subgraph of* Γ *on the set* L *of vertices.*

ii) *Assume* U *is a – not necessarily parabolic – subgroup of* W *with* $W_P \leq U$*. Write* $\overline{w} := w \cdot W_P$ *for* $w \in W$ *and put*

$$\overline{U} := U^P := U/W_P = \{\overline{u} \mid u \in U\},$$
$$\overline{w \cdot U} := w \cdot U^P = \{\overline{w \cdot u} \mid u \in U\} \text{ for } w \in W.$$

For $k \geq 1$ the triple $(W;P;U)$ is called k-regular if the following axiom holds:

(U_k) *For $u_1, u_2 \in U$ with $\overline{u_1} \neq \overline{u_2}$ one has*
$\langle \overline{u_1}, \overline{u_2} \rangle_\Gamma \cap U^P = \{ \overline{w} \in \langle \overline{u_1}, \overline{u_2} \rangle_\Gamma \mid d_\Gamma(\overline{u_1}, \overline{w}) \equiv d_\Gamma(\overline{u_2}, \overline{w}) \equiv 0 \, mod \, k \}$.

iii) *Assume $w \in W$ and the triple $(W;P;U)$ is k-regular. Put*

$$\mathcal{K}_{\overline{w \cdot U}} := \{ \{ \overline{w \cdot u_1}, \overline{w \cdot u_2} \} \mid u_1, u_2 \in U, \, d_\Gamma(\overline{w \cdot u_1}, \overline{w \cdot u_2}) = d_\Gamma(\bar{u}_1, \bar{u}_2) = k \}. \quad (11)$$

Then the $(W;P;w \cdot U)$-graph $\Gamma_{\overline{w \cdot U}} := \left(\overline{w \cdot U}, \mathcal{K}_{\overline{w \cdot U}} \right)$ has $\overline{w \cdot U}$ as its set of vertices and $\mathcal{K}_{\overline{w \cdot U}}$ as its set of edges.
For $\emptyset \neq L \subseteq w \cdot U^P$ let $\Gamma_{\overline{w \cdot U}, L}$ denote the induced subgraph of $\Gamma_{\overline{w \cdot U}}$ on the set L of vertices.

In the sequel, $\Gamma_{\overline{U}}$ will – of course – mean the graph $\Gamma_{\overline{1 \cdot U}}$.

Remarks:

i) Since any $w \in W$ is a finite product of elements in R, the $(W;P)$-graph Γ is trivially connected.

ii) If the triple $(W;P;U)$ is k-regular, then one has $d_\Gamma(\overline{u_1}, \overline{u_2}) \equiv 0 \, mod \, k$ for all $u_1, u_2 \in U$; this follows directly from axiom (U_k).

iii) If $U = W_P$, then by definition the triple $(W;P;U)$ is k-regular for every $k \in \mathbf{N}$. In this case, for any $w \in W$ the $(W;P;w \cdot U)$-graph has only one vertex. However, if $W_P \subsetneqq U$, then the triple $(W;P;U)$ is k-regular for at most one $k \in \mathbf{N}$. The triple $(W;P;W)$ is certainly 1-regular, and one has $\Gamma_{\overline{W}} = \Gamma$.

iv) For $w_1, w_2, w \in W$ one has $\{ w \cdot w_1 \cdot W_P, w \cdot w_2 \cdot W_P \} \in \mathcal{K}$ if and only if $\{ w_1 \cdot W_P, w_2 \cdot W_P \} \in \mathcal{K}$. Therefore, any $w \in W$ induces an automorphism of Γ in the obvious way. Moreover, for a k-regular triple $(W;P;U)$ every $w \in W$ induces an isomorphism from $\Gamma_{\overline{U}}$ onto $\Gamma_{\overline{w \cdot U}}$. If $w, w' \in W$, then $\Gamma_{\overline{w \cdot U}} = \Gamma_{\overline{w' \cdot U}}$ implies $d_\Gamma(\overline{w}, \overline{w'}) \equiv 0 \, mod \, k$. In case $W_P \subsetneqq U$ this means in particular that $[W : U] \geq k$ because there exists a path $(\overline{w_0}, ..., \overline{w_k})$ in Γ with $\overline{w_0} \neq \overline{w_k}$ but $w_0 \cdot U = w_k \cdot U$, and the k cosets $w_1 \cdot U, ..., w_k \cdot U$ of U in W are pairwise distinct. It should be remarked that one has not necessarily $[W : U] = k$ even if $U \neq W_P$; see, for instance, [25, Proposition 3.6].

We have the following

Lemma 3.2. *Assume the triple $(W;P;U)$ is k-regular for some $k \geq 1$, suppose $w \in W$, and put $\Gamma' := \Gamma_{\overline{w \cdot U}}$. Then, for all $w_1, w_2 \in w \cdot U$ one has*

$$k \cdot d_{\Gamma'}(\overline{w_1}, \overline{w_2}) = d_\Gamma(\overline{w_1}, \overline{w_2}), \quad (12)$$

$$\langle \overline{w_1}, \overline{w_2} \rangle_{\Gamma'} = \langle \overline{w_1}, \overline{w_2} \rangle_\Gamma \cap \overline{w \cdot U}. \quad (13)$$

In particular, Γ' is connected.

Proof. This is [25, Lemma 3.4]. □

The following result provides a rather large class of 2-regular triples $(W;P;U)$ and will also encompass the important class of Δ-matroids with coefficients.

Proposition 3.3. *Assume* $W_P \neq W$ *and the* $(W;P)$*-graph* Γ *is bipartite. Let* $W^P = V_1 \dot\cup V_2$ *denote the unique partition of* W^P *with* $\overline{1} \in V_1$ *and* $d_\Gamma(\overline{w_i}, \overline{w_j}) \equiv i-j \bmod 2$ *for* $i,j \in \{1,2\}$, $\overline{w_i} \in V_i$, $\overline{w_j} \in V_j$*. Then the set*

$$U := \{u \in W \mid \overline{u} \in V_1\} = \left\{u \in W \mid d_\Gamma\left(\overline{1}, \overline{u}\right) \equiv 0 \bmod 2\right\} \tag{14}$$

is a subgroup of W *of index* 2*, and the triple* $(W;P;U)$ *is* 2*-regular.*

Proof. This is [25, Proposition 3.5]. □

Examples:

i) Assume $n \in \mathbf{N}$, let $W = W_1 = S_n$ denote the group of permutations of $E = I_n := \{1, ..., n\}$, $R = R_1$ the set of transpositions $(i \quad i+1)$ for $1 \leq i \leq n-1$, choose some fixed m with $0 < m < n$, and put

$$P = P_1 := P_{(m,n)} := R_1 \setminus \{(m \ \ m+1)\}.$$

Then one has

$$W_P = \{w \in W \mid w(I_m) = I_m\}.$$

Therefore, we have a bijection $\chi : W_1^{P_1} \to \binom{E}{m}$ from the set of left cosets of $\langle P_{(m,n)} \rangle$ in S_n onto the m-subsets of E, given by

$$\chi(w \cdot W_P) := w(I_m).$$

For $A \in \binom{E}{m}$ one has $\#(A \cap I_m) = m-1$ if and only if $A = w_0((I_m \setminus \{m\}) \cup \{m+1\})$ for some $w_0 \in W_P$. Thus for $B_1, B_2 \in \binom{E}{m}$ one has $\#(B_1 \cap B_2) = m-1$ if and only if there exist $w \in W$ and $w_0 \in W_P$ with $B_1 = w(I_m)$ and $B_2 = (w \circ w_0 \circ (m \quad m+1))(I_m)$. By Definition 3.1 this means, in view of the identification of $W_1^{P_1}$ with $\binom{E}{m}$ via the bijection χ that the $(W_1; P_1)$-graph is nothing but the graph Γ_1 considered in Example i) following Definition 2.9. Moreover, for an ordinary matroid M defined on E and of rank m with $\mathcal{B}$ as its set of bases the graph Γ_L for $L := \chi^{-1}(\mathcal{B})$ is just the base graph of M.

ii) Assume again $n \in \mathbf{N}$, $E = I_n := \{1, ..., n\}$, and

$$\widetilde{E} := \{1, ..., n\} \dot\cup \{1^*, ..., n^*\};$$

thus $\widetilde{E}$ consists of E together with a disjoint copy of E. For $k \in E$ put $(k^*)^* := k$; then one has $(k^*)^* = k$ for every $k \in \widetilde{E}$. A *transversal* in $\widetilde{E}$ is any subset $T \subseteq \widetilde{E}$ with $\#(\{k, k^*\} \cap T) = 1$ for all $k \in E$.

Let $W = W_2$ denote the group of permutations of $\widetilde{E}$ which map transversals onto transversals, or, equivalently, with $w(k^*) = (w(k))^*$ for every $k \in \widetilde{E}$. Then W is a Coxeter group with

$$R = R_2 := \left\{(i \quad i+1) \circ \left(i^* \quad (i+1)^*\right) \mid 1 \leq i \leq n-1\right\} \dot{\cup} \{(n \quad n^*)\}$$

as an appropriate set of generators. Put

$$P = P_2 := R \setminus \{(n \quad n^*)\}.$$

Then one has

$$W_P = \{w \in W \mid w(E) = E\},$$

and thus we have bijections $\chi' : W_2^{P_2} \to \mathcal{T}$, $\chi'' : W_2^{P_2} \to \mathcal{P}(E)$ onto the set $\mathcal{T}$ of transversals in $\widetilde{E}$ or the power set of E, respectively, given by

$$\begin{aligned} \chi'(w \cdot W_P) &:= w(E), \\ \chi''(w \cdot W_P) &:= w(E) \cap E. \end{aligned}$$

We show that the edge set $\mathcal{K}$ of the $(W;P)$-graph satisfies

$$\mathcal{K} = \left\{\left\{\chi'^{-1}(T_1), \chi'^{-1}(T_2)\right\} \mid T_1, T_2 \in \mathcal{T}, \#(T_1 \triangle T_2) = 2\right\}, \tag{15}$$

$$\mathcal{K} = \left\{\left\{\chi''^{-1}(A), \chi''^{-1}(B)\right\} \mid A, B \subseteq E, \#(A \triangle B) = 1\right\}. \tag{16}$$

It suffices to verify (15), because (15) and (16) are trivially equivalent. Assume first that $T_1, T_2 \in \mathcal{T}$ satisfy $\#(T_1 \triangle T_2) = 2$, and choose some $w \in W$ with $T_1 = w(E)$. Then there exists some $e \in E$ with $T_2 = w(E \triangle \{e, e^*\})$. Choose some $w_0 \in W_P$ with $w_0(n) = e$ and $w_0(n^*) = e^*$. Then one has $T_2 = (w \circ w_0 \circ (n \quad n^*))(E)$ and thus $\left\{\chi'^{-1}(T_1), \chi'^{-1}(T_2)\right\} \in \mathcal{K}$.
If vice versa $T_1, T_2 \in \mathcal{T}$ correspond to neighbours in the $(W;P)$-graph Γ, choose some $w \in W$ and some $w_0 \in W_P$ with $w(E) = T_1$ and $(w \circ w_0 \circ (n \quad n^*))(E) = T_2$. Then one has

$$\#(E \triangle w_0((E \setminus \{n\}) \cup \{n^*\})) = \#(\{w_0(n), w_0(n^*)\}) = 2$$

and thus also $\#(T_1 \triangle T_2) = 2$.
Now (16) shows that the $(W_2;P_2)$-graph is – via the bijection χ'' – nothing but the graph Γ_2 studied in Example ii) following Definition 2.9. Moreover, it is trivial that Γ_2 is a bipartite graph and that with the notations in the example just mentioned any $\{\overline{w_1}, \overline{w_2}\} \in \mathcal{K}$ satisfies $\chi''(\overline{w_1}) \in V(G_i)$, $\chi''(\overline{w_2}) \in V(G_j)$, where $\{i,j\} = \{1,2\}$. If we thus put

$$\begin{aligned} U_\Delta :&= \{w \in W \mid \#(E \setminus w(E)) \equiv 0 \bmod 2\} \\ &= \{w \in W \mid \#(w(E) \cap E) \equiv \#E \bmod 2\}, \end{aligned} \tag{17}$$

then Proposition 3.3 shows that the triple $(W_2;P_2;U_\Delta)$ is 2-regular. It is now also clear that for $U := U_\Delta$ the distinct graphs $\Gamma_{\overline{w \cdot U}}$, $w \in W$, are precisely the graphs G_1

and G_2 and that for an even Δ-matroid $M = (E, \mathcal{F})$ the base graph Γ_M is the graph $\Gamma_{\overline{w \cdot U}, L}$, where L:=$\chi''^{-1}(\mathcal{F})$ and $w \in W$ satisfies $L \subseteq \overline{w \cdot U}$.

For further examples which arise also from combinatorial geometry, but which are not closely related to classical matroid theory, see [25, §3].

