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On Some Influence of the Weak Subalgebra Lattice on the Subalgebra Lattice

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Abstract. In [14] we showed that for each locally finite unary (total) algebra of finite type, its weak subalgebra lattice uniquely determines its (strong) subalgebra lattice. Now we generalize this fact to algebras having also finitely many binary operations (for example, groupoids, semigroups, semilattices). More precisely, we generalize some ideas from [14] to prove: Let $\mathbf{A}$ be a locally finite (total) algebra with m unary operations $k_1^{\mathbf{A}}, \dots, k_m^{\mathbf{A}}$ and n binary operations $f_1^{\mathbf{A}}, \dots, f_n^{\mathbf{A}}$ and let $\mathbf{A}$ satisfy the following formula: for any x, y and $1 \leq i \leq n$, $x \neq y$ implies $f_i(x,y) \neq x$ and $f_i(x,y) \neq y$. Then for every partial algebra $\mathbf{B}$ with m unary and n binary operations, if the weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic, then their (strong) subalgebra lattices are also isomorphic and moreover, $\mathbf{B}$ is total and locally finite and satisfies the same formula.

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1. Introduction

The lattice of (total) subalgebras and connections between (total) algebras and their subalgebra lattices are an important part of universal algebra. For instance, characterizations of subalgebra lattices for algebras in a given variety or of a given type are this kind of problems (see e.g. [10]). Several results (see e.g. [6], [11], [13], [17], [18]) describe algebras or varieties of algebras which have special subalgebra lattices (i.e. modular, distributive, etc.). For example, it is proved in [6] that every subalgebra modular variety is Hamiltonian, so it is Abelian by [11]. Some such results concern also classical algebras (see e.g. [12], [16] or [8], [9]). More precisely, D. Sachs in [16] proved that any Boolean algebra uniquely determines its subalgebra lattice. E. Lukács and P.P. Pálfy showed in [12] that the modularity of the subgroup lattice of the direct square $\mathbf{G} \times \mathbf{G}$ of any group $\mathbf{G}$ implies that $\mathbf{G}$ is commutative.

The theory of partial algebras provides additional tools for such investigations, because several different structures may be considered in this case (see e.g. [3] or [5]). In the present paper, besides the usual subalgebras (which will here be called strong as opposed to other kinds of partial subalgebras) and lattices of strong subalgebras, we consider only the weak subalgebras and the lattices of weak subalgebras. It seems that the weak subalgebra lattice alone, and also together with the strong subalgebra lattice, yields a lot of interesting information on an algebra, also total. For example, [2] contains a characterization of monounary partial algebras uniquely determined in the class of all monounary partial algebras of the same type by their weak subalgebra lattices. In [14] it is shown that for a total and locally finite unary algebra of finite type, its weak subalgebra lattice uniquely determines its strong subalgebra lattice. A complete characterization of the weak subalgebra lattice is given in [1].

We introduced in [15] a hypergraph-algebraic language to investigate partial algebras and their subalgebra lattices. For instance, we have shown in [15] that for partial algebras $\mathbf{A}$ and $\mathbf{B}$ (even of different types), if their directed hypergraphs are isomorphic, then their subalgebra (weak, relative, strong and initial segment) lattices are isomorphic. Moreover, weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic iff their hypergraphs are isomorphic. We also characterized in [15] pairs $\langle \mathbf{L}, \mathbf{A} \rangle$, where $\mathbf{L}$ is a lattice and $\mathbf{A}$ is a partial algebra, such that the weak subalgebra lattice of $\mathbf{A}$ is isomorphic to $\mathbf{L}$.

Now, using this language, we generalize the result from [14] onto algebras with finitely many unary and binary operations. More precisely, let $\mathbf{A}$ be a total, locally finite algebra with unary operations $k_1^{\mathbf{A}}, \dots, k_m^{\mathbf{A}}$ and binary operations $f_1^{\mathbf{A}}, \dots, f_n^{\mathbf{A}}$ such that for $a_1, a_2 \in A$ and $k = 1, \dots, n$, $a_1 \neq a_2$ implies $f_k^{\mathbf{A}}(a_1, a_2) \neq a_1$ and $f_k^{\mathbf{A}}(a_1, a_2) \neq a_2$. Then we prove that for a partial algebra $\mathbf{B}$ with m unary and n binary operations, if the weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ are isomorphic, then their strong subalgebra lattices are isomorphic, and $\mathbf{B}$ is also total, locally finite and satisfies the same formula.

2. Basic results

For basic notions and facts concerning algebras (partial and total) and subalgebras and lattices of subalgebras see e.g. [3], [5] and [7], [10]; concerning hypergraphs see e.g. [4].

Let $\mathbf{A} = \langle A, (k_i^{\mathbf{A}})_{i=1}^{i=m}, (f_i^{\mathbf{A}})_{i=1}^{i=n} \rangle$ and $\mathbf{B} = \langle B, (k_i^{\mathbf{B}})_{i=1}^{i=m}, (f_i^{\mathbf{B}})_{i=1}^{i=n} \rangle$ be partial algebras with m unary and n binary operations. Recall that $\mathbf{B}$ is a *weak subalgebra* of $\mathbf{A}$ iff $B \subseteq A$ and $k_i^{\mathbf{B}} \subseteq k_i^{\mathbf{A}}$, $f_j^{\mathbf{B}} \subseteq f_j^{\mathbf{A}}$ for $i = 1, \dots, m$, $j = 1, \dots, n$. The set of all weak subalgebras of $\mathbf{A}$ forms a complete and algebraic lattice $\mathbf{S}_w(\mathbf{A})$ under (weak subalgebra) inclusion $\leq_w$. Analogously, the algebraic lattice of all strong subalgebras of $\mathbf{A}$ under (strong subalgebra) inclusion $\leq_s$ will be denoted by $\mathbf{S}_s(\mathbf{A})$.

We need some connections between hypergraphs and partial algebras proved in [15]. An (undirected) hypergraph $\mathbf{H} = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{H}} \rangle$ is a triplet (see e.g. [4]) such that $V^{\mathbf{H}}$ and

$E^{\mathbf{H}}$ are sets (of vertices and hyperedges respectively), and $I^{\mathbf{H}}$ is a function of $E^{\mathbf{H}}$ into the family of all finite and non-empty subsets of $V^{\mathbf{H}}$. A dihypergraph (directed hypergraph) $\mathbf{D} = \langle V^{\mathbf{D}}, E^{\mathbf{D}}, I^{\mathbf{D}}\rangle$ is a triplet such that $V^{\mathbf{D}}$ and $E^{\mathbf{D}}$ are sets, and $I^{\mathbf{D}} = \langle I_1^{\mathbf{D}}, I_2^{\mathbf{D}}\rangle$ is a pair, where $I_1^{\mathbf{D}}$ is a function of $E^{\mathbf{D}}$ into the family of all finite subsets of $V^{\mathbf{D}}$, and $I_2^{\mathbf{D}}$ is a function of $E^{\mathbf{D}}$ into $V^{\mathbf{D}}$.

With each dihypergraph $\mathbf{D}$ we can associate the hypergraph $\mathbf{D}^*$ by omitting the orientation of all hyperedges, i.e.

$$V^{\mathbf{D}^*} = V^{\mathbf{D}}, \quad E^{\mathbf{D}^*} = E^{\mathbf{D}} \quad \text{and}$$
$$I^{\mathbf{D}^*}(e) = I_1^{\mathbf{D}}(e) \cup \{I_2^{\mathbf{D}}(e)\} \text{ for each } e \in E^{\mathbf{D}}.$$

Each partial algebra $\mathbf{A} = \big\langle A, (k_i^{\mathbf{A}})_{i=1}^{i=m}, (f_i^{\mathbf{A}})_{i=1}^{i=n}\big\rangle$ with m unary and n binary operations can be represented by a dihypergraph $\mathbf{D}(\mathbf{A})$ as follows (see [15]):

$$\begin{aligned} V^{\mathbf{D}(\mathbf{A})} &= A, \\ E^{\mathbf{D}(\mathbf{A})} &= \big\{\langle a, i, b\rangle \in A \times \{1, \dots, m\} \times A : \ \langle a, b\rangle \in k_i^{\mathbf{A}}\big\} \bigcup \\ &\quad \big\{\langle a_1, a_2, j, b\rangle \in A \times A \times \{1, \dots, n\} \times A : \ \langle a_1, a_2, b\rangle \in f_j^{\mathbf{A}}\big\}, \end{aligned}$$

and for each $\langle a, i, b\rangle, \ \langle a_1, a_2, j, b\rangle \in E^{\mathbf{D}(\mathbf{A})}$,

$$\begin{aligned} &I_1^{\mathbf{D}(\mathbf{A})}\big(\langle a, i, b\rangle\big) = a, \quad I_2^{\mathbf{D}(\mathbf{A})}\big(\langle a, i, b\rangle\big) = b \qquad \text{and} \\ &I_1^{\mathbf{D}(\mathbf{A})}\big(\langle a_1, a_2, j, b\rangle\big) = \{a_1, a_2\}, \quad I_2^{\mathbf{D}(\mathbf{A})}\big(\langle a_1, a_2, j, b\rangle\big) = b. \end{aligned}$$

We can also associate with $\mathbf{A}$ the hypergraph $\mathbf{D}^*(\mathbf{A}) = \big(\mathbf{D}(\mathbf{A})\big)^*$.

