

Rational Summation of Rational Functions

Laura Felicia Matusevich

*University of California at Berkeley
e-mail: laura@math.berkeley.edu*

Abstract. In this article we characterize rational functions for which their indefinite sum is again a rational function.

1. Introduction

Let k be a field of characteristic 0 and let r be a rational function in one variable over k . Pick an integer j_0 such that $r(j)$ is defined for $j \geq j_0$, and consider

$$y_x = \sum_{j=j_0}^x r(j), \quad x > j_0.$$

If there exists a rational function y such that $y(x) = y_x$ for all $x > j_0$, we say that r is *rationally summable*. Notice that r is rationally summable if and only if the difference equation

$$\Delta y = y(x+1) - y(x) = r(x) \tag{1}$$

has a solution in the field $k(x)$ of rational functions in one variable. Of course, this is equivalent to requiring the equation

$$\nabla t = t(x) - t(x-1) = r(x)$$

to have a rational solution.

We consider the question, suggested by Alberto Grünbaum, of finding necessary and sufficient conditions for rational summability. This is accomplished in Theorems 10 and 11.

The rational summation problem has been studied by Abramov ([1]), Paule ([2] and [3]), Pirastu and Strehl ([4] and [5]). These authors give algorithms, based on either the Gosper-Petkovšek representation or the shift saturated representation of a rational function, to decide whether a rational function is rationally summable or not. In the affirmative case, a rational solution to (1) is found. Otherwise, the output is a rational function, closest to a solution of (1) in some suitable sense.

Example 1. Let r be a polynomial of degree d in $k[x]$. We can write $r(n) = \sum_{i=0}^d a_i \binom{n}{i}$, where $a_i \in k$. Define $y(n) = \sum_{i=0}^d a_i \binom{n}{i+1}$. Then $\Delta y = r$.

In view of the division algorithm and Example 1, we assume from now on that $r = p/q$, with p, q polynomials, and $\deg(p) < \deg(q)$.

2. Characterizing rational summability

We define the *forward shift operator*, denoted by σ . That is,

$$(\sigma y)(n) = y(n+1).$$

Definition 2. Let f be a polynomial. The dispersion of f is

$$\begin{aligned} \text{disp } f &= \max\{|l| : l \in \mathbf{Z}, \gcd(f, \sigma^l f) \neq 1\} \\ &= \max\{|a-b| : a, b \in \bar{k} \text{ are roots of } f \text{ and } a-b \in \mathbf{Z}\} \end{aligned}$$

where \bar{k} is an algebraic closure of k . If $\text{disp } f = 0$, then we say that f is *shift-free*.

The concept of dispersion was first introduced by Abramov ([1]).

Lemma 3. Let $y \in k(x)$ and $0 \neq \Delta y = \frac{P}{Q}$, with P and Q not necessarily coprime polynomials. Then $\text{disp } Q > 0$.

Proof. Say $y = f/g$ in lowest terms (that is, f and g are relatively prime polynomials). Let a be a root of g (in an algebraic closure of k), and t a nonnegative integer such that $g(a-t) = 0$, but $g(a+1)$ and $g(a-t-1)$ are nonzero. We claim that $Q(a) = Q(a-t-1) = 0$. Thus $\text{disp } Q \geq t+1 > 0$. To see this, look at

$$\frac{P}{Q} = \Delta \frac{f}{g} = \frac{g(x)f(x+1) - g(x+1)f(x)}{g(x)g(x+1)}$$

The numbers a and $a-t-1$ are roots of $g(x)g(x+1)$, but not of $g(x)f(x+1) - g(x+1)f(x)$, so the factors $(x-a)$, $(x-a+t+1)$ of $g(x)g(x+1)$ could not have been cancelled. \square

Now we investigate the rational summability of functions of the form

$$r = \sum_{i=0}^{l-1} \frac{p_i}{\sigma^{-i}q} ; \quad p_i, q \in k[x]$$

where the numerators and denominators are not necessarily coprime.

Lemma 4. Let $r = \sum_{i=0}^{l-1} p_i/\sigma^{-i}q$. Consider the rational function

$$r^* = \sum_{i=0}^{l-1} \frac{\sigma^i p_i}{q} = \frac{\sum_{i=0}^{l-1} \sigma^i p_i}{q}.$$

Then r is rationally summable if and only if r^* is rationally summable.

Proof. First notice that $r - \sigma^{-l}r^*$ is rationally summable. To see this write:

$$r - \sigma^{-l}r^* = \sum_{i=0}^{l-1} \left(\frac{p_i}{\sigma^{-i}q} - \sigma^{i-l} \left(\frac{p_i}{\sigma^{-i}q} \right) \right).$$

Call $s_i = p_i/\sigma^{-i}q$. The telescoping identity

$$s_i - \sigma^{i-l} s_i = \sum_{j=0}^{l-i-1} (\sigma^{-j} s_i - \sigma^{-j-1} s_i) = \nabla \left(\sum_{j=0}^{l-i-1} \sigma^{-j} s_i \right)$$

implies that each summand of $r - \sigma^{-l}r^*$ is rationally summable, so that $r - \sigma^{-l}r^*$ is rationally summable as well.