I would like to establish a theory in which Maurer's homotopy theory as developed for ordinary matroids in [19, §5] will also hold. To this end, I propose the following

Definition 3.4. *Assume $G = (V(G), \mathcal{K}(G))$ is a graph.*

i) *If $V' \subseteq V(G)$ and $\mathcal{K}' \subseteq \mathcal{K}(G) \cap \binom{V'}{2}$, then the graph $G' := (V', \mathcal{K}')$ is isometric in G, if for all $v_1, v_2 \in V'$ one has $d_{G'}(v_1, v_2) = d_G(v_1, v_2)$.*
ii) *A reentrant[2] path $(v_0, ..., v_\ell)$ is called isometric in G, if $v_i = v_j$ only for $i \equiv j \bmod \ell$ and the cycle $G' = (V', \mathcal{K}')$ with $V' := \{v_0, ..., v_\ell\}$, $\mathcal{K}' := \{\{v_i, v_j\} \mid i \equiv j + 1 \bmod \ell\}$ is isometric in G.*
iii) *The graph G satisfies the Maurer Condition if for any two isometric reentrant paths $(v_0, ..., v_\ell)$, $(v'_0, ..., v'_\ell)$ in G of length $\ell \geq 5$ with $v_i = v'_i$ for $0 \leq i \leq \ell - 2$ one has either $v'_{\ell-1} = v_{\ell-1}$ or $d_G(v_{\ell-1}, v'_{\ell-1}) = 1$.*

We have

Proposition 3.5. *Assume $G = (V(G), \mathcal{K}(G))$ is a graph.*

i) *If G satisfies the Maurer Condition and H is an isometric subgraph in G, then H satisfies the Maurer Condition, too.*
ii) *Suppose the connected components of G are convex-antipodal. Then G satisfies the Maurer Condition if for every odd $\ell \geq 5$ and any two isometric reentrant paths $(v_0, ..., v_\ell), (v'_0, ..., v'_\ell)$ in G of length ℓ with $v_i = v'_i$ for $0 \leq i \leq \ell - 2$ one has either $v'_{\ell-1} = v_{\ell-1}$ or $d_G(v_{\ell-1}, v'_{\ell-1}) = 1$. In particular, every convex-antipodal graph in which each cycle has an even length satisfies the Maurer Condition.*

Proof. This is [25, Proposition 4.2]. i) follows easily from the fact that every reentrant path which is isometric in H is isometric in G, too. ii) holds, because for two isometric reentrant paths $(v_0, ..., v_\ell)$, $(v'_0, ..., v'_\ell)$ in G of even length $\ell = 2h \geq 6$ with $v_i = v'_i$ for $0 \leq i \leq \ell - 2$ one gets:

$$v_{\ell-1} = ap_{\langle v_0, v_h \rangle}(v_{h-1}) = v'_{\ell-1}.$$ □

The next result shows that the base graphs of matroids and – more generally – of even Δ-matroids satisfy the Maurer Condition.

Proposition 3.6:

i) *Let $\widetilde{\Gamma}_2$ denote the graph whose connected components are G_1 and G_2. Then $\widetilde{\Gamma}_2$ satisfies the Maurer Condition.*

[2] "Reentrant" means that $v_0 = v_\ell$.

ii) *The base graph of an even Δ-matroid satisfies the Maurer Condition.*

iii) *The graph Γ_1 as well as all base graphs of ordinary matroids satisfy the Maurer Condition.*

Proof:

i) is [25, Proposition 4.3].

ii) As it follows directly from the strong symmetric exchange condition, the base graph of an even Δ-matroid is isometric in $\widetilde{\Gamma}_2$. Thus ii) follows from i) and Proposition 3.5 i).

iii) is a special case of ii). □

In the sequel, assume $k \geq 1$ and $(W; P; U)$ is a k-regular triple where $W = \langle R \rangle$ as above, and let $\widetilde{\Gamma} = \Gamma^U$ denote the graph with vertex set W^P whose connected components are precisely the distinct graphs $\Gamma_{\overline{w \cdot U}}$; this means that $\widetilde{\Gamma}$ has the same vertex set as the $(W; P)$-graph Γ but $[W : U]$ connected components. For $w_1, w_2 \in W$ with $d_{\widetilde{\Gamma}}(\overline{w_1}, \overline{w_2}) < \infty$ one has of course

$$\langle \overline{w_1}, \overline{w_2} \rangle_{\widetilde{\Gamma}} = \langle \overline{w_1}, \overline{w_2} \rangle_{\Gamma_{\overline{w_1 \cdot U}}} .$$

Moreover, Lemma 3.2 implies trivially the following

Lemma 3.7. *Assume the $(W; P)$-graph Γ is convex-antipodal. Then the connected components of $\widetilde{\Gamma}$ are also convex-antipodal, and for $w \in W$ and $w_1, w_2 \in w \cdot U$ with $\overline{w} \in \langle \overline{w_1}, \overline{w_2} \rangle_{\widetilde{\Gamma}}$ one has*

$$ap_{\langle \overline{w_1}, \overline{w_2} \rangle_{\widetilde{\Gamma}}}(\overline{w}) = ap_{\langle \overline{w_1}, \overline{w_2} \rangle_{\Gamma}}(\overline{w}) .$$

With the intent to unify the Grassmann-Plücker relations and the relations (TP2) for twisted Pfaffian maps, I propose the following

Definition 3.8. *Assume $A_0, ..., A_\ell$ are vertices in $\widetilde{\Gamma}$ with $d_{\widetilde{\Gamma}}(A_i, A_j) = 1$ for $0 \leq i < j \leq \ell$, and put $\mathcal{N} := \{A_0, ..., A_\ell\}$.*

i) *Let $k_0 \in \{0, 1\}$ denote the unique integer satisfying $k_0 \equiv k \bmod 2$. The set $\mathcal{N}$ is called star-shaped in Γ if there exist vertices $A_0', ..., A_\ell' \in W^P$ with $d_\Gamma(A_i, A_i') = \left[\frac{k}{2}\right] = \frac{1}{2} \cdot (k - k_0)$ for $0 \leq i \leq \ell$ and $d_\Gamma\left(A_i', A_j'\right) = k_0$ for $0 \leq i < j \leq \ell$.*

ii) *Assume $B \in W^P$ satisfies $d_{\widetilde{\Gamma}}(A_0, B) < \infty$. Then $\mathcal{N}$ is called a fundamental neighbourhood of A_0 in $\langle A_0, B \rangle_{\widetilde{\Gamma}}$ relative to Γ if $\mathcal{N}$ is – with respect to inclusion – a maximal star-shaped subset of $\langle A_0, B \rangle_{\widetilde{\Gamma}}$.*

In case $k = 1$ any subset $\mathcal{N} = \{A_0, ..., A_\ell\} \subseteq W^P$ with $d_{\widetilde{\Gamma}}(A_i, A_j) = 1$ for $0 \leq i < j \leq \ell$ is star-shaped in Γ, one just has to put $A_i' := A_i$ for $0 \leq i \leq \ell$. In this case we shall also omit the term *relative to* Γ by speaking about fundamental neighbourhoods, because for $k = 1$ and $A, B \in W^P$ with $d_{\widetilde{\Gamma}}(A, B) < \infty$ one has $\langle A, B \rangle_{\widetilde{\Gamma}} = \langle A, B \rangle_\Gamma$ by axiom (U_1). (See Definition 3.1 ii).)

If $k \equiv 0 \bmod 2$, then Definition 3.8 i) means of course that $A_i' = A_j'$ for $0 \leq i < j \leq \ell$. Any subset $\{A_0, A_1\} \subseteq W^P$ with $d_{\widetilde{\Gamma}}(A_0, A_1) = 1$ is certainly star-shaped in Γ.

The following figures illustrate star-shaped sets $\{A_0, A_1, A_2\}$ in the cases $k = 2$ and $k = 3$.

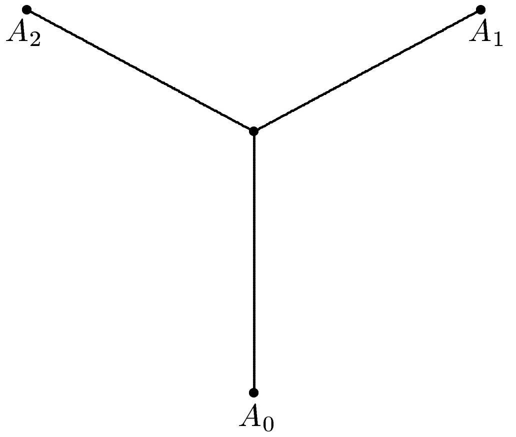

Figure 1

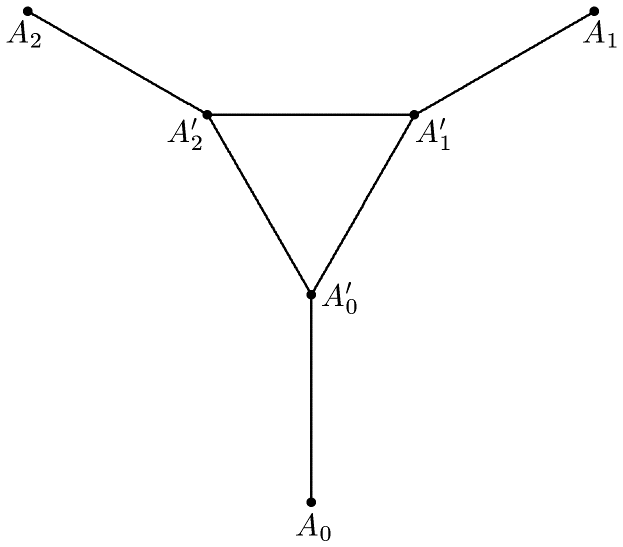

Figure 2

As a last technical preparation we have to consider partitions of the set of triangles in the graph $\widetilde{\Gamma}$. Recall that W operates canonically on Γ as well as on $\widetilde{\Gamma}$.

Definition 3.9. *Assume that the connected components of* $\widetilde{\Gamma} = \Gamma^U$ *are convex-antipodal, put* $d_0 := d_{\widetilde{\Gamma}}$, *let* $\mathcal{K}(\widetilde{\Gamma})$ *denote the set of edges of* $\widetilde{\Gamma}$ *and*

$$\mathcal{D} := \left\{\{A_1, A_2, A_3\} \subseteq W^P \mid d_0(A_i, A_j) = 1 \textit{ for } 1 \leq i < j \leq 3\right\}, \tag{18}$$

$$V := \left\{\begin{matrix} \{A_1, A_2, A_3, A_4\} \subseteq W^P \mid d_0(A_i, A_{i+1}) = 1 \textit{ for } i \bmod 4, \\ d_0(A_1, A_3) = d_0(A_2, A_4) = 2 \end{matrix}\right\} \tag{19}$$

the set of triangles or pure[3] *quadrangles in* $\widetilde{\Gamma}$, *respectively.*

i) *A map* $\eta : \mathcal{K}(\widetilde{\Gamma}) \to \{1, -1\}$ *is called four-balanced if for all* $\{A_1, A_2, A_3, A_4\}$ *as in* (19) *one has*

$$\eta(\{A_1, A_2\}) \cdot \eta(\{A_2, A_3\}) \cdot \eta(\{A_3, A_4\}) \cdot \eta(\{A_4, A_1\}) = 1.$$

ii) *Assume* $\mathcal{D} = \mathcal{D}_1 \dot{\cup} \mathcal{D}_2$ *is a partition of* $\mathcal{D}$ *and put*

$$sign_{(\mathcal{D}_1, \mathcal{D}_2)}(\{A_1, A_2, A_3\}) := \begin{cases} 1 & \textit{for } \{A_1, A_2, A_3\} \in \mathcal{D}_1 \\ -1 & \textit{for } \{A_1, A_2, A_3\} \in \mathcal{D}_2 \end{cases}.$$

The partition $\mathcal{D} = \mathcal{D}_1 \dot{\cup} \mathcal{D}_2$ *is called matroidal if the following three axioms hold:*

---

[3] *Pure* means that the diagonals are no edges.

(D1) *There exists some four-balanced map* $\eta : \mathcal{K}(\widetilde{\Gamma}) \to \{1, -1\}$ *such that for all* $\{A_1, A_2, A_3\} \in \mathcal{D}$ *one has*

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\}) = \eta(\{A_1, A_2\}) \cdot \eta(\{A_2, A_3\}) \cdot \eta(\{A_3, A_1\}).$$

(D2) *If* $A, B \in W^P$ *satisfy* $d_0(A, B) < \infty$ *and if* $A_1, A_2, A_3 \in \langle A, B\rangle := \langle A, B\rangle_{\widetilde{\Gamma}}$ *are the vertices of a triangle in* $\mathcal{D}$ *and* $\widehat{A_i} := ap_{\langle A,B\rangle}(A_i)$ *for* $1 \le i \le 3$, *then one has*

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}\left(\left\{\widehat{A_1}, \widehat{A_2}, \widehat{A_3}\right\}\right) = -sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\}).$$

(D3) *Assume* $r \in R$ *is some involution and* $\{A_1, A_2, A_3\} \in \mathcal{D}$. *If for all* i, j *with* $1 \le i < j \le 3$ *one has* $\{A_i, A_j\} \ne \{rA_i, rA_j\}$, *then*

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{rA_1, rA_2, rA_3\}) = sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\}).$$

If otherwise $\{i, j, h\} = \{1, 2, 3\}$ *and* $\{A_i, A_j\} = \{rA_i, rA_j\}$, *but* $rA_h \ne A_h$, *and* $r_0 : \langle A_i, A_j\rangle_\Gamma \setminus \{A_i, A_j\} \to \langle A_i, A_j\rangle_\Gamma \setminus \{A_i, A_j\}$ *denotes the permutation given by the restriction of* r *on* $\langle A_i, A_j\rangle_\Gamma \setminus \{A_i, A_j\}$, *then one has*

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{rA_1, rA_2, rA_3\}) = sign\ r_0 \cdot sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\}).$$

Remarks:

i) Whether a partition $\mathcal{D} = \mathcal{D}_1 \dot\cup \mathcal{D}_2$ is matroidal or not does, of course, not depend on the order of $\mathcal{D}_1$ and $\mathcal{D}_2$. By definition, one has $sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\}) = -sign_{(\mathcal{D}_2,\mathcal{D}_1)}(\{A_1, A_2, A_3\})$ for all $\{A_1, A_2, A_3\} \in \mathcal{D}$. Thus, by exchanging the rôles of $\mathcal{D}_1$ and $\mathcal{D}_2$ one may replace the map η in (D1) by the map $-\eta$.

ii) Clearly, in case $\mathcal{D} = \emptyset$ the empty partition is matroidal. In (D1) we can choose the constant map $\eta \equiv 1$.
If $\#\mathcal{D} = 1$, then the fact that W operates transitively on W^P implies trivially that $[W : W_P] = 3$, and thus Γ as well as $\widetilde{\Gamma}$ is the cycle of length 3. The partitions $\mathcal{D} = \mathcal{D} \dot\cup \emptyset$ and $\mathcal{D} = \emptyset \dot\cup \mathcal{D}$ are certainly matroidal in this case.

iii) If the triple $(W; P; U)$ is 1-regular, then any map r_0 as in (D3) is the “unique permutation of the empty set” and satisfies $sign\, r_0 = 1$.