First, we use hypergraphs to represent partial algebras, and therefore we do not restrict the cardinality of vertex and hyperedge sets. Secondly, we consider only (partial) algebras $\mathbf{A}$ with unary and binary operations (for example, groupoids, semigroups and semilattices are this kind of algebras). Hence, for each hyperedge e of $\mathbf{D}(\mathbf{A})$, its initial set $I_1^{\mathbf{D}(\mathbf{A})}(e)$ has one or two vertices. Thus we can restrict our attention to dihypergraphs whose hyperedges have one- or two-element initial sets. Then the set of hyperedges of a dihypergraph $\mathbf{D}$ can be divided onto two classes. More precisely, $e \in E^{\mathbf{D}}$ is *a 1-edge*, or simply *an edge*, iff $|I_1^{\mathbf{D}}(e)| = 1$. e is *a 2-edge* iff $|I_1^{\mathbf{D}}(e)| = 2$. The set of all edges and 2-edges are denoted by $E^{\mathbf{D}}(1)$ and $E^{\mathbf{D}}(2)$ respectively. Moreover, $e \in E^{\mathbf{D}}(2)$ is *regular* (*a 2-loop*) iff $I_2^{\mathbf{D}}(e) \notin I_1^{\mathbf{D}}(e)$ $\big(I_2^{\mathbf{D}}(e) \in I_1^{\mathbf{D}}(e)\big)$. Regular edges and loops are analogously defined. The set of all regular edges and 2-edges are denoted by $E_{reg}^{\mathbf{D}}(1)$ and $E_{reg}^{\mathbf{D}}(2)$.

For any finite subset $V \subseteq V^{\mathbf{D}}$ we can define $E_s^{\mathbf{D}}(V) = \big\{e \in E^{\mathbf{D}} : \ I_1^{\mathbf{D}}(e) = V\big\}$, and the cardinal number $s^{\mathbf{D}}(V) = |E_s^{\mathbf{D}}(V)|$. In our case only for one- and two-element sets V, $s^{\mathbf{D}}(V)$ may be non-zero. If $V = \{v_1\}$ or $V = \{v_1, v_2\}$, then we write $E_s^{\mathbf{D}}(v_1)$, $E_s^{\mathbf{D}}(v_1, v_2)$ and $s^{\mathbf{D}}(v_1)$, $s^{\mathbf{D}}(v_1, v_2)$

Since we consider only dihypergraphs $\mathbf{D}$ such that $E^{\mathbf{D}} = E^{\mathbf{D}}(1) \cup E^{\mathbf{D}}(2)$, *the type of dihypergraphs* (defined in [15]) can be represented by a pair of cardinal numbers. More formally, let $\mathbf{D}$ be a dihypergraph and η_1, η_2 cardinal numbers. Then $\mathbf{D}$ is *of type* $\langle \eta_1, \eta_2\rangle$ iff $s^{\mathbf{D}}(v) \leq \eta_1$ for $v \in V^{\mathbf{D}}$ and $s^{\mathbf{D}}(V) \leq \eta_2$ for all two-element $V \subseteq V^{\mathbf{D}}$. $\mathbf{D}$ of type $\langle \eta_1, \eta_2\rangle$ is *total* iff $s^{\mathbf{D}}(v) = \eta_1$ for $v \in V^{\mathbf{D}}$ and $s^{\mathbf{D}}(v_1, v_2) = \eta_2$ for $v_1, v_2 \in V^{\mathbf{D}}$, $v_1 \neq v_2$. A type $\langle \eta_1, \eta_2\rangle$ is *finite* iff η_1 and η_2 are non-negative integers, i.e. $\eta_1, \eta_2 \in \mathbb{N}$.

Proposition 2.1. *Let* $\mathbf{A}$ *be a partial algebra with* m *unary and* n *binary operations. Then*

(a) $\mathbf{D}(\mathbf{A})$ *is a dihypergraph of type* $\langle m+n, 2\cdot n\rangle$.

(b) $\mathbf{A}$ *is total iff* $\mathbf{D}(\mathbf{A})$ *is total.*

It follows from simple observations that for $a,b \in A$ with $a \neq b$, $s^{\mathbf{D}(\mathbf{A})}(a) = \big|\{1 \le i \le m\colon a \in dom(k_i^{\mathbf{A}})\} \cup \{1 \le i \le n\colon \langle a,a\rangle \in dom(f_i^{\mathbf{A}})\}\big|$ and $s^{\mathbf{D}(\mathbf{A})}(a,b) = \big|\{1 \le i \le n\colon \langle a,b\rangle \in dom(f_i^{\mathbf{A}})\}\big| + \big|\{1 \le i \le n\colon \langle b,a\rangle \in dom(f_i^{\mathbf{A}})\}\big|$. Here $k_1^{\mathbf{A}}, \dots, k_m^{\mathbf{A}}$ and $f_1^{\mathbf{A}}, \dots, f_n^{\mathbf{A}}$ are unary and binary operations of $\mathbf{A}$, respectively, and $dom(h^{\mathbf{A}})$ is the domain of any partial function $h^{\mathbf{A}}$.

It is easy to prove (see [15]) that for any $m,n \in \mathbb{N}$ and a dihypergraph $\mathbf{D}$ of type $\langle m+n, 2\cdot n\rangle$ there is a partial algebra $\mathbf{A}$ with m unary and n binary operations such that $\mathbf{D}(\mathbf{A}) \simeq \mathbf{D}$. By Proposition 2.1(b) we obtain also that if $\mathbf{D}$ is a total dihypergraph, then $\mathbf{A}$ is a total algebra.

Two kinds of subdihypergraphs can be defined which represent weak and strong subalgebras (see [15]). Let $\mathbf{D}$ and $\mathbf{G}$ be dihypergraphs. Then $\mathbf{G}$ is *a weak subdihypergraph* of $\mathbf{D}$ ($\mathbf{G} \le_w \mathbf{D}$) iff $V^{\mathbf{G}} \subseteq V^{\mathbf{D}}$, $E^{\mathbf{G}} \subseteq E^{\mathbf{D}}$ and $I^{\mathbf{G}} \subseteq I^{\mathbf{D}}$. $\mathbf{G}$ is *a strong subdihypergraph* of $\mathbf{D}$ ($\mathbf{G} \le_s \mathbf{D}$) iff $\mathbf{G} \le_w \mathbf{D}$ and for $e \in E^{\mathbf{D}}$, $I_1^{\mathbf{D}}(e) \subseteq V^{\mathbf{D}}$ implies $e \in E^{\mathbf{D}}$. Note that a subdihypergraph is called "weak" to stress its relation with weak subalgebras. We call a subdihypergraph "strong" when it represents a strong subalgebra. Since we consider only dihypergraphs with edges and 2-edges, the empty hypergraph $\langle \emptyset, \emptyset, \emptyset\rangle$ is a strong (thus also weak) subdihypergraph of every dihypergraph. Obviously each weak subdihypergraph of a dihypergraph of type $\langle \eta_1, \eta_2\rangle$ is also of this type.

It is proved in [15], in an analogous way as for partial algebras, that the sets of all weak and strong subdihypergraphs of $\mathbf{D}$ with relations $\le_w$ and $\le_s$ respectively, form complete and algebraic lattices $\mathbf{S}_w(\mathbf{D})$ and $\mathbf{S}_s(\mathbf{D})$. The operations of infimum $\bigwedge$ and supremum $\bigvee$ are defined as in the case of partial algebras. In particular, for any set $W \subseteq V^{\mathbf{D}}$, there is the least strong subdihypergraph containing W which will be denoted by $\langle W\rangle_{\mathbf{D}}$. Analogously as for algebras, we say that $\mathbf{D}$ is *locally finite* iff for each finite set $W \subseteq V^{\mathbf{D}}$, $\langle W\rangle_{\mathbf{D}}$ is also finite (i.e. its vertex set $V^{\langle W\rangle_{\mathbf{D}}}$ is finite).

In the same way as above we can define weak subhypergraphs of an (undirected) hypergraph $\mathbf{H}$, and moreover, the set of all weak subhypergraphs of $\mathbf{H}$ also forms an algebraic lattice $\mathbf{S}_w(\mathbf{H})$ (see [15]).

Theorem 2.2. *Let* $\mathbf{A}$ *be a partial algebra. Then*

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w\big(\mathbf{D}(\mathbf{A})\big) \text{ and } \mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s\big(\mathbf{D}(\mathbf{A})\big).$$

Proof. Recall (see [15]) that these isomorphisms are given by φ such that $\varphi(\mathbf{B}) = \mathbf{D}(\mathbf{B})$ for $\mathbf{B} \le_w \mathbf{A}$.

First, for each weak subdihypergraph $\mathbf{H} \le_w \mathbf{D}(\mathbf{A})$, it is not difficult to construct the weak subalgebra $\mathbf{B}$ of $\mathbf{A}$ such that $\mathbf{D}(\mathbf{B}) = \mathbf{H}$. And if $\mathbf{H}$ is a strong subdihypergraph, then $\mathbf{B}$ is a strong subalgebra.

Secondly, it is obtained by a standard verification that for any weak (strong) subalgebras $\mathbf{B}_1, \mathbf{B}_2$ of $\mathbf{A}$, $\mathbf{B}_1 = \mathbf{B}_2$ iff $\mathbf{D}(\mathbf{B}_1) = \mathbf{D}(\mathbf{B}_2)$, and $\mathbf{B}_1 \le_w \mathbf{B}_2$ iff $\mathbf{D}(\mathbf{B}_1) \le_w \mathbf{D}(\mathbf{B}_2)$ ($\mathbf{B}_1 \le_s \mathbf{B}_2$ iff $\mathbf{D}(\mathbf{B}_1) \le_s \mathbf{D}(\mathbf{B}_2)$).

These facts imply that φ is bijective and moreover, φ and its inverse φ^{-1} are order-preserving. Thus φ is an isomorphism of $\mathbf{S}_w(\mathbf{A})$ onto $\mathbf{S}_w\big(\mathbf{D}(\mathbf{A})\big)$. Analogously, φ restricted to the set of all strong subalgebras is an isomorphism between $\mathbf{S}_s(\mathbf{A})$ and $\mathbf{S}_s\big(\mathbf{D}(\mathbf{A})\big)$. □

In exactly the same way we can show that the function ψ such that $\psi(\mathbf{B}) = \mathbf{D}^*(\mathbf{B})$ for each $\mathbf{B} \leq_w \mathbf{A}$ forms an isomorphism of lattices $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w\big(\mathbf{D}^*(\mathbf{A})\big)$ (see also [15]).