Pick a rational function s such that $r - \sigma^{-l}r^* = \Delta s$. If r is rationally summable, then there exists a rational function y with $\Delta y = r$. This means that $\Delta(y - s) = \sigma^{-l}r^*$, and hence $r^* = \Delta \sigma^l(y - s)$. Conversely, if r^* is rationally summable, then there exists a rational function u such that $\Delta u = r^*$. Hence $r = \Delta(s - \sigma^{-l}u)$. □

Proposition 5. Let $r = \sum_{i=0}^{l-1} p_i/\sigma^{-i}q$, and suppose q is shift-free. Then r is rationally summable if and only if $p_0 + \sigma p_1 + \dots + \sigma^{l-1}p_{l-1} = 0$. In that case, the rational function

$$y := \sum_{i=0}^{l-2} \frac{\sum_{j=0}^i \sigma^{j-(i+1)} p_j}{\sigma^{-(i+1)}q}$$

satisfies $\Delta y = r$.

Proof. Assume that r is rationally summable. By Lemma 4, so is r^* , whose denominator is shift-free. From Lemma 3 we conclude that $r^* = 0$, so that $\sum_{i=0}^{l-1} \sigma^i p_i = 0$. The converse follows by direct verification of the formula for y . □

Notice that if we do not assume q to be shift-free, the previous condition is still sufficient, but no longer necessary, as the following example shows.

Example 6. Let

$$r(x) = \frac{-(x+1)(x-2)}{(x+1)x(x-1)(x-2)} + \frac{-3}{x(x-1)(x-2)(x-3)}.$$

The condition of Proposition 5 is not satisfied; however,

$$r = \Delta \left(\frac{1}{x(x-1)(x-2)} + \frac{1}{x(x-1)} \right).$$

The goal now is to break up any given $r \in k(x)$ into summands of the form of Proposition 5. If f is a polynomial, we denote:

$$[f]^l = f(\sigma^{-1}f) \cdots (\sigma^{-(l-1)}f).$$

Definition 7. Two polynomials f and g are called *shift-coprime* if

$$\gcd(\sigma^i f, g) = 1 \quad \forall i \in \mathbf{Z}.$$

Take $r = p/q$ written in lowest terms, with q monic. We want to have control over the factors of q that differ by an integer. To achieve that, we are going to replace q by a polynomial $Q = [Q_1]^l \cdots [Q_t]^l$ such that:

1. q divides Q .
2. The polynomials Q_i are all shift-free and pairwise shift-coprime.

Following Peter Paule, such a Q is called a *shift-saturated* multiple of q . For any polynomial, there is a unique shift-saturated multiple of minimal degree. An algorithm for calculating it was proposed in [2]. Unfortunately, the *RISC* report is not available, and the algorithm does not appear in the published version ([3]). Paule's algorithm has been implemented in Mathematica by Christian Mallinger, and it is from the comments in the code (kindly sent to me by Peter Paule) that the reference to [2] was obtained. Another algorithm can be found in [4].

Example 8. Let

$$q = x(x-10)^{15}(x-17)(x-25)^2(x-1/2)(x-3/2)(x-1/3)(x-4/3)^2(x-1/5).$$

The polynomials

$$Q = [x-1/5][(x-1/2)(x-1/3)^2]^2[x^{15}]^{26}$$

and

$$Q' = [x-1/5][((x-1/2)(x-1/3))^2]^5[(x-1/7)x^{15}]^{26}$$

are shift-saturated multiples of q . Among all such multiples, Q has minimal degree.

Now suppose $r = p/q$. Compute $Q = [Q_1]^l \cdots [Q_t]^l$ a shift-saturation of q . Then $r = H/Q$ for $H = pQ/q$. Since the Q_i are pairwise shift-coprime, we can write (using the extended gcd):

$$r = \sum_{i=1}^l \frac{h_i}{[Q_i]^l}$$

for some polynomials h_i .

Proposition 9. An r as above is rationally summable if and only if all $h_i/[Q_i]^l$ are rationally summable.

Proof. The following claim gives a slightly stronger statement than what we need for an induction on l .

Let $R = R_1 + R_2 \in k(x)$; $R_i = p_i/q_i$, with q_1 and q_2 shift coprime. Then R is rationally summable if and only if R_1 and R_2 are rationally summable.