As a direct consequence of axiom (D3), the last Remark, and the fact that R generates W we get the following

Lemma 3.10. *Assume the partition* $\mathcal{D} = \mathcal{D}_1 \dot\cup \mathcal{D}_2$ *is matroidal.*

i) *If the triple* $(W; P; U)$ *is 1-regular, then for all* $\{A_1, A_2, A_3\} \in \mathcal{D}$ *and all* $w \in W$ *one has*

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{wA_1, wA_2, wA_3\}) = sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_1, A_2, A_3\});$$

that is, $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *are unions of* W*-orbits under the canonical operation of* W *on* $\mathcal{D}$.

ii) *If W operates transitively on $\mathcal{D}$, then $\mathcal{D}=\mathcal{D}_1\dot\cup\mathcal{D}_2$ and $\mathcal{D}=\mathcal{D}_2\dot\cup\mathcal{D}_1$ are the only matroidal partitions of $\mathcal{D}$.*

Definition 3.11. *Assume the partition $\mathcal{D}=\mathcal{D}_1\dot\cup\mathcal{D}_2$ is matroidal.*

i) *A map $\eta:\mathcal{K}\big(\widetilde{\Gamma}\big)\to\{1,-1\}$ fits for the pair $(\mathcal{D}_1,\mathcal{D}_2)$, if η is four-balanced and satisfies axiom* (D1) *in Definition* 3.9.

ii) *Assume $K{=}(K;+;\cdot;\varepsilon;K_0)$ is a fuzzy ring, and $\eta:\mathcal{K}\big(\widetilde{\Gamma}\big)\to\{1,-1\}$ fits for $(\mathcal{D}_1,\mathcal{D}_2)$. Then we define $\overline{\eta}=\overline{\eta_K}:\mathcal{K}\big(\widetilde{\Gamma}\big)\to\{1,\varepsilon\}\subseteq K^*$ by*

$$\overline{\eta}\left(\{A,B\}\right)=\overline{\eta_K}\left(\{A,B\}\right):=\begin{cases}1 & \text{if } \eta\left(\{A,B\}\right)=1\\ \varepsilon & \text{if } \eta\left(\{A,B\}\right)=-1.\end{cases}\tag{20}$$

Now, we are ready to state the following basic

Definition 3.12. *Suppose the triple $(W;P;U)$ is k-regular for some $k\geq 1$, and assume the graph $\widetilde{\Gamma}=\Gamma^U$ satisfies the Maurer Condition. Suppose the connected components of $\widetilde{\Gamma}$ are convex-antipodal, and for any $A,B\in W^P$ with $d_{\widetilde{\Gamma}}(A,B)<\infty$ the map $ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}:\langle A,B\rangle_{\widetilde{\Gamma}}\to\langle A,B\rangle_{\widetilde{\Gamma}}$ can be extended to an automorphism $\alpha_{A,B}$ of the induced subgraph of Γ on the vertex set $\langle A,B\rangle_{\Gamma}$. Let $\mathcal{D}=\mathcal{D}_1\dot\cup\mathcal{D}_2$ denote a matroidal partition of the set of triangles in $\widetilde{\Gamma}$ and put*

$$Q:=(W;P;U;(\mathcal{D}_1,\mathcal{D}_2)).$$

Assume $\mathcal{B}$ is some non-empty subset of W^P contained in some connected component of $\widetilde{\Gamma}$. Finally, suppose that $K=(K;+;\cdot;\varepsilon;K_0)$ is a fuzzy ring.

i) *Assume $\eta:\mathcal{K}\big(\widetilde{\Gamma}\big)\to\{1,-1\}$ fits for $(\mathcal{D}_1,\mathcal{D}_2)$, and $f:W^P\to K^*\dot\cup\{0\}$ is a map. Then the pair (η,f) defines a $(W;P;U)$-matroid M with coefficients in K with $\mathcal{B}$ as its set of bases if the following two axioms hold:*

   (M1) *For $A\in W^P$ one has $f(A)\neq 0$ if and only if $A\in\mathcal{B}$.*

   (M2) *If $A,B\in W^P$ satisfy $0<d_{\widetilde{\Gamma}}(A,B)<\infty$ and $\mathcal{N}$ is a fundamental neighbourhood of A in $\langle A,B\rangle_{\widetilde{\Gamma}}$ relative to Γ, then one has*

$$\begin{aligned}&\kappa_{\eta,f}(A,B;\mathcal{N}):=\varepsilon\cdot f(A)\cdot f(B)+\\ &\sum_{C\in\mathcal{N}\setminus\{A\}}\overline{\eta}(\{C,A\})\cdot\overline{\eta}(\{ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}(C),B\})\cdot f(C)\cdot f(ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}(C))\in K_0.\end{aligned}\tag{21}$$

ii) *Two pairs (η,f), (η',f') as in* i) *where both η and η' fit for $(\mathcal{D}_1,\mathcal{D}_2)$, define the same $(W;P;U)$-matroid with coefficients in K and base set $\mathcal{B}$ if for all $A\in W^P$ and $B\in\mathcal{B}$ with $\{A,B\}\in\mathcal{K}\big(\widetilde{\Gamma}\big)$ one has*

$$f(A)\cdot f(B)^{-1}\cdot\overline{\eta}\left(\{A,B\}\right)=f'(A)\cdot f'(B)^{-1}\cdot\overline{\eta'}\left(\{A,B\}\right).\tag{22}$$

iii) $\underline{M} := (W; P; U; (\mathcal{D}_1, \mathcal{D}_2); \mathcal{B}) = (Q; \mathcal{B})$ *is called a combinatorial* $(W; P; U)$*-geometry with* $\mathcal{B}$ *as its set of bases if the following strong exchange axiom holds:*

For all $A, B \in \mathcal{B}$ *with* $A \neq B$ *and any fundamental neighbourhood* $\mathcal{N}$ *of* A *in* $\langle A, B\rangle_{\widetilde{\Gamma}}$ *relative to* Γ *there exists some* $C \in \mathcal{N} \setminus \{A\}$ *with* $C \in \mathcal{B}$ *and* $ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}(C) \in \mathcal{B}$.

Remarks:

i) In [26, Definition 3.15], $(W; P; U)$-matroids (or, in short, $(W; P)$-matroids) with coefficients in a fuzzy ring K are defined in terms of some map

$$g: \quad \left\{(A, B) \in W^P \times \mathcal{B} \mid \{A, B\} \in \mathcal{K}\big(\widetilde{\Gamma}\big)\right\} \to K^* \dot{\cup} \{0\}$$

satisfying certain axioms. It then turns out (cf. [26, Theorem 5.22]) that Definition 3.12 is equivalent to [26, Definition 3.15]. More precisely, a map g as in the definition just cited corresponds to a pair (η, f) as in Definition 3.12 if and only if for all $A \in W^P$ and $B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}\big(\widetilde{\Gamma}\big)$ one has

$$g(A, B) = f(A) \cdot f(B)^{-1} \cdot \bar{\eta}(\{A, B\}),$$

thereby motivating Definition 3.12 ii).

ii) By Definition 2.9, any fundamental neighbourhood $\mathcal{N}$ as in axiom (M2) is finite; thus the sum in (21) is well-defined in any case.
The existence of the automorphisms $\alpha_{A,B}$ for $d_{\widetilde{\Gamma}}(A, B) < \infty$ as required implies that for a fundamental neighbourhood $\{A_0, ..., A_\ell\}$ of $A = A_0$ in $\langle A, B\rangle_{\widetilde{\Gamma}}$ relative to Γ the set $\left\{ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}(A_0), ..., ap_{\langle A,B\rangle_{\widetilde{\Gamma}}}(A_\ell)\right\}$ is a fundamental neighbourhood of B in $\langle A, B\rangle_{\widetilde{\Gamma}}$ relative to Γ. Therefore, axiom (M2) is symmetric in A and B. A sufficient condition for the existence of the automorphisms $\alpha_{A,B}$ is of course that the $(W; P)$-graph Γ is convex-antipodal. Then, by Lemma 3.7 one just may put $\alpha_{A,B}(C) := ap_{\langle A,B\rangle_\Gamma}(C)$ for $C \in \langle A, B\rangle_\Gamma$. Consider, however, the following example (see Figure 3):
Let $W := \big\langle r, s \mid (rs)^4 = r^2 = s^2 = 1\big\rangle$ denote the Dieder group of order 8, and let U denote the subgroup generated by $r \cdot s$. Then the triple $(W; \emptyset; U)$ is 2-regular because axiom (U_2) is trivially fulfilled. The $(W; \emptyset)$-graph Γ is the cycle of length 8 and thus not <u>convex</u>-antipodal, but $\widetilde{\Gamma}$ consists of two cycles of length 4 which are certainly convex-antipodal.
The graph $\widetilde{\Gamma}$ satisfies all the assumptions in Definition 3.12. For two vertices A, B of Γ with $d_\Gamma(A, B) = 2$ there exists a unique vertex C of Γ with $\langle A, B\rangle_\Gamma = \{A, B, C\}$, and the map $\alpha = \alpha_{A,B} : \langle A, B\rangle_\Gamma \to \langle A, B\rangle_\Gamma$, defined by $\alpha(A) := B$, $\alpha(B) := A$, $\alpha(C) := C$ satisfies what we want.

iii) One might probably ask for a connection between the concept of a (W, P)-matroid as introduced by I. M. Gelfand and V. V. Serganova in [14], [15] and the above Definition 3.12. In [25, §4] it is verified that for many important classes of Coxeter groups $W = \langle R\rangle$ and $P \subseteq R$ we consider the same structures. Nevertheless, it is also shown that there are (W, P)-matroids in the sense of [14], [15] which do not give rise to

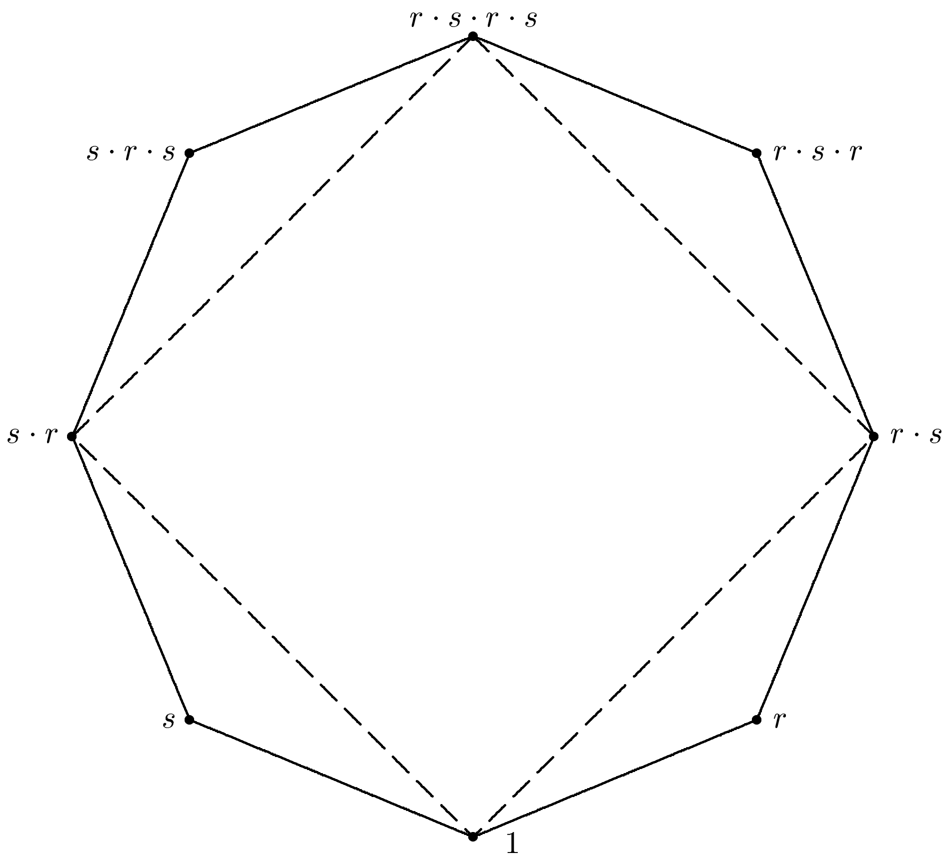

Figure 3

combinatorial $(W;P;U)$-geometries, and vice versa. The condition for the graphs $\Gamma_{\overline{w \cdot U}}$ to be convex-antipodal is a rather strong one, but I was led to require this condition in view of the formulas (GP2) and (TP2) for Grassmann-Plücker maps and twisted Pfaffian maps, respectively.

iv) For a combinatorial $(W;P;U)$-geometry $(W;P;U;(\mathcal{D}_1,\mathcal{D}_2);\mathcal{B})$ it seems superfluous to consider the matroidal partition $\mathcal{D} = \mathcal{D}_1 \dot{\cup} \mathcal{D}_2$. However, this partition will be of great significance when studying the concept of the Tutte group of a combinatorial $(W;P;U)$-geometry (cf. the forthcoming paper [27]). Note that the pair $(\mathcal{D}_1,\mathcal{D}_2)$ is uniquely determined by any map $\eta : \mathcal{K}\big(\widetilde{\Gamma}\big) \to \{1,-1\}$ which satisfies axiom (D1); indeed, for $\{A_1,A_2,A_3\} \in \mathcal{D}$ one has

$$\{A_1,A_2,A_3\} \in \begin{cases} \mathcal{D}_1, & \text{if } \eta(\{A_1,A_2\}) \cdot \eta(\{A_2,A_3\}) \cdot \eta(\{A_3,A_1\}) = 1 \\ \mathcal{D}_2, & \text{if } \eta(\{A_1,A_2\}) \cdot \eta(\{A_2,A_3\}) \cdot \eta(\{A_3,A_1\}) = -1. \end{cases}$$

The following first result connects $(W;P;U)$-matroids with coefficients and combinatorial $(W;P;U)$-geometries.