Proposition 2.3. *A partial algebra* $\mathbf{A}$ *is locally finite iff* $\mathbf{D}(\mathbf{A})$ *is a locally finite dihypergraph.*

From the proof of Theorem 2.2 (see also [15]) it follows that for a partial algebra $\mathbf{A}$ and $B \subseteq A$, $\mathbf{D}\big(\langle B\rangle_{\mathbf{A}}\big) = \langle B\rangle_{\mathbf{D}(\mathbf{A})}$ (where $\langle B\rangle_{\mathbf{A}}$ is *the strong subalgebra of* $\mathbf{A}$ *generated by* B).

Theorem 2.4. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be partial algebras (which can be of arbitrary and different types). Then*

$$\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w(\mathbf{B}) \quad \textit{iff} \quad \mathbf{D}^*(\mathbf{A}) \simeq \mathbf{D}^*(\mathbf{B}).$$

Proof. It is also proved in [15]. Now we sketch only main steps of this proof. $\Leftarrow$ is obvious, since $\mathbf{S}_w(\mathbf{A}) \simeq \mathbf{S}_w\big(\mathbf{D}^*(\mathbf{A})\big)$, $\mathbf{S}_w(\mathbf{B}) \simeq \mathbf{S}_w\big(\mathbf{D}^*(\mathbf{B})\big)$ and, of course, $\mathbf{S}_w\big(\mathbf{D}^*(\mathbf{A})\big) \simeq \mathbf{S}_w\big(\mathbf{D}^*(\mathbf{B})\big)$.

Let $\mathbf{S}_w(\mathbf{A})$ and $\mathbf{S}_w(\mathbf{B})$ be isomorphic. Then there is also an isomorphism Φ of $\mathbf{S}_w\big(\mathbf{D}^*(\mathbf{A})\big)$ onto $\mathbf{S}_w\big(\mathbf{D}^*(\mathbf{B})\big)$. In particular, we have bijections Φ_a and Φ_i between the sets of all atoms and the sets of all non-atomic join-irreducible elements (of these lattices) respectively. A non-zero element i of a lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ is *join-irreducible* iff for $k, l \in L$, $k \vee l = i$ implies $k = i$ or $l = i$ (see e.g. [10]).

Let $\mathbf{H}$ be a hypergraph and $\mathbf{M}$ a weak subhypergraph. It is easy to show (in the same way as for partial algebras, see [1]) that: $\mathbf{M}$ is an atom in $\mathbf{S}_w(\mathbf{H})$ iff $\mathbf{M}$ has exactly one vertex and none hyperedges. $\mathbf{M}$ is a non-atomic join-irreducible element in $\mathbf{S}_w(\mathbf{H})$ iff $\mathbf{M}$ has exactly one hyperedge and its endpoints as the set of vertices.

These facts imply that Φ_a and Φ_i induce the bijections φ and ψ between $V^{\mathbf{D}^*(\mathbf{A})}$, $V^{\mathbf{D}^*(\mathbf{B})}$ and $E^{\mathbf{D}^*(\mathbf{A})}$, $E^{\mathbf{D}^*(\mathbf{B})}$ respectively. Moreover, using these facts, since Φ and Φ^{-1} preserve $\leq_w$, it can be obtained (by standard, but long verification) $\varphi\big(I^{\mathbf{D}^*(\mathbf{A})}(e)\big) = I^{\mathbf{D}^*(\mathbf{B})}\big(\psi(e)\big)$ for all $e \in E^{\mathbf{D}^*(\mathbf{A})}$. Thus $\langle\varphi, \psi\rangle$ is an isomorphism of $\mathbf{D}^*(\mathbf{A})$ and $\mathbf{D}^*(\mathbf{B})$. □

Finally, observe that our formula: $\forall_{x,y}\forall_{1\leq i\leq n}\ x \neq y \Rightarrow f_i(x,y) \neq x \wedge f_i(x,y) \neq y$, has the following dihypergraph interpretation (obtained by a simple verification):

Proposition 2.5. *Let* $\mathbf{A}$ *be a partial algebra with* $(k_i^{\mathbf{A}})_{i=1}^{i=m}$ *unary and* $(f_i^{\mathbf{A}})_{i=1}^{i=n}$ *binary operations. Then* $\mathbf{D}(\mathbf{A})$ *has no 2-loops iff for all* $a, b \in A$ *and* $1 \leq i \leq n$, $a \neq b$ *implies* $f_i^{\mathbf{A}}(a,b) \neq a$ *and* $f_i^{\mathbf{A}}(a,b) \neq b$.

3. Main results

Results and definitions of the previous section reduce our algebraic problem to some dihypergraph question. More precisely, partial algebras $\mathbf{A}$ and $\mathbf{B}$ with m unary and n binary operations can be replaced by dihypergraphs $\mathbf{D}$ and $\mathbf{G}$ with edges and 2-edges. Assumptions on $\mathbf{A}$ are translated onto the hypergraph language as follows: $\mathbf{D}$ is a total dihypergraph of finite type $\langle m+n, 2n\rangle$ and locally finite and without 2-loops. Moreover, the condition about the weak subalgebra lattices of $\mathbf{A}$ and $\mathbf{B}$ is equivalent to the property that $\mathbf{D}^*$ and $\mathbf{G}^*$ are isomorphic. Thus to prove our algebraic result we must show: Let $\mathbf{D}$ be a total dihypergraph of finite type $\langle n_1, n_2\rangle$ and locally finite and without 2-loops, and let $\mathbf{G}$ be a dihypergraph of type $\langle n_1, n_2\rangle$ such that $\mathbf{D}^* \simeq \mathbf{G}^*$. Then the strong subdihypergraph lattices of $\mathbf{D}$ and $\mathbf{G}$ are isomorphic, and $\mathbf{G}$ is also total, locally finite and without 2-loops.

We start from simple facts describing dihypergraphs with the same (up to isomorphism) undirected hypergraphs. First, we can form, in a similar way as for graphs, new dihypergraphs by inverting the orientation of some regular edges. More precisely, let $\mathbf{D}$ be a dihypergraph and $E \subseteq E^{\mathbf{D}}_{reg}(1)$. Then $\mathbf{D}(E)$ is a dihypergraph defined as follows: $V^{\mathbf{D}(E)} = V^{\mathbf{D}}$, $E^{\mathbf{D}(E)} = E^{\mathbf{D}}$, $I^{\mathbf{D}(E)}(e) = I^{\mathbf{D}}(e)$ for $e \in E^{\mathbf{D}} \setminus E$, and $I_1^{\mathbf{D}(E)}(f) = \{I_2^{\mathbf{D}}(f)\}$, $\{I_2^{\mathbf{D}(E)}(f)\} = I_1^{\mathbf{D}}(f)$ for $f \in E$. Secondly, we can generalize this construction on sets of regular 2-edges. More formally, let $F \subseteq E^{\mathbf{D}}_{reg}(2)$ and let $U = \{u_f \colon f \in F\}$ be a set of vertices such that $u_f \in I_1^{\mathbf{D}}(f)$ for $f \in F$. Then $\mathbf{D}(F,U)$ is a dihypergraph obtained from $\mathbf{D}$ as follows: $V^{\mathbf{D}(F,U)} = V^{\mathbf{D}}$, $E^{\mathbf{D}(F,U)} = E^{\mathbf{D}}$, $I^{\mathbf{D}(F,U)}(e) = I^{\mathbf{D}}(e)$ for $e \in E^{\mathbf{D}} \setminus F$, and $I_1^{\mathbf{D}(F,U)}(f) = \big(I_1^{\mathbf{D}}(f) \setminus \{u_f\}\big) \cup \{I_2^{\mathbf{D}}(f)\}$, $I_2^{\mathbf{D}(F,U)}(f) = u_f$ for $f \in F$. We can apply both these construction to $\mathbf{D}$, and, of course, it does not matter which is first, because E and F are always disjoint. Thus we can denote $\mathbf{D}\big(E; \langle F, U\rangle\big) = \mathbf{D}(F,U)(E) = \mathbf{D}(E)(F,U)$.