The “if” implication is trivial. For the “only if”, we assume that R_i are in lowest terms (cancellation does not effect shift-coprimeness). Let $y = f/g$, with f, g coprime, such that $\Delta y = R$. Then $g = ABC$, where $A = \prod (x - \alpha)^{e_\alpha}$, over the α that are the roots of g (in \bar{k}) and of some integer shift of q_1 ; analogously for B and q_2 , and C is the leftover. Notice that A, B, C are polynomials over k , because $A = \gcd(g, ([q_1]^{t+1})^N)$, for $t = \text{disp } q_1$, and N a sufficiently large integer. Likewise for B .

As q_1 and q_2 are shift coprime, A, B and C are pairwise relatively prime. Then we can write (using the extended gcd) $y = \frac{A_1}{A} + \frac{B_1}{B} + \frac{C_1}{C}$, for some polynomials A_1, B_1, C_1 , and then

$$0 = \Delta y - R = \left(\Delta \frac{A_1}{A} - R_1 \right) + \left(\Delta \frac{B_1}{B} - R_2 \right) + \Delta \frac{C_1}{C}.$$

So we have three rational functions that add up to zero, and whose denominators are pairwise coprime. Taking into account that the denominators of the first two summands are nonconstant (since the q_i are nonconstant) we see that this cannot happen unless those rational functions are zero. In particular $R_1 = \Delta \frac{A_1}{A}$ and $R_2 = \Delta \frac{B_1}{B}$. □

Theorem 10. Let $r = \sum_{i=1}^l h_i/[Q_i]^i$ in the form of Proposition 9. In particular, the Q_i are shift free, so that the $\sigma^{-j}Q_i$ are relatively prime. Using the extended gcd, write

$$r = \sum_{i=1}^l \sum_{j=0}^{i-1} \frac{p_{i,j}}{\sigma^{-j}Q_i}$$

Then r is rational summable if and only if $p_{i,0} + \sigma p_{i,1} + \dots + \sigma^{i-1}p_{i,i-1} = 0$ for all i . In that case, the rational function:

$$y := \sum_{i=1}^l \sum_{t=0}^{i-2} \frac{\sum_{j=0}^t \sigma^{j-(t+1)} p_{i,j}}{\sigma^{-(t+1)}Q_i}$$

satisfies $\Delta y = r$.

Proof. Propositions 5 and 9. □

3. Remarks

- Given $r = p/q$, it is possible (even symbolically) to find a shift free multiple of q , so that multiplying and dividing by an adequate polynomial, Proposition 5 applies. However, the denominator in such a representation of r will have much larger degree than the one we get by using Theorem 10.
- If we know the complete partial fraction decomposition of our rational function, the previous results can be restated as follows:

Theorem 11. *Suppose we are given:*

$$r(x) = \sum_{n=1}^l \sum_{i=0}^s \sum_{j=1}^e \frac{A_{n,i,j}}{(x - \alpha_n - i)^j}$$

where $\alpha_n - \alpha_m$ is not an integer for $n \neq m$ and the $A_{n,i,j}$ are (not necessarily nonzero) constants. Then r is rational summable if and only if $\sum_{i=0}^s A_{n,i,j} = 0$ for all j, n . In that case, the rational function

$$y(x) = \sum_{n=1}^l \sum_{j=1}^e \sum_{t=1}^s \frac{\sum_{i=0}^{t-1} A_{n,i,j}}{(x - \alpha_n - t)^j}$$

satisfies $\Delta y = r$.

Proof. The difficult implication is a straightforward consequence of Propositions 5 and 9. \square

Acknowledgements. I am very grateful to Alberto Grünbaum, Peter Paule, Marko Petkovšek, Michael Singer and Bernd Sturmfels for their help and encouragement, and to Alicia Dickenstein, for her advice and comments. I would also like to thank the reviewer, who suggested Lemma 4, and greatly improved the presentation.

This work was partially supported by the David and Lucile Packard Foundation.

References

- [1] Abramov, S. A.: *The Rational Component of the Solution of a First-Order Linear Recurrence Relation with a Rational Right Side*. Zh. vychisl. Mat. mat. fis. no. 4, **15** (1975), 40–42.
- [2] Paule, Peter: *Greatest Factorial Factorization and Symbolic Summation 1*. RISC-Linz Report Series No. 93-02 (1993).
- [3] Paule, Peter: *Greatest Factorial Factorization and Symbolic Summation*. J. Symbolic Computation **20** (1995), 235–268.
- [4] Pirastu, Roberto: *Algorithmen zur Summation rationaler Funktionen* (in German). Diploma thesis, University of Erlangen-Nürnberg 1992.
- [5] Pirastu, Roberto; Strehl, Volker: *Rational Summation and the Gosper-Petkovšek Representation*. J. Symbolic Computation **20** (1995), 617–635.

Received June 29, 1999