Proposition 3.13:

i) *Assume the pair* (η, f) *defines a* $(W;P;U)$*-matroid* M *with coefficients in some fuzzy ring* K *with* $\mathcal{B}$ *as its set of bases, and* η *fits for* $(\mathcal{D}_1, \mathcal{D}_2)$*.*
Then $\underline{M} := (W;P;U;(\mathcal{D}_1,\mathcal{D}_2);\mathcal{B})$ *is a combinatorial* $(W;P;U)$*-geometry.*

ii) *The combinatorial* $(W;P;U)$*-geometries are in a canonical one-to-one correspondence to the* $(W;P;U)$*-matroids with coefficients in the fuzzy ring* $K_c = \mathbf{R}/\mathbf{R}^*$*.*

Proof:

i) follows trivially from the axioms (M1) and (M2), because for all $A, B \in \mathcal{B}$ with $A \neq B$ one has $d_{\widetilde{\Gamma}}(A,B) < \infty$ and $\varepsilon \cdot f(A) \cdot f(B) \notin K_0$.

ii) For $K = K_c$ one has $K^* = \{1\}$. Since $1 + ... + 1 \in K_0$ whenever the sum contains at least two summands, it is trivial that a non-empty subset $\mathcal{B}$ of W^P, which is contained in some connected component of $\widetilde{\Gamma}$, satisfies the strong exchange axiom if and only if the maps $\overline{\eta} \equiv 1$ and $f : W^P \to K^* \dot\cup \{0\}$ defined by

$$f(A) := \begin{cases} 1 & \text{for} \quad A \in \mathcal{B} \\ 0 & \text{for} \quad A \notin \mathcal{B} \end{cases}$$

satisfy axiom (M2). (Note that for any two maps $\eta, \eta' : \mathcal{K}(\widetilde{\Gamma}) \to \{1,-1\}$, which fit for one and the same matroidal partition $(\mathcal{D}_1, \mathcal{D}_2)$, the pairs (η, f) and (η', f) define the same $(W;P;U)$-matroid with coefficients in K_c, because one has $\overline{\eta} \equiv \overline{\eta'} \equiv 1$.) $\square$

If M and $\underline{M}$ are as in Proposition 3.13 i), then $\underline{M}$ is called the *underlying combinatorial* $(W;P;U)$*-geometry* of M.

If $Q = (W;P;U;(\mathcal{D}_1,\mathcal{D}_2))$ for some k-regular triple $(W;P;U)$ and some matroidal partition $\mathcal{D} = \mathcal{D}_1 \dot\cup \mathcal{D}_2$, then we put

$$Q^* := (W;P;U;(\mathcal{D}_2,\mathcal{D}_1)).$$

We have the following, almost trivial, but basic

Proposition 3.14. *Assume* K *is a fuzzy ring, and the pair* (η, f) *defines a* $(W;P;U)$*-matroid* M *with coefficients in* K *and* $\mathcal{B}$ *as its set of bases. Then one has*

i) *The pair* $(-\eta, f)$ *defines a* $(W;P;U)$*-matroid* M^* *with coefficients in* K *and the same base set* $\mathcal{B}$*.*

ii) *One has* $M^{**} = M$*.*

iii) *If* $\underline{M} = (Q;\mathcal{B})$*, then one has* $\underline{M^*} = (Q^*;\mathcal{B})$*.*

Proof:

i) follows directly from the axioms (M1) and (M2).
ii) is trivial.
iii) If η fits for $(\mathcal{D}_1,\mathcal{D}_2)$, then $-\eta$ fits for $(\mathcal{D}_2,\mathcal{D}_1)$. $\square$

Definition 3.15. *If M and M^* are as in Proposition* 3.14*, then M^* is called the dual of M. The combinatorial $(W;P;U)$-geometry $\underline{M}^*$ is called the dual of $\underline{M}$.*

At this point, it is not obvious that the definition of the dual of a combinatorial $(W;P;U)$-geometry covers the classical concept of the dual of a matroid. However, as in Example i) following Proposition 3.3, it is shown in [25, Theorem 4.8] that for $W = W_1$ and $P = P_1$ the combinatorial $(W_1;P_1;W_1)$-geometries are precisely the ordinary matroids and that dualizing in the classical sense amounts to change the rôles of $\mathcal{D}_1$ and $\mathcal{D}_2$; one just has to identify the cosets $w \cdot \langle P_1 \rangle$ for $w \in W_1$ with $E \setminus w\,(I_m)$ instead of $w\,(I_m)$ by studying $\underline{M}^*$.

4. Equivalent Representations of Combinatorial $(W;P;U)$-Geometries

Throughout this section assume that $\underline{M} = (W;P;U;(\mathcal{D}_1,\mathcal{D}_2);\mathcal{B})$ is a combinatorial $(W;P;U)$-geometry. We want to study more thoroughly when two pairs $(\eta, f), (\eta', f')$ as in Definition 3.12 give rise to the same $(W;P;U)$-matroid with coefficients in some fixed fuzzy ring K.

Definition 4.1.

i) *Suppose the pair (η, f) defines a $(W;P;U)$-matroid M_1 with coefficients in the fuzzy ring K. Then (η, f) is a representation of $\underline{M}$ over K if η fits for $(\mathcal{D}_1,\mathcal{D}_2)$ and $\mathcal{B} = f^{-1}(K^*)$, that is, $\underline{M}$ is the underlying combinatorial $(W;P;U)$-geometry of M_1.*

ii) *Two representations (η, f) and (η', f') of $\underline{M}$ are equivalent if they define one and the same $(W;P;U)$-matroid with coefficients in K, that is, equation* (22) *holds for all $A \in W^P$ and $B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma})$.*

Note that by the definitions the pair (η, f) can be a representation of $\underline{M} = (W;P;U;(\mathcal{D}_1,\mathcal{D}_2);\mathcal{B})$ only if η fits for $(\mathcal{D}_1,\mathcal{D}_2)$ even in case $\varepsilon = 1$.

In order to compare equivalent representations of $\underline{M}$ we shall now introduce the concept of the base graph.

Definition 4.2. *The base graph $\Gamma_{\underline{M}} = \Gamma_{\mathcal{B}}$ of $\underline{M}$ is the induced subgraph $\Gamma_{\overline{w \cdot U}, \mathcal{B}}$ of $\widetilde{\Gamma}$ (or of $\Gamma_{\overline{w \cdot U}}$) on the set $\mathcal{B}$ of vertices, where $w \in W$ satisfies $\mathcal{B} \subseteq \overline{w \cdot U}$. If M_1 is a $(W;P;U)$-matroid with coefficients in K, then the base graph Γ_{M_1} of M_1 is the base graph $\Gamma_{\underline{M_1}}$ of its underlying combinatorial $(W;P;U)$-geometry.*

The Examples i), ii) following Proposition 3.3 show that Definition 4.2 recovers the concepts of the base graph of a matroid and of an even Δ-matroid.

We have the following

Proposition 4.3. *The base graph $\Gamma_{\mathcal{B}}$ of $\underline{M}$ is an isometric subgraph of the connected graph $\Gamma_{\overline{w \cdot U}}$, where $w \in W$ satisfies $\mathcal{B} \subseteq \overline{w \cdot U}$. In particular, $\Gamma_{\mathcal{B}}$ is also connected.*

Proof. This is [25, Proposition 5.2]. The strong exchange axiom implies easily that for $\Gamma' := \Gamma_{\overline{w \cdot U}}$ and $A, B \in \mathcal{B}$ one has $d_{\Gamma'}(A, B) = d_{\Gamma_{\mathcal{B}}}(A, B)$; actually, one only has to make use of the existence of some $C \in \langle A,B\rangle_{\Gamma'} \cap \mathcal{B}$ with $d_{\Gamma'}(A,C) = 1$, and then one continues by induction. $\square$

Convention. *For simplicity, the metric $d_{\Gamma_{\mathcal{B}}} : \mathcal{B} \times \mathcal{B} \to \mathbf{N}_0$ is from now on denoted by $d_{\mathcal{B}}$. Thus, for $A, B \in \mathcal{B}$ one has*

$$d_{\mathcal{B}}(A,B) = d_{\Gamma_{\mathcal{B}}}(A,B) = d_{\widetilde{\Gamma}}(A,B).$$

In the sequel, let $\Gamma' = \Gamma_{\overline{w \cdot U}}$ denote the uniquely determined connected component of the graph $\widetilde{\Gamma} = \Gamma^U$ which contains the base graph $\Gamma_{\mathcal{B}}$, and let $V(\Gamma')$ denote the vertex set of Γ'. Remember that the graph Γ' is convex-antipodal. Finally, for $A, B \in V(\Gamma')$ we put

$$\langle A,B\rangle := \langle A,B\rangle_{\Gamma'} \ . \tag{23}$$

We have the following, quite useful

Lemma 4.4. *Assume $K = (K;+;\cdot;\varepsilon;K_0)$ is a fuzzy ring, let $\eta : \mathcal{K}(\widetilde{\Gamma}) \to \{1,-1\}$ denote a map which fits for $(\mathcal{D}_1, \mathcal{D}_2)$, and suppose that the map $f : W^P \to K^* \dot{\cup} \{0\}$ satisfies axiom* (M1) *in Definition* 3.12. *Assume $A_0, B_0 \in V(\Gamma')$ are distinct, and let $\mathcal{N}$ denote a fundamental neighbourhood of A_0 in $\langle A_0, B_0\rangle$ relative to Γ. Moreover, suppose $A_1 \in \mathcal{N} \setminus \{A_0\}$ and put $B_1 := ap_{\langle A_0,B_0\rangle}(A_1)$. Then one has*

$$\kappa_{\eta,f}(A_1,B_1;\mathcal{N}) = \varepsilon \cdot \overline{\eta}(\{A_0,A_1\}) \cdot \overline{\eta}(\{B_0,B_1\}) \cdot \kappa_{\eta,f}(A_0,B_0;\mathcal{N}) \ . \tag{24}$$

Thus, the relation $\kappa_{\eta,f}(A_1,B_1;\mathcal{N}) \in K_0$ holds if and only if $\kappa_{\eta,f}(A_0,B_0;\mathcal{N}) \in K_0$. In particular, the pair (η, f) defines a $(W;P;U)$-matroid with coefficients in K if axiom (M2) *holds at least for all $A, B \in \mathcal{B}$ with $A \neq B$.*

Proof. The second assertion is a trivial consequence of the first one. The last assertion follows from the second one, because one has $\kappa_{\eta,f}(A_0,B_0;\mathcal{N}) = 0$ if there does not exist any $A \in \mathcal{N}$ with $A \in \mathcal{B}$ and $ap_{\langle A_0,B_0\rangle}(A) \in \mathcal{B}$. It remains to verify (24).