Observe that the above construction preserves *, i.e. $\mathbf{D}\big(E; \langle F, U\rangle\big)^* = \mathbf{D}^*$. Unfortunately, the inverse fact is not true. More precisely, there are dihypergraphs $\mathbf{D}$ and $\mathbf{H}$ such that $\mathbf{H}^* \simeq \mathbf{D}^*$ and $\mathbf{H}$ is not isomorphic to $\mathbf{D}\big(E; \langle F, U\rangle\big)$ for any sets E, F, U (satisfying suitable conditions). It follows from the fact that images of 2-loops and regular edges under * are (undirected) hyperedges with exactly two vertices, so a 2-loop in $\mathbf{D}$ may correspond to a regular edge in $\mathbf{H}$, and conversely. But if we additionally assume that $\mathbf{D}$ and $\mathbf{H}$ have no 2-loops, then this inverse result holds. More precisely,

Lemma 3.1. *Let $\mathbf{D}$, $\mathbf{H}$ be dihypergraphs without 2-loops and $\mathbf{D}^* \simeq \mathbf{H}^*$. Then there are $F_1 \subseteq E^{\mathbf{D}}_{reg}(1)$, $F_2 \subseteq E^{\mathbf{D}}_{reg}(2)$, $U = \{u_f \colon f \in F_2\} \subseteq V^{\mathbf{D}}$ such that $u_f \in I_1^{\mathbf{D}}(f)$ for all $f \in F_2$ and $\mathbf{H} \simeq \mathbf{D}\big(F_1; \langle F_2, U\rangle\big)$.*

Proof. Let $\varphi = \langle \varphi_V, \varphi_E\rangle$ be an isomorphism of $\mathbf{D}^*$ and $\mathbf{H}^*$. First, each hyperedge of $\mathbf{D}^*$ with exactly one or exactly two endpoints is the image of an edge of $\mathbf{D}$ under *, because $\mathbf{D}$ has no 2-loops. Of course, for $\mathbf{H}$ the analogous fact holds. Thus φ_E restricted to $E^{\mathbf{D}}(1)$ is a bijection onto $E^{\mathbf{H}}(1)$. Moreover, φ_E induces a bijective correspondence between the sets of all loops of $\mathbf{D}$ and $\mathbf{H}$. Secondly, each 2-edge of $\mathbf{D}$ or $\mathbf{H}$ is a hyperedge of $\mathbf{D}^*$ or $\mathbf{H}^*$ respectively, with exactly three endpoints (because $\mathbf{D}$ and $\mathbf{H}$ have only regular 2-edges). Hence, φ_E induces also a bijection between sets of all 2-edges of $\mathbf{D}$ and $\mathbf{H}$. Now take the set $F_1 \subseteq E^{\mathbf{D}}_{reg}(1)$ ($F_2 \subseteq E^{\mathbf{D}}_{reg}(2)$) of regular edges (2-edges) e such that $\varphi_V\big(I_2^{\mathbf{D}}(e)\big) \neq I_2^{\mathbf{H}}\big(\varphi_E(e)\big)$, and let $U = \{u_f \colon f \in F_2\}$ with $u_f = \varphi_V^{-1}\big(I_2^{\mathbf{H}}(\varphi_E(f))\big)$. Then it is easily shown that φ is the desired isomorphism of $\mathbf{D}\big(F_1; \langle F_2, U\rangle\big)$ and $\mathbf{H}$. It follows from the definition of $\mathbf{D}\big(F_1; \langle F_2, U\rangle\big)$ and equalities: $\varphi_V\big(I_1^{\mathbf{D}}(e)\big) \cup \{\varphi_V\big(I_2^{\mathbf{D}}(e)\big)\} = \varphi_V\big(I^{\mathbf{D}^*}(e)\big) = I^{\mathbf{H}^*}\big(\varphi_E(e)\big) = I_1^{\mathbf{H}}\big(\varphi_E(e)\big) \cup \{I_2^{\mathbf{H}}\big(\varphi_E(e)\big)\}$ for $e \in E^{\mathbf{D}}$. □

By the above proposition we have that for a given dihypergraph $\mathbf{D}$ without 2-loops, each dihypergraph also without 2-loops with the same (undirected) hypergraph can be obtained from $\mathbf{D}$ (up to isomorphism) by changing the orientation of some hyperedges. Unfortunately, in the contradiction to edges, for regular 2-edges it is not sufficient to know which 2-edges are inverting, but we must also know how it is done, i.e. which vertices form new final vertices of these 2-edges. Therefore we now introduce a concept of labeled 2-edges. More formally, let $\mathbf{D}$ be a dihypergraph and $e \in E^{\mathbf{D}}_{reg}(2)$ and let v be a vertex such that $v \in I^{\mathbf{D}}_1(e)$. Then the pair $\langle e, v\rangle$ will be said a labeled 2-edge. Of course, each regular 2-edge e can be labeled onto exactly two ways by choosing a vertex in its initial set $I^{\mathbf{D}}_1(e)$.

Now we generalize the concepts of chains, paths and cycles in graphs. Let $\mathbf{D}$ be a dihypergraph and $r = \big(\langle e_1, v_1\rangle, \dots, \langle e_m, v_m\rangle\big)$, where $m \geq 1$, a sequence of labeled 2-edges of $\mathbf{D}$. Then r is a (directed) hyperchain iff r satisfies the following two conditions: $\big(I^{\mathbf{D}}_1(e_i) \setminus \{v_i\}\big) \cup \big\{I^{\mathbf{D}}_2(e_i)\big\} = I^{\mathbf{D}}_1(e_{i+1})$ for any $1 \leq i \leq m-1$, and for each $1 \leq i \neq j \leq m$, $e_i = e_j$ implies $v_i = v_j$ (i.e. r does not contain a 2-edge labeled onto two different ways); a hyperchain r is a (directed) hyperpath iff $e_1, \dots, e_m$ are pairwise different; a hyperchain (hyperpath) r is a hypercycle (simple hypercycle) iff $\big(I^{\mathbf{D}}_1(e_m) \setminus \{v_m\}\big) \cup \big\{I^{\mathbf{D}}_2(e_m)\big\} = I^{\mathbf{D}}_1(e_1)$. Let $E^r = \{e_1, \dots, e_m\}$ and $V^r = \{v_1, \dots, v_m\}$.

We also use usual chains, cycles and simple cycles in dihypergraphs defined as for graphs. Recall only that a sequence $(f_1., \dots, f_m)$ of edges is a simple cycle iff it is a chain with at least two vertices and its final vertex is equal to its initial vertex and all its edges are regular and pairwise different.

Observe that the construction of new dihypergraphs from a given dihypergraph $\mathbf{D}$ by changing the orientation of 2-edges in a set $F \subseteq E^{\mathbf{D}}_{reg}(2)$ according to U (where $U = \{u_f \colon f \in F\}$ is a set of vertices such that $u_f \in I^{\mathbf{D}}_1(f)$) can be formulated in terms of labeled 2-edges. Because each such pair of sets $\langle F, U\rangle$ uniquely determines the set of labeled 2-edges, and conversely, any set of labeled 2-edges (we must, of course, assume that each 2-edge in this set is labeled in exactly one way) uniquely forms such a pair of sets. In particular, we can apply these notes to families of hyperchains. More precisely, let R_1 be a family of chains and R_2 a family of pairwise hyperedge-disjoint hyperchains, i.e. for all $r, p \in R_2$, $r \neq p$ implies $E^r \cap E^p = \emptyset$. Then $\mathbf{D}(R_1, R_2) = \mathbf{D}\big(E^{R_1}; \langle E^{R_2}, V^{R_2}\rangle\big)$, where $E^{R_1} = \bigcup_{r \in R_1} E^r$ and $E^{R_2} = \bigcup_{r \in R_2} E^r$, $V^{R_2} = \bigcup_{r \in R_2} V^r$. Observe that the second condition of the definition of hyperchains and the assumption that hyperchains in R_2 are pairwise hyperedge-disjoint imply that each 2-edge in E^{R_2} is labeled in exactly one way, so this construction is correctly defined. Of course, for arbitrary families of chains and hyperchains, dihypergraphs so obtained have quite different properties, but now we prove that for families of cycles and hypercycles, this construction preserves the strong subdihypergraph lattices. We start with the following auxiliary:

Proposition 3.2. *Let $\mathbf{D}$ be a dihypergraph, R_1, R_2 families of cycles and pairwise hyperedge-disjoint hypercycles respectively. Let $\mathbf{H} \leq_w \mathbf{D}$, $\mathbf{K} \leq_w \mathbf{D}(R_1, R_2)$ be such that $V^{\mathbf{H}} = V^{\mathbf{K}}$, $E^{\mathbf{H}} = E^{\mathbf{K}}$. Then:*

$$\mathbf{H} \leq_s \mathbf{D} \quad \textit{iff} \quad \mathbf{K} \leq_s \mathbf{D}(R_1, R_2).$$

Proof. Let $\mathbf{G}$ be a dihypergraph and $\mathbf{C}$ a strong subdihypergraph of $\mathbf{G}$. By a simple induction we obtain that for any cycle r of $\mathbf{G}$, if r and $\mathbf{C}$ have a common vertex, then r is contained in $\mathbf{C}$. Moreover, an analogous result holds for hypercycles, i.e. if $r = \big(\langle f_1, v_1\rangle, \dots, \langle f_m, v_m\rangle\big)$ is a hypercycle such that $I^{\mathbf{G}}_1(f_i) \subseteq V^{\mathbf{C}}$ for some $1 \leq i \leq m$, then $E^r \subseteq E^{\mathbf{C}}$. This also follows by an induction, since $I_1^{\mathbf{G}}(f_i) \subseteq V^{\mathbf{C}}$ implies $I_2^{\mathbf{G}}(f_i) \in V^{\mathbf{C}}$, so $I_1^{\mathbf{G}}(f_{i+1}) = \big(I_1^{\mathbf{G}}(f_i)\setminus\{v_i\}\big)\cup\big\{I_2^{\mathbf{G}}(f_i)\big\} \subseteq V^{\mathbf{C}}$, where $f_{m+1} = f_1$, and so on.

$\Rightarrow$: Assume that $\mathbf{H}$ is a strong subdihypergraph of $\mathbf{D}$ and take $e \in E^{\mathbf{D}}$ such that $I_1^{\mathbf{M}}(e) \subseteq V^{\mathbf{K}} = V^{\mathbf{H}}$, where $\mathbf{M} = \mathbf{D}(R_1, R_2)$. If $e \notin E^{R_1} \cup E^{R_2}$, then $I_1^{\mathbf{M}}(e) = I_1^{\mathbf{D}}(e)$ so $e \in E^{\mathbf{H}} = E^{\mathbf{K}}$, because $\mathbf{H} \leq_s \mathbf{D}$. If $e \in E^{R_1}$, then $\{I_2^{\mathbf{D}}(e)\} = I_1^{\mathbf{M}}(e) \subseteq V^{\mathbf{K}}$ and there is a cycle $r \in R_1$ such that $e \in E^r$. Hence, $E^r \subseteq E^{\mathbf{H}} = E^{\mathbf{K}}$, in particular $e \in E^{\mathbf{H}}$. Assume now that $e \in E^{R_2}$, i.e. there is a hypercycle $r = \big(\langle f_1, v_1\rangle, \dots, \langle f_m, v_m\rangle\big)$ such that $e = f_i$ for some $i = 1, \dots, m$. Then $I_1^{\mathbf{D}}(f_{i+1}) = \big(I_1^{\mathbf{D}}(f_i) \setminus \{v_i\}\big) \cup \big\{I_2^{\mathbf{D}}(f_i)\big\} = I_1^{\mathbf{M}}(f_i) \subseteq V^{\mathbf{K}}$, where $f_{m+1} = f_1$, so $I_1^{\mathbf{D}}(f_{i+1}) \subseteq V^{\mathbf{H}}$. Hence, $e \in E^r \subseteq E^{\mathbf{H}} = E^{\mathbf{K}}$, because $\mathbf{H} \leq_s \mathbf{D}$. This completes the proof that $\mathbf{K}$ is a strong subdihypergraph.

$\Leftarrow$: Observe first that for any cycle $r = (f_1, \dots, f_m) \in R_1$, $\overline{r} = (f_m, \dots, f_1)$ is a cycle in $\mathbf{M} = \mathbf{D}(R_1, R_2)$. Let $\overline{R}_1$ be the set of all such cycles in $\mathbf{M}$. Secondly, an analogous result also holds for hypercycles in R_2. More precisely, it is easily obtained by the definition of $\mathbf{M}$ that for any $r = \big(\langle f_1, v_1\rangle, \dots, \langle f_m, v_m\rangle\big) \in R_2$, $\overline{r} = \big(\langle f_m, \overline{v}_m\rangle, \dots, \langle f_1, \overline{v}_1\rangle\big)$, where $\overline{v}_i = I_2^{\mathbf{D}}(f_i)$ for $i = 1, \dots, m$, is a hypercycle in $\mathbf{M}$. Let $\overline{R}_2$ be the set of all such cycles in $\mathbf{M}$. Thirdly, it is easy to see $\mathbf{M}(\overline{R}_1, \overline{R}_2) = \mathbf{D}$.

Assume now that $\mathbf{K}$ is a strong subdihypergraph of $\mathbf{M}$. Then by the above facts and the proved implication $\Rightarrow$ (applying to $\mathbf{M}$ and $\overline{R}_1$,$\overline{R}_2$) we obtain that $\mathbf{H}$ is a strong subdihypergraph of $\mathbf{D}$. $\square$

Theorem 3.3. *Let $\mathbf{D}$ be a dihypergraph, R_1 a family of cycles and R_2 a family of pairwise hyperedge-disjoint hypercycles. Then* $\mathbf{S}_s(\mathbf{D}) \simeq \mathbf{S}_s\big(\mathbf{D}(R_1, R_2)\big)$.

Proof. Let φ be a function of the set of all strong subdihypergraphs of $\mathbf{D}$ into the set of all strong subdihypergraphs of $\mathbf{M} = \mathbf{D}(R_1, R_2)$ such that $\varphi(\mathbf{H}) = \langle V^{\mathbf{H}}, E^{\mathbf{H}}, I^{\mathbf{M}}|_{E^{\mathbf{H}}}\rangle$ for each $\mathbf{H} \leq_s \mathbf{D}$. First, it is easy to see that $\varphi(\mathbf{H})$ is a well-defined weak subdihypergraph of $\mathbf{M}$, so by Proposition 3.2 we have that $\varphi(\mathbf{H}) \leq_s \mathbf{M}$. Thus φ is correctly defined. Secondly, by Proposition 3.2 φ is surjective. More formally, take $\mathbf{K} \leq_s \mathbf{M}$, and let $\mathbf{H} = \langle V^{\mathbf{K}}, E^{\mathbf{K}}, I^{\mathbf{D}}|_{E^{\mathbf{K}}}\rangle$. Then obviously $\mathbf{H}$ is a weak subdihypergraph of $\mathbf{D}$, so it is also strong by Proposition 3.2. Moreover, $\varphi(\mathbf{H}) = \mathbf{K}$.

Now observe that for two arbitrary strong subdihypergraphs $\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{D}$ (and analogously for $\mathbf{H}_1, \mathbf{H}_2 \leq_s \mathbf{M}$), $\mathbf{H}_1 \leq_s \mathbf{H}_2$ iff $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$; $\mathbf{H}_1 = \mathbf{H}_2$ iff $V^{\mathbf{H}_1} = V^{\mathbf{H}_2}$. These facts easily follow from the definition of strong subdihypergraphs. Hence we deduce that φ is also injective. Moreover, we obtain that $\mathbf{H}_1 \leq_s \mathbf{H}_2$ iff $V^{\mathbf{H}_1} \subseteq V^{\mathbf{H}_2}$ iff $V^{\varphi(\mathbf{H}_1)} \subseteq V^{\varphi(\mathbf{H}_2)}$ iff $\varphi(\mathbf{H}_1) \leq_s \varphi(\mathbf{H}_2)$. Thus φ and its inverse φ^{-1} preserve the (strong subdihypergraph) inclusion, so φ is the desired lattice isomorphism. $\square$

Corollary 3.4. *Let $\mathbf{D}$ be a dihypergraph, R_1 a family of cycles and R_2 a family of pairwise hyperedge-disjoint hypercycles. Then for each $W \subseteq V^{\mathbf{D}}$, $V^{\langle W\rangle^s_{\mathbf{D}}} = V^{\langle W\rangle^s_{\mathbf{D}(R_1,R_2)}}$ and $E^{\langle W\rangle^s_{\mathbf{D}}} = E^{\langle W\rangle^s_{\mathbf{D}(R_1,R_2)}}$.*

Proof. Let φ be the lattice isomorphism of $\mathbf{S}_s(\mathbf{D})$ and $\mathbf{S}_s\big(\mathbf{D}(R_1, R_2)\big)$ from the proof of Theorem 3.3. Then, of course, to the family of all strong subdihypergraphs of $\mathbf{D}$ containing W is assigned under φ the family of all strong subdihypergraphs of $\mathbf{D}(R_1, R_2)$ also containing W. Hence, $\varphi\big(\langle W\rangle_{\mathbf{D}}\big) = \langle W\rangle_{\mathbf{D}(R_1,R_2)}$, because φ preserves all meets and joins. This completes the proof. $\square$

Proposition 3.5. *Let* $\mathbf{D}$ *be a dihypergraph and* R_1, R_2 *families of hyperedge-disjoint simple cycles and hypercycles of* $\mathbf{D}$ *respectively. Then for each one- or two-element* $V \subseteq V^{\mathbf{D}}$, $s^{\mathbf{D}(R_1,R_2)}(V) = s^{\mathbf{D}}(V)$.

Proof. Take a two-element set $V \subseteq V^{\mathbf{D}}$ and a simple hypercycle $r = \big(\langle f_1, v_1\rangle, \dots, \langle f_m, v_m\rangle\big) \in R_2$. Observe that if $\langle f_i, v_i\rangle$ is a labeled 2-edge starting from V, then $\langle f_{i-1}, v_{i-1}\rangle$, where $f_0 = f_m$, is a labeled 2-edge ending in V, i.e. $\big(I_1^{\mathbf{D}}(f_{i-1}) \setminus \{v_{i-1}\}\big) \cup \big\{I_2^{\mathbf{D}}(f_{i-1})\big\} = V$. Conversely, if $\langle f_i, v_i\rangle$ ends in V, then $\langle f_{i+1}, v_{i+1}\rangle$, where $f_{m+1} = f_1$, starts from V. Moreover, $f_1, \dots, f_m$ are pairwise different. These facts imply that the number of all labeled 2-edges of r starting from V is equal to the number of all labeled 2-edges of r ending in V. Hence we infer that the number of all labeled 2-edges in E^{R_2} starting from V is equal to the number of all labeled 2-edges in E^{R_2} ending in V, because hypercycles in R_2 are pairwise 2-edge-disjoint. This implies $s^{\mathbf{D}(R_1,R_2)}(V) = s^{\mathbf{D}}(V)$.

Using the definition of simple cycles and the assumption that cycles in R_1 are pairwise edge-disjoint we can show, in a similar way, that for any $v \in V^{\mathbf{D}}$, $s^{\mathbf{D}(R_1,R_2)}(v) = s^{\mathbf{D}}(v)$. □

A simple consequence of the above fact is the following

Corollary 3.6. *Let* $\mathbf{D}$ *be a total dihypergraph of finite type* $\langle n_1, n_2\rangle$ *(where* $n_1, n_2 \in \mathbb{N}$*) and let* R_1, R_2 *be families of hyperedge-disjoint simple cycles and simple hypercycles of* $\mathbf{D}$ *respectively. Then* $\mathbf{D}(R_1, R_2)$ *is also a total dihypergraph of finite type* $\langle n_1, n_2\rangle$.

Now we must prove several, in general, difficult and rather technical results on dihypergraphs, which will be needed in the proof of our main result. For example we formulate an analogue of Euler's Theorem for dihypergraphs. To this purpose we consider in the sequel dihypergraphs having only regular 2-edges and such that each of its hyperedges is labeled; such dihypergraphs will be called labeled dihypergraphs. In other words, let $\mathbf{D}$ be a dihypergraph without edges and 2-loops (i.e. $E^{\mathbf{D}} = E^{\mathbf{D}}_{reg}(2)$) and let $U \subseteq V^{\mathbf{D}}$ be a set of vertices such that $|U \cap I_1^{\mathbf{D}}(e)| = 1$ for each $e \in E^{\mathbf{D}}$. Then the pair $\langle \mathbf{D}, U\rangle$ will be called a labeled dihypergraph (or more formally, a dihypergraph labeled by U). Moreover, for any (regular) 2-edge e of $\mathbf{D}$, the exactly one element of U belonging to $I_1^{\mathbf{D}}(e)$ will be denoted by $u(e)$, and the other vertex of $I_1^{\mathbf{D}}(e)$ will be denoted by $u^{ot}(e)$, i.e. $\{u^{ot}(e)\} = I_1^{\mathbf{D}}(e) \setminus \{u(e)\}$.