Axiom (D2) of Definition 3.9 implies for any $C \in \mathcal{N} \setminus \{A_0, A_1\}$:

$$sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_0,A_1,C\}) = -sign_{(\mathcal{D}_1,\mathcal{D}_2)}\left(\left\{B_0,B_1,ap_{\langle A_0,B_0\rangle}(C)\right\}\right).$$

Since η fits for $(\mathcal{D}_1, \mathcal{D}_2)$, this means

$$\begin{aligned} &\overline{\eta}(\{A_0,A_1\}) \cdot \overline{\eta}(\{A_1,C\}) \cdot \overline{\eta}(\{C,A_0\}) \\ &= \varepsilon \cdot \overline{\eta}(\{B_0,B_1\}) \cdot \overline{\eta}\left(\left\{B_1, ap_{\langle A_0,B_0\rangle}(C)\right\}\right) \cdot \overline{\eta}\left(\left\{ap_{\langle A_0,B_0\rangle}(C), B_0\right\}\right). \end{aligned}$$

Together with axiom (FR2) in Definition 2.1 and the Remark following Definition 2.9 we get

$$\begin{aligned}
&\kappa_{\eta,f}\left(A_1,B_1;\mathcal{N}\right)\\
&=\varepsilon\cdot f\left(A_1\right)\cdot f\left(B_1\right)+\overline{\eta}\left(\{A_0,A_1\}\right)\cdot\overline{\eta}\left(\{B_0,B_1\}\right)\cdot f\left(A_0\right)\cdot f\left(B_0\right)\\
&+\sum_{C\in\mathcal{N}\setminus\{A_0,A_1\}}\overline{\eta}\left(\{C,A_1\}\right)\cdot\overline{\eta}\left(\left\{ap_{\langle A_0,B_0\rangle}\left(C\right),B_1\right\}\right)\cdot f\left(C\right)\cdot f\left(ap_{\langle A_0,B_0\rangle}\left(C\right)\right)\\
&=\varepsilon\cdot\overline{\eta}\left(\{A_0,A_1\}\right)\cdot\overline{\eta}\left(\{B_0,B_1\}\right)\\
&\qquad\cdot\left(\overline{\eta}\left(\{A_1,A_0\}\right)\cdot\overline{\eta}\left(\{B_1,B_0\}\right)\cdot f\left(A_1\right)\cdot f\left(B_1\right)+\varepsilon\cdot f\left(A_0\right)\cdot f\left(B_0\right)\right.\\
&+\left.\sum_{C\in\mathcal{N}\setminus\{A_0,A_1\}}\overline{\eta}\left(\{C,A_0\}\right)\cdot\overline{\eta}\left(\left\{ap_{\langle A_0,B_0\rangle}\left(C\right),B_0\right\}\right)\cdot f\left(C\right)\cdot f\left(ap_{\langle A_0,B_0\rangle}\left(C\right)\right)\right)\\
&=\varepsilon\cdot\overline{\eta}\left(\{A_0,A_1\}\right)\cdot\overline{\eta}\left(\{B_0,B_1\}\right)\cdot\kappa_{\eta,f}\left(A_0,B_0;\mathcal{N}\right).
\end{aligned}$$

□

We show now

Proposition 4.5. *Assume K is a fuzzy ring, let $f,f':W^P\to K^*\dot\cup\{0\}$ denote maps with $f^{-1}(K^*)=f'^{-1}(K^*)=\mathcal{B}$, and suppose that the maps $\eta,\eta':\mathcal{K}\big(\widetilde{\Gamma}\big)\to\{1,-1\}$ fit for $(\mathcal{D}_1,\mathcal{D}_2)$. Moreover, assume that equation (22) holds for all $A\in W^P$ and $B\in\mathcal{B}$ with $\{A,B\}\in\mathcal{K}\big(\widetilde{\Gamma}\big)$. Then, for $A,B\in\mathcal{B}$ with $A\neq B$ and any fundamental neighbourhood $\mathcal{N}$ of A in $\langle A,B\rangle$ relative to Γ one has*

$$\kappa_{\eta',f'}\left(A,B;\mathcal{N}\right)=f'\left(A\right)\cdot f'\left(B\right)\cdot f\left(A\right)^{-1}\cdot f\left(B\right)^{-1}\cdot\kappa_{\eta,f}\left(A,B;\mathcal{N}\right). \tag{25}$$

In particular, (η',f') is a representation of $\underline{M}$ if and only if (η,f) is.

Proof. By applying the last assertion in Lemma 4.4 to (η,f) and (η',f'), we see that it suffices to verify (25). By (22) one gets for all $C\in\mathcal{N}\setminus\{A\}$:

$$f\left(A\right)\cdot f'\left(C\right)\cdot\overline{\eta'}\left(\{A,C\}\right)=f'\left(A\right)\cdot f\left(C\right)\cdot\overline{\eta}\left(\{A,C\}\right), \tag{26}$$

$$\begin{aligned}
&f\left(B\right)\cdot f'\left(ap_{\langle A,B\rangle}\left(C\right)\right)\cdot\overline{\eta'}\left(\left\{B,ap_{\langle A,B\rangle}\left(C\right)\right\}\right)\\
&\qquad=f'\left(B\right)\cdot f\left(ap_{\langle A,B\rangle}\left(C\right)\right)\cdot\overline{\eta}\left(\left\{B,ap_{\langle A,B\rangle}\left(C\right)\right\}\right).
\end{aligned} \tag{27}$$

Thus, we get by axiom (FR2) in view of $f(A),f(B),f'(A),f'(B)\in K^*$:

$$\begin{aligned}
&f\left(A\right)\cdot f\left(B\right)\cdot\kappa_{\eta',f'}\left(A,B;\mathcal{N}\right)\\
&=\varepsilon\cdot f\left(A\right)\cdot f\left(B\right)\cdot f'\left(A\right)\cdot f'\left(B\right)\\
&+\sum_{C\in\mathcal{N}\setminus\{A\}}f\left(A\right)\cdot f'\left(C\right)\cdot\overline{\eta'}\left(\{A,C\}\right)\cdot f\left(B\right)\cdot f'\left(ap_{\langle A,B\rangle}\left(C\right)\right)\cdot\overline{\eta'}\left(\left\{ap_{\langle A,B\rangle}\left(C\right),B\right\}\right)\\
&=\varepsilon\cdot f'\left(A\right)\cdot f'\left(B\right)\cdot f\left(A\right)\cdot f\left(B\right)\\
&+\sum_{C\in\mathcal{N}\setminus\{A\}}f'\left(A\right)\cdot f\left(C\right)\cdot\overline{\eta}\left(\{A,C\}\right)\cdot f'\left(B\right)\cdot f\left(ap_{\langle A,B\rangle}\left(C\right)\right)\cdot\overline{\eta}\left(\left\{ap_{\langle A,B\rangle}\left(C\right),B\right\}\right)\\
&=f'\left(A\right)\cdot f'\left(B\right)\cdot\kappa_{\eta,f}\left(A,B;\mathcal{N}\right),
\end{aligned}$$

and (25) follows. □

The next result answers the question when two representations (η, f) and (η, f_1) of $\underline{M}$, which coincide in the first component, are equivalent.

Proposition 4.6. *Suppose (η, f) is a representation of $\underline{M}$ over the fuzzy ring K; moreover, let $f_1 : W^P \to K^* \dot\cup \{0\}$ denote some map. Then (η, f_1) is a representation of $\underline{M}$ and equivalent to (η, f) if and only if there exists some $c \in K^*$ with $f_1(A) = c \cdot f(A)$ for all $A \in W^P$.*

Proof. If there exists some $c \in K^*$ such that $f_1(A) = c \cdot f(A)$ holds for all $A \in W^P$, then by Definition 3.12 i,ii) it is trivial that (η, f_1) is a representation of $\underline{M}$ which is equivalent to (η, f).

Now, assume that, vice versa, (η, f_1) is a representation of $\underline{M}$ such that (22) holds for all $A \in W^P$ and $B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma})$. Then axiom (M1) implies $f_1^{-1}(K^*) = \mathcal{B}$ and $f_1^{-1}(\{0\}) = W^P \setminus \mathcal{B}$. Moreover, (22) implies for all $A, B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma})$:

$$f_1(A) \cdot f(A)^{-1} = f_1(B) \cdot f(B)^{-1}. \tag{28}$$

Since the base graph $\Gamma_{\mathcal{B}}$ is connected, it follows that equation (28) holds for all $A, B \in \mathcal{B}$ as claimed. □

Now, we want to prove the following, more general

Proposition 4.7. *Suppose (η, f) is a representation of $\underline{M}$ over the fuzzy ring K and $\eta' : \mathcal{K}(\widetilde{\Gamma}) \to \{1, -1\}$ fits for $(\mathcal{D}_1, \mathcal{D}_2)$. Then there exists some map $f' : W^P \to K^* \dot\cup \{0\}$ such that (η', f') is also a representation of $\underline{M}$ which is equivalent to (η, f). The map f' is uniquely determined up to a scalar factor.*

Proof. The last assertion follows from Proposition 4.6. To show the first one, we define for any path $Z = (B_0, ..., B_\ell)$ in the base graph $\Gamma_{\mathcal{B}}$ the value $\kappa(Z) = \kappa(B_0, ..., B_\ell) \in K^*$ by

$$\begin{aligned}\kappa(Z) &= \prod_{i=1}^{\ell} \Big(f(B_i) \cdot f(B_{i-1})^{-1} \cdot \overline{\eta}(\{B_{i-1}, B_i\}) \cdot \overline{\eta'}(\{B_{i-1}, B_i\}) \Big) \\ &= f(B_\ell) \cdot f(B_0)^{-1} \cdot \prod_{i=1}^{\ell} \big(\overline{\eta}(\{B_{i-1}, B_i\}) \cdot \overline{\eta'}(\{B_{i-1}, B_i\}) \big).\end{aligned} \tag{29}$$

Clearly, for $A, B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma})$ one has $\kappa(A, B) = \kappa(B, A)^{-1}$, and for any path $Z = (B_0, ..., B_\ell)$ in $\Gamma_{\mathcal{B}}$ we have

$$\kappa(Z) = \prod_{i=1}^{\ell} \kappa(B_{i-1}, B_i).$$

We want to show that for any path $Z = (B_0, ..., B_\ell)$ in $\Gamma_{\mathcal{B}}$ the value $\kappa(Z)$ depends only on the bases B_0 and B_ℓ. This amounts to prove that for any reentrant path Z in $\Gamma_{\mathcal{B}}$ one has $\kappa(Z) = 1$. By [25, Theorem 5.7], which generalizes Maurer's homotopy theorem for ordinary matroids, we have only to verify the following two claims:

(I) If $Z = (A_0, A_1, A_2, A_0)$ is a reentrant path in $\Gamma_{\mathcal{B}}$ of length 3, then one has $\kappa(Z) = 1$.

(II) If $Z = (A_0, A_1, A_2, A_3, A_0)$ is an isometric reentrant path in $\Gamma_{\mathcal{B}}$ of length 4, then one has $\kappa(Z) = 1$.

Note that (29) implies for any reentrant path $Z = (B_0, ..., B_\ell)$ in $\Gamma_{\mathcal{B}}$ the equation

$$\kappa(Z) = \prod_{i=1}^{\ell} \left(\overline{\eta}(\{B_{i-1}, B_i\}) \cdot \overline{\eta'}(\{B_{i-1}, B_i\})\right).$$

Thus, (I) follows from the identity $\left(sign_{(\mathcal{D}_1,\mathcal{D}_2)}(\{A_0, A_1, A_2\})\right)^2 = 1$, because both η and η' fit for $(\mathcal{D}_1, \mathcal{D}_2)$. (II) follows from the fact that η and η' are four-balanced (cf. Definitions 3.9 i) and 3.11 i)). Now, by what we have already proved we may define $f' : W^P \to K^* \dot{\cup} \{0\}$ as follows:

Choose some fixed $B_0 \in \mathcal{B}$, put $f'(B_0) = 1$, and for any $B \in \mathcal{B}$ choose some path $Z = (B_0, ..., B_\ell)$ in $\Gamma_{\mathcal{B}}$ with $B_\ell = B$ and put $f'(B) := \kappa(Z)$. Furthermore, of course, we put $f'(A) := 0$ for $A \in W^P \setminus \mathcal{B}$. By (29) and the definition of f' it is trivial that (22) is fulfilled for all $A \in W^P$ and $B \in \mathcal{B}$ with $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma})$. Thus Proposition 4.5 implies that (η', f') is also a representation of $\underline{M}$, which, in view of (22), is equivalent to (η, f).$\square$

In the forthcoming paper [27], we shall use the extension of Maurer's homotopy theory and the concept of the Tutte group of a combinatorial $(W; P; U)$-geometry in order to derive a second equivalent definition of $(W; P; U)$-matroids with coefficients (cf. also [26, Definition 3.15]). While Definition 3.12 will be used in the next section when studying examples of $(W; P; U)$-matroids with coefficients, the alternative definition is more convenient for many theoretical considerations.

5. Examples of $(W; P; U)$-Matroids with Coefficients in a Fuzzy Ring

In this section, we want to show that with the notations as in the Examples i), ii) following Proposition 3.3 the ordinary matroids with coefficients are precisely the $(W_1; P_1; W_1)$-matroids with coefficients and that the Δ-matroids with coefficients are the $(W_2; P_2; U_\Delta)$-matroids with coefficients.

In the sequel, assume that m, n are fixed integers with $0 \leq m \leq n$, put $E := \{1, ..., n\}$, and identify $W_1^{P_1}$ via χ with $\binom{E}{m}$ in the Example i) just mentioned, and as above let Γ_1 denote the $(W_1; P_1)$-graph. Furthermore, put

$$\mathcal{D}_0 := \left\{\{A_1, A_2, A_3\} \subseteq \tbinom{E}{m} \mid d_{\Gamma_1}(A_i, A_j) = 1 \text{ for } 1 \leq i < j \leq 3 \right\}, \tag{30}$$

$$\mathcal{D}_{0,1} := \left\{\{A \cup \{a_1\}, A \cup \{a_2\}, A \cup \{a_3\}\} \mid A \in \tbinom{E}{m-1},\ a_i \in E \setminus A \text{ and } a_i \neq a_j \text{ for } i \neq j \right\}, \tag{31}$$

$$\mathcal{D}_{0,2} := \left\{\{A \setminus \{a_1\}, A \setminus \{a_2\}, A \setminus \{a_3\}\} \mid A \in \tbinom{E}{m+1},\ a_i \in A \text{ and } a_i \neq a_j \text{ for } i \neq j \right\}. \tag{32}$$

Clearly, $\mathcal{D}_0$ is the set of triangles in Γ_1, and we have $\mathcal{D}_0 = \mathcal{D}_{0,1} \dot{\cup} \mathcal{D}_{0,2}$. Now, we recall the following combinatorial result.