We say that a labeled dihypergraph $\langle \mathbf{D}, U\rangle$ is labeled-connected (or more formally, that $\mathbf{D}$ is labeled-connected with respect to U) iff each vertex of $\mathbf{D}$ is an endpoint of some hyperedge (i.e. $V^{\mathbf{D}} = \bigcup_{e\in E^{\mathbf{D}}} \big(I_1^{\mathbf{D}}(e) \cup \{I_2^{\mathbf{D}}(e)\}\big)$) and for any two 2-edges $f, g \in E^{\mathbf{D}}$, $I_1^{\mathbf{D}}(f) = I_1^{\mathbf{D}}(g)$ or there is a sequence $\big(\langle e_1, u(e_1)\rangle, \dots, \langle e_m, u(e_m)\rangle\big)$ of labeled 2-edges such that

(HC.1) $I_1^{\mathbf{D}}(e_1) = I_1^{\mathbf{D}}(f)$ or $\big\{u^{ot}(e_1), I_2^{\mathbf{D}}(e_1)\big\} = I_1^{\mathbf{D}}(f)$,

(HC.2) $I_1^{\mathbf{D}}(e_m) = I_1^{\mathbf{D}}(g)$ or $\big\{u^{ot}(e_m), I_2^{\mathbf{D}}(e_m)\big\} = I_1^{\mathbf{D}}(g)$,

(HC.3) for $1 \le i \le m-1$, one of the following holds:
$\big\{u^{ot}(e_i), I_2^{\mathbf{D}}(e_i)\big\} = I_1^{\mathbf{D}}(e_{i+1})$ or $\big\{u^{ot}(e_i), I_2^{\mathbf{D}}(e_i)\big\} = \big\{u^{ot}(e_{i+1}), I_2^{\mathbf{D}}(e_{i+1})\big\}$
or $I_1^{\mathbf{D}}(e_i) = \big\{u^{ot}(e_{i+1}), I_2^{\mathbf{D}}(e_{i+1})\big\}$ or $I_1^{\mathbf{D}}(e_i) = I_1^{\mathbf{D}}(e_{i+1})$.

Having this definition we can take the relation $\sim$ on $E^{\mathbf{D}}$ such that for $f, g \in E^{\mathbf{D}}$, $f \sim g$ iff $I_1^{\mathbf{D}}(f) = I_1^{\mathbf{D}}(g)$ or there is $\big(\langle e_1, u(e_1)\rangle, \dots, \langle e_m, u(e_m)\rangle\big)$ satisfying (HC.1)–(HC.3) for f and g. Then it is easy to see that $\sim$ is an equivalence relation on $E^{\mathbf{D}}$, so we can take the family $\{E_i\}_{i\in I}$ of equivalence classes of $\sim$, and next subdihypergraphs $\{\mathbf{D}_{i\in I}\}$ such that $\mathbf{D}_i$

consists of E_i and all endpoints of hyperedges of E_i, for each $i \in I$. Observe also that U uniquely labels each of such dihypergraphs, it is sufficient to take $U_i = U \cap V^{\mathbf{D}_i}$ for any $i \in I$. Moreover, it is obvious that each of these labeled dihypergraphs is labeled-connected and a maximal labeled subdihypergraph of $\langle \mathbf{D}, U \rangle$ with this property. The labeled subdihypergraphs $\{\langle \mathbf{D}_i, U_i \rangle\}_{i \in I}$ of $\mathbf{D}$ so obtained will be called labeled-connected components of $\langle \mathbf{D}, U \rangle$. Obviously, labeled-connected components are pairwise hyperedge-disjoint (i.e. $E^{\mathbf{D}_i} \cap E^{\mathbf{D}_j} = \emptyset$ for $i \neq j$). But their vertex sets do not need be disjoint (they can even be equal). For instance, take the dihypergraph $\mathbf{D}$ with two 2-edges e_1, e_2 and four vertices v_1, v_2, v_3, v_4 and $I^{\mathbf{D}}(e_1) = \langle \{v_1, v_2\}, v_3 \rangle$, $I^{\mathbf{D}}(e_2) = \langle \{v_3, v_4\}, v_2 \rangle$ and $u(e_1) = v_1$, $u(e_2) = v_3$. Then obviously e_1 and e_2 belong to two different labeled-connected components of $\langle \mathbf{D}, U \rangle$, i.e. e_1, v_1, v_2, v_3 form one labeled-connected component, and e_2, v_1, v_2, v_3 form the other. Note also that the connectivity of labeled dihypergraphs (and thus also the decomposition onto labeled-connected components) depends on U. For instance, $\langle \mathbf{D}, U \rangle$ in the above example is not labeled-connected, but if we take $U_1 = \{u(e_1), u(e_2)\}$ such that $u(e_1) = v_1$ and $u(e_2) = v_4$, then $\langle \mathbf{D}, U_1 \rangle$ is labeled-connected.

For directed graphs the concept of labeled graphs is of no interest, since then U must be the set of the initial vertices of all edges). Thus $\langle \mathbf{D}, U \rangle$ is equivalent to $\mathbf{D}$. Moreover, $\langle \mathbf{D}, U \rangle$ is labeled-connected iff the subgraph of $\mathbf{D}$ consisting of all edges and their endpoints is connected in the usual sense, because then (HC.1)–(HC.3) are reduced to the existence of an undirected chain. Hence, labeled-connected components of $\langle \mathbf{D}, U \rangle$ are just connected components of $\mathbf{D}$ with at least two vertices.

Let $\langle \mathbf{D}, U \rangle$ be any labeled dihypergraph and $V \subseteq V^{\mathbf{D}}$ a two-element set. Then

$$E_k^{\langle \mathbf{D}, U \rangle}(V) = \{e \in E^{\mathbf{D}} \colon \ \{u^{ot}(e), I_2^{\mathbf{D}}(e)\} = V\} \quad \text{and} \quad k^{\langle \mathbf{D}, U \rangle}(V) = \left|E_k^{\langle \mathbf{D}, U \rangle}(V)\right|.$$

It easily follows from the definition of labeled-connected components by a simple verification:

Lemma 3.7. *Let $\langle \mathbf{D}, U \rangle$ be a labeled dihypergraph and $\langle \mathbf{H}, W \rangle$ a labeled-connected component of $\langle \mathbf{D}, U \rangle$. Then for each $e \in E^{\mathbf{H}}$,*

$$E_s^{\mathbf{H}}(V) = E_s^{\mathbf{D}}(V) \quad \textit{and} \quad E_k^{\langle \mathbf{H}, W \rangle}(V) = E_k^{\langle \mathbf{D}, U \rangle}(V),$$

where $V = I_1^{\mathbf{D}}(e)$ or $V = \{u^{ot}(e), I_2^{\mathbf{D}}(e)\}$.

Now observe that for any dihypergraph $\mathbf{D}$, $E^{\mathbf{D}} = \bigcup_{v_1, v_2 \in V^{\mathbf{D}}} E_s^{\mathbf{D}}(v_1, v_2)$. This equality implies that if $\mathbf{D}$ is finite (i.e. $V^{\mathbf{D}}$ is finite) and of finite type $\langle n_1, n_2 \rangle$, then

$$|E^{\mathbf{D}}| \leq \sum_{v_1, v_2 \in V^{\mathbf{D}}} s^{\mathbf{D}}(v_1, v_2) = \sum_{v \in V^{\mathbf{D}}} s^{\mathbf{D}}(v) + \sum_{v_1, v_2 \in V^{\mathbf{D}}, v_1 \neq v_2} s^{\mathbf{D}}(v_1, v_2) \leq$$
$$n_1 \cdot |V^{\mathbf{D}}| + n_2 \cdot \left|\{\{v_1, v_2\} \subseteq V^{\mathbf{D}} \colon v_1 \neq v_2\}\right|,$$

so $E^{\mathbf{D}}$ is also finite. Moreover, if $\mathbf{D}$ has no edges, then $2 \cdot |E^{\mathbf{D}}| = \sum_{v_1, v_2 \in V^{\mathbf{D}}, v_1 \neq v_2} s^{\mathbf{D}}(v_1, v_2)$.

For a dihypergraph $\mathbf{D}$ without edges, the concept of the dihypergraph type can be a little simplified, because $s^{\mathbf{D}}(v) = 0$ for $v \in V^{\mathbf{D}}$. More precisely, a dihypergraph $\mathbf{D}$ without edges (i.e. $E^{\mathbf{D}} = E^{\mathbf{D}}(2)$) is of 2-type η (where η is a cardinal number) iff $\mathbf{D}$ is of type $\langle 0, \eta \rangle$.

A labeled dihypergraph $\langle \mathbf{D}, U\rangle$ is of 2-type η iff $\mathbf{D}$ is of 2-type η. Analogously, a directed graph $\mathbf{D}$ is of 1-type η iff $\mathbf{D}$ is of type $\langle \eta, 0\rangle$.