Proposition 5.1.

i) *Assume* A_0 *and* B *are distinct vertices of the graph* Γ_1. *Then a subset* $\mathcal{N} \subseteq \langle A_0, B\rangle_{\Gamma_1}$ *is a fundamental neighbourhood of* A_0 *in* $\langle A_0, B\rangle_{\Gamma_1}$ *if and only if there exists some* $a_0 \in A_0 \setminus B$ *with*

$$\mathcal{N} = \{A_0\} \cup \{(A_0 \setminus \{a_0\}) \cup \{b\} \mid b \in B \setminus A_0\} \tag{33}$$

or some $b_0 \in B \setminus A_0$ *with*

$$\mathcal{N} = \{A_0\} \cup \{(A_0 \setminus \{a\}) \cup \{b_0\} \mid a \in A_0 \setminus B\}. \tag{34}$$

ii) *The partition* $\mathcal{D}_0 = \mathcal{D}_{0,1} \dot{\cup} \mathcal{D}_{0,2}$ *is matroidal, and this partition is – apart from the order – the only matroidal partition of* $\mathcal{D}_0$. *Moreover, the map* $\eta_1 : \mathcal{K}(\Gamma_1) \to \{1, -1\}$ *defined by*

$$\begin{aligned} &\eta_1 (\{D \cup \{e\}, D \cup \{f\}\}) := (-1)^{\#\{d \in D \mid e < d < f\}} \\ &\quad \textit{for } D \cup \{e, f\} \subseteq E \textit{ with } \#(D \cup \{e, f\}) = \#D + 2 = m + 1 \textit{ and } e < f \end{aligned} \tag{35}$$

fits for $(\mathcal{D}_{0,1}, \mathcal{D}_{0,2})$.

iii) *The ordinary matroids defined on* E *and of rank* m *are in a canonical one-to-one correspondence to the combinatorial* $(W_1; P_1; W_1)$*-geometries* $(W_1; P_1; W_1; (\mathcal{D}_{0,1}, \mathcal{D}_{0,2}); \mathcal{B})$, *where only* $\mathcal{B}$ *is varying, and any combinatorial* $(W_1; P_1; W_1)$*-geometry or its dual is of this type.*

Proof.
i) is [25, Proposition 4.1 i)], ii) is [25, Proposition 4.7], and iii) is [25, Theorem 4.8 i)]. □

Assume $\sigma \in W_1 = S_n$. Since for $i \in \{1, 2\}$ and any $\{A_1, A_2, A_3\} \in \mathcal{D}_{0,i}$ one has also $\{\sigma(A_1), \sigma(A_2), \sigma(A_3)\} \in \mathcal{D}_{0,i}$, it follows trivially that the map $\eta_\sigma : \mathcal{K}(\Gamma_1) \to \{1, -1\}$, defined by

$$\eta_\sigma (\{A, B\}) := \eta_1 (\{\sigma(A), \sigma(B)\}) \tag{36}$$

fits for $(\mathcal{D}_{0,1}, \mathcal{D}_{0,2})$, too.

In order to relate ordinary matroids with coefficients to $(W_1; P_1; W_1)$-matroids with coefficients, we prove the following

Lemma 5.2. *Assume* $K = (K; +; \cdot; \varepsilon; K_0)$ *is a fuzzy ring, suppose* $\sigma \in W_1$, *and let* $\widetilde{b}, \widetilde{b}_\sigma : \binom{E}{m} \to K^* \dot{\cup} \{0\}$ *and* $b : E^m \to K^* \dot{\cup} \{0\}$ *denote maps which are related by the formulas*

$$\widetilde{b}(\{a_1, ..., a_m\}) = b(a_1, ..., a_m) \quad \textit{whenever } a_i < a_j \textit{ for } 1 \le i < j \le m, \tag{37}$$

$$\widetilde{b}_\sigma(\{a_1, ..., a_m\}) = b(a_1, ..., a_m) \quad \textit{whenever } \sigma(a_i) < \sigma(a_j) \textit{ for } 1 \le i < j \le m. \tag{38}$$

Moreover, suppose b satisfies axiom (GP1) *in Definition* 2.2. *Then the following statements hold:*

i) *Assume $a_1, ..., a_{m-1}, e, f \in E$ are pairwise distinct, put $D := \{a_1, ..., a_{m-1}\}$, $A := D \cup \{e\}$, $B := D \cup \{f\}$, and suppose $\widetilde{b}(B) \neq 0$. Then one has*

$$\begin{aligned}\widetilde{b}(A) \cdot \widetilde{b}(B)^{-1} \cdot \overline{\eta_1}(\{A, B\}) &= b(a_1, ..., a_{m-1}, e) \cdot b(a_1, ..., a_{m-1}, f)^{-1} \\ &= \widetilde{b}_\sigma(A) \cdot \widetilde{b}_\sigma(B)^{-1} \cdot \overline{\eta}_\sigma(\{A, B\}).\end{aligned}$$

ii) *Assume $1 \leq \ell \leq m$, and $e_0, ..., e_m, f_2, ..., f_m \in E$ satisfying $e_i = f_i$ for $\ell + 1 \leq i \leq m$, but $e_0, ..., e_m, f_2, ..., f_\ell$ are pairwise distinct. Suppose $A_0, B_0 \in \binom{E}{m}$ satisfy*

$$A_0 = \{e_1, ..., e_m\}, \quad B_0 = \{e_0, f_2, ..., f_m\}.$$

Then $\mathcal{N} := \{(A_0 \setminus \{e_i\}) \cup \{e_0\} \mid 1 \leq i \leq \ell\} \dot\cup \{A_0\}$ is a fundamental neighbourhood of A_0 in $\langle A_0, B_0 \rangle := \langle A_0, B_0 \rangle_{\Gamma_1}$. If σ satisfies $\sigma(e_{i_1}) < \sigma(e_{i_2}) < \sigma(f_{i_3}) \leq \sigma(f_{i_4})$ for all i_1, i_2, i_3, i_4 with $0 \leq i_1 < i_2 \leq \ell$ and $2 \leq i_3 \leq i_4 \leq m$, then one has

$$\sum_{i=0}^{\ell} \varepsilon^i \cdot b(e_0, ..., \hat{e}_i, ..., e_m) \cdot b(e_i, f_2, ..., f_m) = \varepsilon \cdot \kappa_{\eta_\sigma, \widetilde{b}_\sigma}(A_0, B_0; \mathcal{N}). \tag{39}$$

Proof:

i) Suppose $D = \{d_1, ..., d_{m-1}\} = \{d'_1, ..., d'_{m-1}\}$, where $d_i < d_j$ and $\sigma(d'_i) < \sigma(d'_j)$ for $1 \leq i < j \leq m-1$. Furthermore, put

$$\begin{aligned}&r_1 := \#\{d \in D \mid e < d\} &&, \quad r_2 := \#\{d \in D \mid f < d\}, \\ &r_3 := \#\{d \in D \mid \sigma(e) < \sigma(d)\} &&, \quad r_4 := \#\{d \in D \mid \sigma(f) < \sigma(d)\}.\end{aligned}$$

Then, we get by (37), (38) and a repeated application of axiom (GP1):

$$\begin{aligned}&\widetilde{b}(D \cup \{e\}) \cdot \widetilde{b}(D \cup \{f\})^{-1} \cdot \overline{\eta}_1(\{D \cup \{e\}, D \cup \{f\}\}) \\ &= (b(d_1, ..., d_{m-1}, e) \cdot \varepsilon^{r_1}) \cdot (b(d_1, ..., d_{m-1}, f) \cdot \varepsilon^{r_2})^{-1} \cdot \varepsilon^{r_2 - r_1} \\ &= b(a_1, ..., a_{m-1}, e) \cdot b(a_1, ..., a_{m-1}, f)^{-1} \\ &= b(d'_1, ..., d'_{m-1}, e) \cdot b(d'_1, ..., d'_{m-1}, f)^{-1} \\ &= \left(\widetilde{b}_\sigma(D \cup \{e\}) \cdot \varepsilon^{r_3}\right) \cdot \left(\widetilde{b}_\sigma(D \cup \{f\}) \cdot \varepsilon^{r_4}\right)^{-1} \\ &= \widetilde{b}_\sigma(D \cup \{e\}) \cdot \widetilde{b}_\sigma(D \cup \{f\})^{-1} \cdot \overline{\eta}_\sigma(\{D \cup \{e\}, D \cup \{f\}\}).\end{aligned}$$

ii) By Proposition 5.1 i) we already know that $\mathcal{N}$ is a fundamental neighbourhood of A_0 in $\langle A_0, B_0 \rangle$. By (38) and our assumption concerning σ and the elements $e_0, ..., e_m, f_2, ..., f_m$, we have

$$\begin{aligned}
&b\left(e_0, ..., \hat{e}_i, ..., e_m\right) = \widetilde{b}_\sigma\left(\left(A_0 \cup \{e_0\}\right) \setminus \{e_i\}\right) && \text{for } 0 \le i \le \ell, \\
&b\left(e_i, f_2, ..., f_m\right) = \widetilde{b}_\sigma\left(\left(B_0 \setminus \{e_0\}\right) \cup \{e_i\}\right) && \text{for } 0 \le i \le \ell, \\
&\overline{\eta}_\sigma\left(\left\{\left(A_0 \setminus \{e_i\}\right) \cup \{e_0\}, A_0\right\}\right) = \varepsilon^{i-1} && \text{for } 1 \le i \le \ell, \\
&\overline{\eta}_\sigma\left(\left\{\left(B_0 \setminus \{e_0\}\right) \cup \{e_i\}, B_0\right\}\right) = 1 && \text{for } 1 \le i \le \ell.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
&\sum_{i=0}^{\ell} \varepsilon^i \cdot b\left(e_0, ..., \hat{e}_i, ..., e_m\right) \cdot b\left(e_i, f_2, ..., f_m\right) \\
&= \widetilde{b}_\sigma\left(A_0\right) \cdot \widetilde{b}_\sigma\left(B_0\right) + \sum_{i=1}^{\ell} \varepsilon^i \cdot \widetilde{b}_\sigma\left(\left(A_0 \setminus \{e_i\}\right) \cup \{e_0\}\right) \cdot \widetilde{b}_\sigma\left(\left(B_0 \setminus \{e_0\}\right) \cup \{e_i\}\right) \\
&= \widetilde{b}_\sigma\left(A_0\right) \cdot \widetilde{b}_\sigma\left(B_0\right) \\
&+ \sum_{C \in \mathcal{N} \setminus \{A_0\}} \varepsilon \cdot \overline{\eta}_\sigma\left(\{C, A_0\}\right) \cdot \overline{\eta}_\sigma\left(\left\{ap_{\langle A_0, B_0\rangle}\left(C\right), B_0\right\}\right) \cdot \widetilde{b}_\sigma\left(C\right) \cdot \widetilde{b}_\sigma\left(ap_{\langle A_0, B_0\rangle}\left(C\right)\right) \\
&= \varepsilon \cdot \kappa_{\eta_\sigma, \widetilde{b}_\sigma}\left(A_0, B_0; \mathcal{N}\right). \qquad \square
\end{aligned}$$

If $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring, then as a technical notation we put for any permutation τ of some finite set

$$sign_\varepsilon \tau := \begin{cases} \varepsilon & \text{if } \tau \text{ is odd,} \\ 1 & \text{if } \tau \text{ is even,} \end{cases}$$

and for pairwise distinct $e_1, ..., e_m \in E$ let $\tau = \tau\left(e_1, ..., e_m\right)$ denote the unique permutation of $\{e_1, ..., e_m\}$ with $\tau\left(e_i\right) < \tau\left(e_j\right)$ for $1 \le i < j \le m$.

Now, we are able to state and to prove the following

Theorem 5.3. *Assume $K = (K; +; \cdot; \varepsilon; K_0)$ is a fuzzy ring. Then there is a canonical one-to-one correspondence between the ordinary matroids with coefficients in K, defined on E and of rank m, and the $(W_1; P_1; W_1)$-matroids with coefficients in K endowed with the ordered matroidal partition $\mathcal{D}_0 = \mathcal{D}_{0,1} \dot\cup \mathcal{D}_{0,2}$. The correspondence is as follows:*

If $\mathcal{B}$ is the set of bases of a $(W_1; P_1; W_1)$-matroid with coefficients in K and $\widetilde{b} : \binom{E}{m} \to K^* \dot\cup \{0\}$ is a map such that $\left(\eta_1, \widetilde{b}\right)$ is a representation of $\left(W_1; P_1; W_1; \left(\mathcal{D}_{0,1}, \mathcal{D}_{0,2}\right); \mathcal{B}\right)$, then the map $b : E^m \to K^* \dot\cup \{0\}$ defined by

$$b\left(e_1, ..., e_m\right) := \begin{cases} sign_\varepsilon \tau\left(e_1, ..., e_m\right) \cdot \widetilde{b}\left(\{e_1, ..., e_m\}\right) & \textit{if } e_1, ..., e_m \in E \\ & \textit{are pairwise distinct} \\ 0 & \textit{otherwise} \end{cases} \tag{40}$$

is a corresponding Grassmann-Plücker map.