Now we generalize two well-known results of graph theory onto labeled dihypergraphs with finitely many vertices and 2-edges. Since we know that each finite labeled dihypergraph $\langle \mathbf{D}, U\rangle$ (i.e. $V^{\mathbf{D}}$ is finite) of finite 2-type has only finitely many 2-edges, we can formulate the first of them as follows:

Lemma 3.8. *Let $\langle \mathbf{D}, U\rangle$ be a finite labeled dihypergraph of finite 2-type n (where $n \in \mathbb{N}$). Then*

$$\sum_{v_1,v_2\in V^{\mathbf{D}},v_1\neq v_2} s^{\mathbf{D}}(v_1,v_2) \quad = \sum_{v_1,v_2\in V^{\mathbf{D}},v_1\neq v_2} k^{\langle \mathbf{D},U\rangle}(v_1,v_2).$$

Proof. Since the left side of this equality is equal to the number $2\cdot|E^{\mathbf{D}}|$ of all hyperedges of $\mathbf{D}$, it is sufficient to prove that the right side is also $2\cdot|E^{\mathbf{D}}|$. But this is trivial, because for each labeled 2-edge e there exist exactly two pairs $\langle v_1, v_2\rangle$ of vertices such that $e \in E_k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$. $\square$

Lemma 3.9. *Let $\langle \mathbf{D}, U\rangle$ be a labeled-connected and finite labeled dihypergraph of finite 2-type n with at least one 2-edge. Then the following conditions are equivalent:*

(a) *there is a simple hypercycle $\big(\langle f_1, u(f_1)\rangle, \dots, \langle f_m, u(f_m)\rangle\big)$ such that $E^{\mathbf{D}} = \{f_1,\dots,f_m\}$.*

(b) $s^{\mathbf{D}}(v_1,v_2) = k^{\langle \mathbf{D},U\rangle}(v_1,v_2)$ *for each $v_1, v_2 \in V^{\mathbf{D}}$, $v_1 \neq v_2$.*

Of course, this result (and its proof) is some generalization of Euler's Theorem.

Proof. (a) $\Rightarrow$ (b): Take $v_1, v_2 \in V^{\mathbf{D}}$. Observe that if $\langle f_i, u(f_i)\rangle$ is a labeled 2-edge starting from $\{v_1, v_2\}$, then $\langle f_{i-1}, u(f_{i-1})\rangle$, where $f_0 = f_m$, is a labeled 2-edge ending in $\{v_1, v_2\}$ (i.e. $\{u^{ot}(f_{i-1}), I_2^{\mathbf{D}}(f_{i-1})\} = \{v_1, v_2\}$). Conversely, if $\langle f_i, u(f_i)\rangle \in E_k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$, then $\langle f_{i+1}, u(f_{i+1})\rangle$, where $f_{m+1} = f_1$, starts from $\{v_1, v_2\}$. Moreover, $f_1, \dots, f_m$ are pairwise different. These facts imply that $s^{\mathbf{D}}(v_1, v_2) = k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$.

(b) $\Rightarrow$ (a): We apply induction on $E^{\mathbf{D}}$. Basis: If $\langle \mathbf{D}, U\rangle$ has at least one 2-edge and satisfies (b), then $\langle \mathbf{D}, U\rangle$ must have at least two 2-edges. Let $\langle \mathbf{D}, U\rangle$ have exactly two, $\langle e, u(e)\rangle$, $\langle f, u(f)\rangle$. Then $\{u^{ot}(e), I_2^{\mathbf{D}}(e)\} = I_1^{\mathbf{D}}(f)$ and $\{u^{ot}(f), I_2^{\mathbf{D}}(f)\} = I_1^{\mathbf{D}}(e)$. Hence, $\big(\langle e, u(e)\rangle, \langle f, u(f)\rangle\big)$ is a simple hypercycle containing $E^{\mathbf{D}}$.

Induction step: Let $|E^{\mathbf{D}}| \geq 2$ and assume that our thesis is true for all labeled dihypergraphs having at least one hyperedge and not greater than $|E^{\mathbf{D}}| - 1$.

Take $A = \{a_1, a_2\} = I_1^{\mathbf{D}}(e)$ for some $e \in E^{\mathbf{D}}$ and observe first that there are hyperpaths starting from A, because, for example, $\langle e, u(e)\rangle$ forms such a hyperpath. Secondly, there are only finitely many hyperpaths starting from A, since $E^{\mathbf{D}}$ is finite. These facts imply that there is a hyperpath $r = \big(\langle e_1, u(e_1)\rangle, \dots, \langle e_m, u(e_m)\rangle\big)$ starting from A (i.e. $I_1^{\mathbf{D}}(e_1) = A$) with the greatest length $m \geq 1$.

Assume now that $B \neq A$, where $B = \{u^{ot}(e_m), I_2^{\mathbf{D}}(e_m)\}$. Then it is not difficult to see (in a similar way as in the proof of (a)$\Rightarrow$(b)) that $|E_k^{\langle \mathbf{D},U\rangle}(B) \cap E^r| = |E_s^{\mathbf{D}}(B) \cap E^r| + 1$ (because $\langle e_m, u(e_m)\rangle$ ends in B, but $\langle e_1, u(e_1)\rangle$ does not start from B, by our assumption). This implies that there is a labeled 2-edge $\langle f, u(f)\rangle$ such that $f \notin \{e_1, \dots, e_m\}$ and $I_1^{\mathbf{D}}(f) = B$, since $s^{\mathbf{D}}(B) = k^{\langle \mathbf{D},U\rangle}(B)$. Thus we obtain another hyperpath starting from A such that its length is equal $m+1$. This contradiction implies that r is a simple hypercycle (i.e. $\{u^{ot}(e_m), I_2^{\mathbf{D}}(e_m)\} = I_1^{\mathbf{D}}(e_1)$).

Now it is sufficient to show that r contains all hyperedges of $\mathbf{D}$. Assume otherwise that r does not contain all hyperedges of $\mathbf{D}$. Then first, the labeled dihypergraph $\langle\overline{\mathbf{D}}, \overline{U}\rangle$ obtained from $\langle\mathbf{D}, U\rangle$ by omitting all labeled 2-edges of r has hyperedges. Secondly, by the proof of (a)⇒(b) we know that for any two-element set $\{v_1, v_2\} \subseteq V^{\mathbf{D}}$, $|E_s^{\mathbf{D}}(v_1, v_2) \cap E^r| = |E_k^{\langle\mathbf{D},U\rangle}(v_1, v_2) \cap E^r|$. This fact implies

$$s^{\overline{\mathbf{D}}}(v_1, v_2) = k^{\langle\overline{\mathbf{D}},\overline{U}\rangle}(v_1, v_2) \quad \text{for each } v_1, v_2 \in V^{\mathbf{D}},\ v_1 \neq v_2. \tag{1}$$

Now let $\langle\mathbf{H}, W\rangle$ be a labeled-connected component of $\langle\overline{\mathbf{D}}, \overline{U}\rangle$ and take $v_1, v_2 \in V^{\mathbf{H}}, v_1 \neq v_2$. If there is no labeled 2-edge of $\langle\mathbf{H}, W\rangle$ ending or starting from $\{v_1, v_2\}$, then $s^{\mathbf{H}}(v_1, v_2) = 0 = k^{\langle\mathbf{H},W\rangle}(v_1, v_2)$. If there is a labeled 2-edge $\langle e, u(e)\rangle$ of $\langle\mathbf{H}, W\rangle$ such that $I_1^{\mathbf{D}}(e) = \{v_1, v_2\}$ or $\{u^{ot}(e), I_2^{\mathbf{D}}(e)\} = \{v_1, v_2\}$, then by Lemma 3.7 we have $s^{\mathbf{H}}(v_1, v_2) = s^{\overline{\mathbf{D}}}(v_1, v_2)$ and $k^{\langle\mathbf{H},W\rangle}(v_1, v_2) = k^{\langle\overline{\mathbf{D}},\overline{U}\rangle}(v_1, v_2)$. Thus by (1)

$$s^{\mathbf{H}}(v_1, v_2) = k^{\langle\mathbf{H},W\rangle}(v_1, v_2) \quad \text{for each } v_1, v_2 \in V^{\mathbf{H}},\ v_1 \neq v_2. \tag{2}$$

Now we show that there is a labeled 2-edge $\langle h, u(h)\rangle$ of $\langle\mathbf{H}, W\rangle$ such that $I_1^{\mathbf{D}}(h) = I_1^{\mathbf{D}}(e_p)$ for some $p = 1, \dots, m$. Take $\langle g, u(g)\rangle$ of $\langle\mathbf{H}, W\rangle$. If $I_1^{\mathbf{D}}(g) = I_1^{\mathbf{D}}(e_1)$, then $\langle g, u(g)\rangle$ is the desired labeled 2-edge. Thus we can assume $I_1^{\mathbf{D}}(g) \neq I_1^{\mathbf{D}}(e_1)$. Then there is a sequence $(\langle f_1, u(f_1)\rangle, \dots, \langle f_l, u(f_l)\rangle)$ satisfying (HC.1)–(HC.3) for $\langle g, u(g)\rangle$ and $\langle e_1, u(e_1)\rangle$, because $\langle\mathbf{D}, U\rangle$ is labeled-connected. We have three cases: $f_1 = e_i$ for some $1 \leq i \leq m$; or there is $2 \leq j \leq l$ such that $f_j = e_i$ for some $1 \leq i \leq m$ and $\{f_1, \dots, f_{j-1}\} \cap \{e_1, \dots, e_m\} = \emptyset$; or $\{f_1, \dots, f_l\} \cap \{e_1, \dots, e_m\} = \emptyset$. Let $\langle\overline{h}, u(\overline{h})\rangle = \langle g, u(g)\rangle$ or $\langle\overline{h}, u(\overline{h})\rangle = \langle f_{j-1}, u(f_{j-1})\rangle$ or $\langle\overline{h}, u(\overline{h})\rangle = \langle f_l, u(f_l)\rangle$, respectively. Then $I_1^{\mathbf{D}}(\overline{h}) = I_1^{\mathbf{D}}(e_i)$ or $I_1^{\mathbf{D}}(\overline{h}) = \{u^{ot}(e_i), I_2^{\mathbf{D}}(e_i)\} = I_1^{\mathbf{D}}(e_{i+1})$ (where $e_{m+1} = e_1$) or $\{u^{ot}(\overline{h}), I_2^{\mathbf{D}}(\overline{h})\} = I_1^{\mathbf{D}}(e_i)$ or $\{u^{ot}(\overline{h}), I_2^{\mathbf{D}}(\overline{h})\} = \{u^{ot}(e_i), I_2^{\mathbf{D}}(e_i)\} = I_1^{\mathbf{D}}(e_{i+1})$. Hence, in the first two cases, $\langle\overline{h}, u(\overline{h})\rangle$ is the required labeled 2-edge. In the last two cases by (2) there is a labeled 2-edge $\langle h, u(h)\rangle$ such that $I_1^{\mathbf{D}}(h) = \{u^{ot}(\overline{h}), I_2^{\mathbf{D}}(\overline{h})\}$; recall $\langle e_1, u(e_1)\rangle, \dots, \langle e_m, u(e_m)\rangle$ do not belong to $\langle\mathbf{H}, W\rangle$. Obviously $\langle h, u(h)\rangle$ is the desired labeled 2-edge.