Vice versa, if $b : E^m \to K^* \dot\cup \{0\}$ *is a Grassmann-Plücker map and* $\widetilde{b} : \binom{E}{m} \to K^* \dot\cup \{0\}$ *is defined by*

$$\widetilde{b}\left(\{e_1, ..., e_m\}\right) := b\left(e_1, ..., e_m\right) \text{ whenever } e_i < e_j \text{ for } 1 \le i < j \le m, \tag{41}$$

then $\left(\eta_1, \widetilde{b}\right)$ *is a corresponding representation.*

Proof. Assume first that $\left(\eta_1, \widetilde{b}\right)$ defines a $(W_1; P_1; W_1)$-matroid M with coefficients in K and $\mathcal{B}$ as its set of bases. Then the map $b : E^m \to K^* \dot\cup \{0\}$ as defined in (40) satisfies trivially axiom (GP1) while axiom (GP0) follows from axiom (M1) and $\mathcal{B} \neq \emptyset$. We still have to verify axiom (GP2).

Assume $e_0, ..., e_m, f_2, ..., f_m \in E$ and put

$$\kappa_0 := \sum_{i=0}^{m} \varepsilon^i \cdot b\left(e_0, ..., \hat{e}_i, ..., e_m\right) \cdot b\left(e_i, f_2, ..., f_m\right).$$

We must show that $\kappa_0 \in K_0$.

If there exist i_1, i_2 with $2 \le i_1 < i_2 \le m$ and $f_{i_1} = f_{i_2}$, then one has trivially $\kappa_0 = 0$. Moreover, if $e_{i_1} = e_{i_2}$ for some i_1, i_2 with $0 \le i_1 < i_2 \le m$, then one gets

$$\kappa_0 = b\left(e_0, ..., \hat{e}_{i_2}, ..., e_m\right) \cdot b\left(e_{i_2}, f_2, ..., f_m\right) \cdot \left(\varepsilon^{i_1} \cdot \varepsilon^{i_2 - i_1 - 1} + \varepsilon^{i_2}\right) \in K_0.$$

Thus, from now on we assume that $\#\{e_0, ..., e_m\} = m+1$ and $\#\{f_2, ..., f_m\} = m-1$. Suppose i_0, j_0 are such that $0 \le i_0 \le m-1$ and $2 \le j_0 \le m-1$, and put $e'_{i_0} := e_{i_0+1}$, $e'_{i_0+1} := e_{i_0}$, $e'_i := e_i$ for $i \in \{0, ..., m\} \setminus \{i_0\,,\, i_0+1\}$ and $f'_{j_0} := f_{j_0+1}$, $f'_{j_0+1} := f_{j_0}$, $f'_i := f_i$ for $i \in \{2, ..., m\} \setminus \{j_0\,,\, j_0+1\}$. Then one has

$$\begin{aligned}\kappa_0 &= \varepsilon \cdot \sum_{i=0}^{m} \varepsilon^i \cdot b\left(e_0, ..., \hat{e}_i, ..., e_m\right) \cdot b\left(e_i, f'_2, ..., f'_m\right) \\ &= \varepsilon \cdot \sum_{i=0}^{m} \varepsilon^i \cdot b\left(e'_0, ..., \hat{e}'_i, ..., e'_m\right) \cdot b\left(e'_i, f_2, ..., f_m\right).\end{aligned}$$

Thus, we may also assume that there exists some ℓ with $1 \le \ell \le m$ and $e_i = f_i$ for $\ell+1 \le i \le m$ but that $e_0, ..., e_m, f_2, ..., f_\ell$ are pairwise distinct. Let $\sigma : E \to E$ denote some bijection with $\sigma\left(e_{i_1}\right) < \sigma\left(e_{i_2}\right) < \sigma\left(f_{i_3}\right) \le \sigma\left(f_{i_4}\right)$ for all i_1, i_2, i_3, i_4 with $0 \le i_1 < i_2 \le \ell$ and $2 \le i_3 \le i_4 \le m$, and let $\widetilde{b}_\sigma : \binom{E}{m} \to K^* \dot\cup \{0\}$ denote the unique map satisfying (38). Now, by Lemma 5.2 i) and Proposition 4.5 we conclude that $\left(\eta_\sigma, \widetilde{b}_\sigma\right)$ is also a representation of M. Thus, Lemma 5.2 ii) yields that $\kappa_0 \in K_0$.

Now, assume that vice versa $b : E^m \to K^* \dot\cup \{0\}$ is a Grassmann-Plücker map. Then, by Proposition 5.1 iii), we know already that the set $\mathcal{B}$ of bases of M_b defines a combinatorial $(W_1; P_1; W_1)$-geometry. Moreover, the map $\widetilde{b} : \binom{E}{m} \to K^* \dot\cup \{0\}$ as defined in (41) satisfies axiom (M1) by definition. To verify that $\widetilde{b}$ also satisfies (M2), assume that $A_0, B_0 \in \binom{E}{m}$

are distinct and $\mathcal{N}$ is a fundamental neighbourhood of A_0 in $\langle A_0, B_0\rangle$. We must prove that $\kappa_{\eta_1,\widetilde{b}}(A_0, B_0; \mathcal{N}) \in K_0$.

By Lemma 4.4, we may assume that $A_0, B_0 \in \mathcal{B}$. Furthermore, by Proposition 5.1 i) and the trivial equation $\kappa_{\eta_1,\widetilde{b}}(A_0, B_0; \mathcal{N}) = \kappa_{\eta_1,\widetilde{b}}(B_0, A_0; \mathcal{N}')$ for $\mathcal{N}' := \{ap_{\langle A_0,B_0\rangle}(C) \mid C \in \mathcal{N}\}$ we may suppose that there exists some $e_0 \in B_0 \setminus A_0$ such that

$$\mathcal{N} = \{A_0\} \cup \{(A_0 \setminus \{e\}) \cup \{e_0\} \mid e \in A_0 \setminus B_0\}.$$

Put $\ell := \#(B_0 \setminus A_0)$, and assume $A_0 = \{e_1, ..., e_m\}$, $B_0 = \{e_0, f_2, ..., f_m\}$, where $e_i = f_i$ for $\ell + 1 \le i \le m$. Then $e_0, e_1, ..., e_m, f_2, ..., f_\ell$ are pairwise distinct, and as above we may choose some bijection $\sigma : E \to E$ with $\sigma(e_{i_1}) < \sigma(e_{i_2}) < \sigma(f_{i_3}) \le \sigma(f_{i_4})$ for all i_1, i_2, i_3, i_4 with $0 \le i_1 < i_2 \le \ell$ and $2 \le i_3 \le i_4 \le m$. Then for the unique map $\widetilde{b}_\sigma : \binom{E}{m} \to K^*\dot\cup\{0\}$ which satisfies (38) one has $\kappa_{\eta_\sigma,\widetilde{b}_\sigma}(A_0, B_0; \mathcal{N}) \in K_0$ by Lemma 5.2 ii), because b satisfies axiom (GP2). Now, Lemma 5.2 i) and Proposition 4.5 imply also $\kappa_{\eta_1,\widetilde{b}}(A_0, B_0; \mathcal{N}) \in K_0$.

Proposition 4.6 implies that two Grassmann-Plücker maps $b, b' : E^m \to K^*\dot\cup\{0\}$ are equivalent if and only if the corresponding maps $\widetilde{b}, \widetilde{b}' : \binom{E}{m} \to K^*\dot\cup\{0\}$ give rise to equivalent representations $(\eta_1, \widetilde{b})$ and $(\eta_1, \widetilde{b}')$. Since by Proposition 4.7 any representation of a combinatorial $(W_1; P_1; W_1)$-geometry is equivalent to some representation of the form $(\eta_1, \widetilde{b})$, the proof is complete. □

Now, we turn to Δ-matroids with coefficients. As in Example ii) following Proposition 3.3 identify $W_2^{P_2}$ via χ'' with $\mathcal{P}(E)$, and – as above – let $\widetilde{\Gamma}_2 = \Gamma_2^{U_\Delta} = \left(V(\widetilde{\Gamma}_2), \mathcal{K}(\widetilde{\Gamma}_2)\right)$ denote the graph with $V(\widetilde{\Gamma}_2) = E$ and $\{A, B\} \in \mathcal{K}(\widetilde{\Gamma}_2)$ if and only if $\#(A\Delta B) = 2$. Furthermore, put

$$\widetilde{\mathcal{D}}_0 := \Big\{\{A_1, A_2, A_3\} \subseteq \mathcal{P}(E) \;\Big|\; d_{\widetilde{\Gamma}_2}(A_i, A_j) = 1 \text{ for } 1 \le i < j \le 3\Big\}, \tag{42}$$

$$\widetilde{\mathcal{D}}_1 := \Big\{\{A\Delta\{a_1\}, A\Delta\{a_2\}, A\Delta\{a_3\}\} \;\Big|\; A \subseteq E,\ 1 \le a_1 < a_2 < a_3 \le n,\ a_2 \notin A\Big\}, \tag{43}$$

$$\widetilde{\mathcal{D}}_2 := \Big\{\{A\Delta\{a_1\}, A\Delta\{a_2\}, A\Delta\{a_3\}\} \;\Big|\; A \subseteq E,\ 1 \le a_1 < a_2 < a_3 \le n,\ a_2 \in A\Big\}. \tag{44}$$

It is easily verified that $\widetilde{\mathcal{D}}_0 = \widetilde{\mathcal{D}}_1 \dot\cup \widetilde{\mathcal{D}}_2$ (cf. also [25, Lemma 4.4]). Moreover, we have the following

Proposition 5.4.

i) *Let A_0, B denote distinct vertices of the graph $\widetilde{\Gamma}_2$ with $\#A_0 \equiv \#B \bmod 2$. Then a subset $\mathcal{N} \subseteq \langle A_0, B\rangle_{\widetilde{\Gamma}_2}$ is a fundamental neighbourhood of A_0 in $\langle A_0, B\rangle_{\widetilde{\Gamma}_2}$ relative to Γ_2 if and only if there exists some $c_0 \in A_0\Delta B$ such that*

$$\mathcal{N} = \{A_0\Delta\{c_0\}\Delta\{d\} \mid d \in A_0\Delta B\}. \tag{45}$$

ii) *The partition* $\widetilde{\mathcal{D}}_0 = \widetilde{\mathcal{D}}_1 \dot{\cup} \widetilde{\mathcal{D}}_2$ *is matroidal, and this partition is – apart from the order – the only matroidal partition of* $\widetilde{\mathcal{D}}_0$*. Moreover, the map* $\eta_2 : \mathcal{K}\big(\widetilde{\Gamma}_2\big) \to \{1, -1\}$ *defined by*

$$\eta_2 \left(\{A\Delta \{a\}, A\Delta \{b\}\}\right) := (-1)^{\#\{i\in A|a<i<b\}} \quad \textit{for } A \subseteq E, 1 \leq a < b \leq n \tag{46}$$

fits for $\big(\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2\big)$.

iii) *The even* Δ*-matroids defined on* E *are in a canonical one-to-one correspondence to the combinatorial* $(W_2; P_2; U_\Delta)$*-geometries* $\Big(W_2; P_2; U_\Delta; \big(\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2\big); \mathcal{B}\Big)$*, where only* $\mathcal{B}$ *is varying, and any combinatorial* $(W_2; P_2; U_\Delta)$*-geometry or its dual is of this type.*

Proof.
i) is [25, Proposition 4.1 ii)], ii) is [25, Proposition 4.5], and iii) is [25, Theorem 4.8 ii)]. □

We shall now prove the following

Theorem 5.5. *Assume* $K = (K; +; \cdot; \varepsilon; K_0)$ *is a fuzzy ring. Then there is a canonical one-to-one correspondence between the* Δ*-matroids with coefficients in* K*, defined on* E*, and the* $(W_2; P_2; U_\Delta)$*-matroids with coefficients in* K *endowed with the ordered matroidal partition* $\widetilde{\mathcal{D}}_0 = \widetilde{\mathcal{D}}_1 \dot{\cup} \widetilde{\mathcal{D}}_2$*. More precisely, for an even* Δ*-matroid* $(E, \mathcal{B})$ *a map* $p : \mathcal{P}(E) \to K^* \dot{\cup} \{0\}$ *with* $p^{-1}(K^*) = \mathcal{B}$ *is a twisted Pfaffian map if and only if* (η_2, p) *is a representation of* $\Big(W_2; P_2; U_\Delta; \big(\widetilde{\mathcal{D}}_1, \widetilde{\mathcal{D}}_2\big); \mathcal{B}\Big)$.

Proof. Assume that $(E, \mathcal{B})$ is an even Δ-matroid and $p : \mathcal{P}(E) \to K^* \dot{\cup} \{0\}$ is a map with $\mathcal{B} = p^{-1}(K^*)$. Then one has $\mathcal{B} \neq \emptyset$ and $\#B_1 \equiv \#B_2 \, mod\, 2$ for all $B_1, B_2 \in \mathcal{B}$. Thus, it is trivial that p satisfies the axioms (TP0), (TP1) and (M1). It remains to show that (TP2) and (M2) are equivalent. Assume $A, B \subseteq E$ and $A\Delta B = \{i_1, ..., i_\ell\}$ with $\ell \geq 2$, $\ell \equiv 0 \; mod 2, i_j < i_{j+1}$ for $1 \leq j \leq \ell - 1$, and for some fixed ν with $1 \leq \nu \leq \ell$ put $A' := A\Delta \{i_\nu\}$, $B' := B\Delta \{i_\nu\}$ and $\mathcal{N} := \{A'\Delta \{i_j\} \mid 1 \leq j \leq \ell\}$. By Proposition 5.4 i) the equivalence of the axioms (TP2) and (M2) follows once it is shown that

$$\kappa_{\eta_2, p}(A, B; \mathcal{N}) = \varepsilon^{\nu - 1} \cdot \sum_{j=1}^{\ell} \varepsilon^j \cdot p\left(A'\Delta \{i_j\}\right) \cdot p\left(B'\Delta \{i_j\}\right). \tag{47}$$

We get

$$\begin{aligned} &\kappa_{\eta_2, p}(A, B; \mathcal{N}) \\ &= \varepsilon \cdot p(A) \cdot p(B) + \sum_{C \in \mathcal{N} \setminus \{A\}} \overline{\eta}_2(\{C, A\}) \cdot \overline{\eta}_2(\{A\Delta B\Delta C, B\}) \cdot p(C) \cdot p(A\Delta B\Delta C) = \end{aligned}$$

$$= \varepsilon \cdot p\left(A' \Delta \left\{i_\nu\right\}\right) \cdot p\left(B' \Delta \left\{i_\nu\right\}\right)$$
$$+ \sum_{j=1, j\neq\nu}^{\ell} \overline{\eta}_2 \left(\left\{A' \Delta \left\{i_j\right\}, A' \Delta \left\{i_\nu\right\}\right\}\right) \cdot \overline{\eta}_2 \left(\left\{B' \Delta \left\{i_j\right\}, B' \Delta \left\{i_\nu\right\}\right\}\right) \cdot p\left(A' \Delta \left\{i_j\right\}\right) \cdot p\left(B' \Delta \left\{i_j\right\}\right)$$

$$= \varepsilon \cdot p\left(A' \Delta \left\{i_\nu\right\}\right) \cdot p\left(B' \Delta \left\{i_\nu\right\}\right)$$
$$+ \sum_{j=1}^{\nu-1} \varepsilon^{\#\left\{e\in A' \mid i_j < e < i_\nu\right\} + \#\left\{e \in B' \mid i_j < e < i_\nu\right\}} \cdot p\left(A' \Delta \left\{i_j\right\}\right) \cdot p\left(B' \Delta \left\{i_j\right\}\right)$$
$$+ \sum_{j=\nu+1}^{\ell} \varepsilon^{\#\left\{e \in A' \Delta B' \mid i_\nu < e < i_j\right\}} \cdot p\left(A' \Delta \left\{i_j\right\}\right) \cdot p\left(B' \Delta \left\{i_j\right\}\right)$$
$$= \varepsilon \cdot p\left(A' \Delta \left\{i_\nu\right\}\right) \cdot p\left(B' \Delta \left\{i_\nu\right\}\right) \ + \sum_{j=1, j\neq\nu}^{\ell} \varepsilon^{j-\nu-1} \cdot p\left(A' \Delta \left\{i_j\right\}\right) \cdot p\left(B' \Delta \left\{i_j\right\}\right)$$
$$= \varepsilon^{\nu-1} \cdot \sum_{j=1}^{\ell} \varepsilon^{j} \cdot p\left(A' \Delta \left\{i_j\right\}\right) \cdot p\left(B' \Delta \left\{i_j\right\}\right),$$

and (47) is proved.

Finally, Propositions 4.6 and 4.7 show that the equivalence classes of twisted Pfaffian maps are in a one-to-one correspondence to the equivalence classes of representations of combinatorial $(W_2; P_2; U_\Delta)$-geometries. $\square$

Remark. If $p : \mathcal{P}(E) \to K^* \dot\cup \{0\}$ is a twisted Pfaffian map, then by [23, Definition 3.4] a twisted Pfaffian map $p^* : \mathcal{P}(E) \to K^* \dot\cup \{0\}$ of the dual of M_p is given by

$$p^*(F) := \varepsilon^{\sum_{e\in F} e} \cdot p(E \setminus F). \tag{48}$$

Assume $A \subseteq E$ and $e, f \in E$ such that $e \neq f$ and $p^*(A \Delta \{f\}) \neq 0$. Then we get

$$p^*(A\Delta\{e\}) \cdot p^*(A\Delta\{f\})^{-1} \cdot \overline{\eta}_2(\{A\Delta\{e\}, A\Delta\{f\}\})$$
$$= \varepsilon^{f-e} \cdot p(E \setminus (A\Delta\{e\})) \cdot p(E \setminus (A\Delta\{f\}))^{-1}$$
$$\cdot \varepsilon^{f-e-1} \cdot \overline{\eta}_2(\{E \setminus (A\Delta\{e\}), E \setminus (A\Delta\{f\})\})$$
$$= p(E \setminus (A\Delta\{e\})) \cdot p(E \setminus (A\Delta\{f\}))^{-1} \cdot (\varepsilon \cdot \overline{\eta}_2(\{E \setminus (A\Delta\{e\}), E \setminus (A\Delta\{f\})\})).$$

Thus – if we identify the cosets $w \cdot \langle P_2 \rangle$ for $w \in W_2$ with $E \setminus w(E)$ instead of $E \cap w(E)$ when studying M_{p*} – we see that Definition 3.15 recovers the definition of the dual of a Δ-matroid with coefficients. Similarly, it follows that Definition 3.15 is consistent with the definition of the dual of a matroid with coefficients (cf. also [8, §5], [26, §6]), but this is now also a consequence of [23, §4], where it is shown that the matroids with coefficients are precisely the Δ-matroids with coefficients and equicardinal feasible sets and that for a matroid M with coefficients its dual M^* means the same in matroid theory as in Δ-matroid theory.

At the end of this paper, we want to study $(W;P;W)$-matroids with coefficients, where $W = \langle R \rangle$ is an abelian Coxeter group, that is, an $\mathbf{F}_2$-vector space with a distinguished base. For any $w \in W$ let $R(w)$ denote the uniquely determined subset of R with

$$w = \prod_{r \in R(w)} r. \tag{49}$$

Let Γ denote the $(W;P)$-graph. It is obvious that Γ is convex-antipodal; for $w_1, w_2 \in W$ and $w \cdot W_P \in \langle \overline{w}_1, \overline{w}_2 \rangle := \langle w_1 \cdot W_P, w_2 \cdot W_P \rangle_\Gamma$ one has

$$ap_{\langle \overline{w}_1, \overline{w}_2 \rangle}(w \cdot W_P) = w_1 \cdot w_2 \cdot w \cdot W_P.$$

Moreover, for $w_0, w_1, w_2 \in W$ and $r \in R \backslash P$ one has $r \cdot w_1 \cdot w_0 \cdot W_P \in \langle w_1 \cdot w_0 \cdot W_P, w_2 \cdot w_0 \cdot W_P \rangle_\Gamma$ if and only if $r \cdot W_P \in \langle W_P, w_1 \cdot w_2 \cdot W_P \rangle_\Gamma$ if and only if $r \in (R \setminus P) \cap R(w_1 \cdot w_2)$.

Let $\eta_0 : \mathcal{K}(\Gamma) \to \{1, -1\}$ denote the constant map given by

$$\eta_0(\{r \cdot w \cdot W_P, w \cdot W_P\}) := 1 \text{ for all } w \in W \text{ and } r \in R \setminus P. \tag{50}$$

In the $(W;P)$-graph Γ the length of any reentrant path is trivially even, and thus Γ does not contain any triangle. This means in particular that the map η_0 fits for $(\emptyset, \emptyset)$.

Now, we have the following

Proposition 5.6. *Assume $W = \langle R \rangle$ is an abelian Coxeter group, $P \subseteq R$, $\mathcal{B} \subseteq W^P$, and let K denote a fuzzy ring. Then one has*

i) *The following four conditions are equivalent:*
   (a) *The tuple $\underline{M} := (W;P;W;(\emptyset,\emptyset);\mathcal{B})$ is a combinatorial $(W;P;W)$-geometry.*
   (b) *The triple $(W,P,\mathcal{B})$ is a (W,P)-matroid in the sense of* [14], [15].
   (c) *$\mathcal{B}$ is a non-empty convex subset of $\left(W^P, d_\Gamma\right)$.*
   (d) *There exists some $w_0 \in W$ and some parabolic subgroup $W' = \langle P' \rangle$ of W with $P \subseteq P' \subseteq R$ and*

$$W' = \{w \in W \mid w \cdot w_0 \cdot W_P \in \mathcal{B}\}. \tag{51}$$

ii) *Assume $\mathcal{B}$ satisfies the four equivalent conditions in i), and suppose in particular that w_0 and $W' = \langle P' \rangle$ are as in* i) (d). *Then for a map $f : W^P \to K^* \dot{\cup} \{0\}$ the pair (η_0, f) is a representation of $\underline{M}$ if and only if there exists some $\kappa \in K^*$ and some map $\psi : P' \setminus P \to K^*$ such that for $w \in \langle R \setminus P \rangle$ one has*

$$f(w \cdot w_0 \cdot W_P) = \begin{cases} \kappa \cdot \prod\limits_{r \in R(w)} \psi(r) & \text{if} \quad w \in \langle P' \setminus P \rangle \\ 0 & \text{if} \quad w \in W \setminus W'. \end{cases} \tag{52}$$

In particular, for every fuzzy ring K there exists at least one $(W;P;W)$-matroid with coefficients in K and $\mathcal{B}$ as its set of bases.

Proof:

i) is [25, Proposition 4.10].

ii) Assume first that f is as in (52). Then by (51) we have

$$f^{-1}(K^*) = \{w \cdot w_0 \cdot W_P \mid w \in W'\} = \mathcal{B};$$

thus axiom (M1) holds. To verify (M2) assume – in view of Lemma 4.4 – that $A, B \in \mathcal{B}$ are distinct, and $\mathcal{N}$ is a fundamental neighbourhood of A in $\langle A, B\rangle := \langle A, B\rangle_\Gamma$. Since Γ does not contain any triangles, we have $\mathcal{N} = \{A, C\}$ for some $C \in W^P$ with $1 = d_\Gamma(A, C) = d_\Gamma(A, B) - d_\Gamma(C, B)$. By (51) there exist $w_1, w_2 \in W'$ with $A = w_1 \cdot w_0 \cdot W_P$ and $B = w_2 \cdot w_0 \cdot W_P$, and we have $C = r \cdot w_1 \cdot w_0 \cdot W_P$ for some $r \in (P' \setminus P) \cap R(w_1 \cdot w_2)$. Then we get $ap_{\langle A,B\rangle}(C) = r \cdot w_2 \cdot w_0 \cdot W_P$ and either $\{r\} = R(r \cdot w_1) \setminus R(w_1) = R(w_2) \setminus R(r \cdot w_2)$ or $\{r\} = R(r \cdot w_2) \setminus R(w_2) = R(w_1) \setminus R(r \cdot w_1)$.
By (52) this means in any case

$$\begin{aligned}
&\kappa_{\eta_0,f}(A, B; \mathcal{N})\\
&= \varepsilon \cdot f(w_1 \cdot w_0 \cdot W_P) \cdot f(w_2 \cdot w_0 \cdot W_P) + f(r \cdot w_1 \cdot w_0 \cdot W_P) \cdot f(r \cdot w_2 \cdot w_0 \cdot W_P)\\
&= f(w_1 \cdot w_0 \cdot W_P) \cdot f(w_2 \cdot w_0 \cdot W_P) \cdot \left(\varepsilon + \psi(r) \cdot \psi(r)^{-1}\right) \in K_0.
\end{aligned}$$

Now suppose that vice versa (η_0, f) is a representation of $\underline{M}$. We prove that for $w_1, w_2 \in W'$ and $r \in P' \setminus P$ with $R(r \cdot w_i) = R(w_i) \dot\cup \{r\}$ for $i \in \{1, 2\}$ one has

$$\begin{aligned}
&f(r \cdot w_1 \cdot w_0 \cdot W_P) \cdot f(w_1 \cdot w_0 \cdot W_P)^{-1}\\
&= f(r \cdot w_2 \cdot w_0 \cdot W_P) \cdot f(w_2 \cdot w_0 \cdot W_P)^{-1}.
\end{aligned} \tag{53}$$

Put $A := w_1 \cdot w_0 \cdot W_P$, $B := r \cdot w_2 \cdot w_0 \cdot W_P$, $C := r \cdot w_1 \cdot w_0 \cdot W_P$. The relation $r \notin R(w_1) \cup R(w_2)$ implies $C \in \langle A, B\rangle$ and $ap_{\langle A,B\rangle}(C) = w_2 \cdot w_0 \cdot W_P$. Since $\{A, C\}$ is a fundamental neighbourhood of A in $\langle A, B\rangle$, axiom (M2) implies

$$\varepsilon \cdot f(A) \cdot f(B) + f(C) \cdot f\left(ap_{\langle A,B\rangle}(C)\right) \in K_0,$$

and thus (53) follows from (FR5').
By what we have just shown we get a well-defined map $\psi : P' \setminus P \to K^*$ by

$$\psi(r) := f(r \cdot w \cdot w_0 \cdot W_P) \cdot f(w \cdot w_0 \cdot W_P)^{-1} \quad \text{for } w \in W' \text{ with } r \notin R(w).$$

Thus (52) follows for $\kappa := f(w_0 \cdot W_P)$. □

So far, we have considered examples of $(W;P;U)$-matroids with coefficients for fixed k-regular triples $(W;P;U)$. In forthcoming papers we shall also study $(W;P;U)$-matroids with coefficients in a fixed fuzzy ring K thereby using the theory of the Tutte group of a combinatorial $(W;P;U)$-geometry which will be developed in [27].

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Received: June 7, 1994, revised version September 29, 1995