Now we apply the induction hypothesis.
First, we know that $\langle\overline{\mathbf{D}}, \overline{U}\rangle$, thus also $\langle\mathbf{H}, W\rangle$, has at least one hyperedge and less than $\mathbf{D}$. Moreover, $\langle\mathbf{H}, W\rangle$ is labeled-connected and satisfies (2). Thus by the induction hypothesis there is a simple hypercycle $(\langle f_1, u(f_1)\rangle, \dots, \langle f_k, u(f_k)\rangle)$ containing all hyperedges of $\mathbf{H}$; of course, we can assume $f_1 = h$. Then $(\langle e_1, u(e_1)\rangle, \dots, \langle e_p, u(e_p)\rangle, \langle f_1, u(f_1)\rangle, \dots, \langle f_k, u(f_k)\rangle, \langle e_{p+1}, u(e_{p+1})\rangle, \dots, \langle e_m, u(e_m)\rangle)$ is also a simple hypercycle of $\langle\mathbf{D}, U\rangle$. But its length is equal to $m + l \geq m + 1$, which contradicts our assumption that r has the greatest length. This completes the proof of the induction step, and consequently, the implication (a) ⇒ (b). □

Let $\langle\mathbf{D}, U\rangle$ be a labeled dihypergraph and let V be an arbitrary subset of $V^{\mathbf{D}}$. Then the strong subdihypergraph $\langle V\rangle_{\mathbf{D}}$ generated by V can also be labeled by U; more precisely, by $\overline{U} \subseteq U$ such that $\overline{U} = \{u(e) \in U : \ e \in E^{\langle V\rangle_{\mathbf{D}}}\}$. The labeled dihypergraph so obtained will be denoted by $\langle V\rangle_{\langle\mathbf{D},U\rangle}$.

Lemma 3.10. *Let a labeled-connected dihypergraph $\langle \mathbf{D}, U\rangle$ and $V \subseteq V^{\mathbf{D}}$ satisfy the following:*

$(*)$ $V = I_1^{\mathbf{D}}(e)$ *for some* $e \in E^{\mathbf{D}}$.

$(**)$ $E_k^{\langle V\rangle_{\langle \mathbf{D},U\rangle}}(w_1, w_2) = E_k^{\langle \mathbf{D},U\rangle}(w_1, w_2)$ *for each* $w_1, w_2 \in V^{\langle V\rangle_{\mathbf{D}}}$, $w_1, \neq w_2$.

Then $\langle \mathbf{D}, U\rangle = \langle V\rangle_{\langle \mathbf{D},U\rangle}$.

Proof. It is sufficient to show $E^{\mathbf{D}} = E^{\langle V\rangle_{\mathbf{D}}}$, because $\langle \mathbf{D}, U\rangle$ is labeled-connected and $V^{\langle V\rangle_{\mathbf{D}}} \subseteq V^{\mathbf{D}}$. Obviously $E^{\mathbf{D}} \supseteq E^{\langle V\rangle_{\mathbf{D}}}$. Moreover, the definition of strong subdihypergraphs easily implies

$$(1) \qquad E_s^{\langle V\rangle_{\mathbf{D}}}(w_1, w_2) = E_s^{\mathbf{D}}(w_1, w_2) \quad \text{for any} \quad w_1, w_2 \in V^{\langle V\rangle_{\mathbf{D}}}.$$

Now take $f \in E^{\mathbf{D}}$. If $I_1^{\mathbf{D}}(f) = I_1^{\mathbf{D}}(e) = V$, then $f \in E^{\langle V\rangle_{\mathbf{D}}}$, by (1). Thus we can assume $I_1^{\mathbf{D}}(f) \neq V$.
Then, since $\langle \mathbf{D}, U\rangle$ is labeled-connected, there is a sequence $\big(\langle f_1, u(f_1)\rangle, \dots, \langle f_l, u(f_l)\rangle\big)$ satisfying (HC.1)–(HC.3) for e and f (see $(**)$). Then $I_1^{\mathbf{D}}(f_1) = I_1^{\mathbf{D}}(e) = V$ or $\{u^{ot}(f_1), I_2^{\mathbf{D}}(f_1)\} = V$. By (1) and $(**)$ we obtain in both cases $f_1 \in E^{\langle V\rangle_{\mathbf{D}}}$, so also $u(f_1), u^{ot}(f_1), I_2^{\mathbf{D}}(f_1) \in V^{\langle V\rangle_{\mathbf{D}}}$. Hence and by $(**)$, (HC.3) we deduce $I_1^{\mathbf{D}}(f_2) \subseteq V^{\langle V\rangle_{\mathbf{D}}}$ or $\{u^{ot}(f_2), I_2^{\mathbf{D}}(f_2)\} \subseteq V^{\langle V\rangle_{\mathbf{D}}}$, so again $f_2 \in E^{\langle V\rangle_{\mathbf{D}}}$. Thus by a simple induction we obtain $\{f_1, \dots, f_l\} \subseteq E^{\langle V\rangle_{\mathbf{D}}}$, in particular $u(f_l), u^{ot}(f_l), I_2^{\mathbf{D}}(f_l) \in V^{\langle V\rangle_{\mathbf{D}}}$. This fact and (HC.2) imply $I_1^{\mathbf{D}}(f) \subseteq V^{\langle V\rangle_{\mathbf{D}}}$, so $f \in E^{\langle V\rangle_{\mathbf{D}}}$. □

Lemma 3.11. *Let a labeled dihypergraph $\langle \mathbf{D}, U\rangle$ of finite 2-type n (where $n \in \mathbb{N}$) satisfy the following:*

$(*)$ *$\langle \mathbf{D}, U\rangle$ is labeled-connected and locally finite.*

$(**)$ $s^{\mathbf{D}}(w_1, w_2) = k^{\langle \mathbf{D},U\rangle}(w_1, w_2)$ *for each* $w_1, w_2 \in V^{\mathbf{D}}$, $w_1 \neq w_2$.

Then $\langle \mathbf{D}, U\rangle = \big\langle I_1^{\mathbf{D}}(e)\big\rangle_{\langle \mathbf{D},U\rangle}$ *for any* $e \in E^{\mathbf{D}}$*; in particular* $\langle \mathbf{D}, U\rangle$ *is finite.*

Proof. Take $e \in E^{\mathbf{D}}$ and $W = I_1^{\mathbf{D}}(e)$. Let $v_1, v_2 \in V^{\langle W\rangle_{\mathbf{D}}}$ be vertices such that $v_1 \neq v_2$. Then

$$(1) \qquad s^{\langle W\rangle_{\mathbf{D}}}(v_1, v_2) = s^{\mathbf{D}}(v_1, v_2)$$

(see (1) in the previous proof) and, of course, $k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(v_1, v_2) \leq k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$. Hence and by $(**)$,

$$s^{\langle W\rangle_{\mathbf{D}}}(w_1, w_2) \geq k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(w_1, w_2) \quad \text{for all } w_1, w_2 \in V^{\langle W\rangle_{\mathbf{D}}},\ w_1 \neq w_2.$$

Moreover, $\langle W\rangle_{\langle \mathbf{D},U\rangle}$ is of finite 2-type n, and is finite, by $(*)$. Thus by Lemma 3.8 we obtain

$$\sum_{w_1, w_2 \in V^{\langle W\rangle_{\mathbf{D}}}, w_1 \neq w_2} s^{\langle W\rangle_{\mathbf{D}}}(w_1, w_2) = \sum_{w_1, w_2 \in V^{\langle W\rangle_{\mathbf{D}}}, w_1 \neq w_2} k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(w_1, w_2).$$

These two facts imply $s^{\langle W\rangle_{\mathbf{D}}}(v_1, v_2) = k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(v_1, v_2)$, so by (1) and $(**)$

$$(2) \qquad k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(v_1, v_2) = k^{\langle \mathbf{D},U\rangle}(v_1, v_2).$$

Since $E_k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$ is finite ($k^{\langle \mathbf{D},U\rangle}(v_1, v_2) \leq n$, by $(**)$) and $E_k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(v_1, v_2) \subseteq E_k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$, we have by (2) that $E_k^{\langle W\rangle_{\langle \mathbf{D},U\rangle}}(v_1, v_2) = E_k^{\langle \mathbf{D},U\rangle}(v_1, v_2)$. This fact (because v_1, v_2 are arbitrary), $(*)$ and Lemma 3.10 imply $\langle W\rangle_{\langle \mathbf{D},U\rangle} = \langle \mathbf{D}, U\rangle$. □

Lemma 3.12. *Let a dihypergraph* $\mathbf{D}$ *without edges and sets* $\emptyset \neq F \subseteq E^{\mathbf{D}}$, $U = \{u(f)\colon f \in F\} \subseteq V^{\mathbf{D}}$ *(where* $u(f) \in I_1^{\mathbf{D}}(f)$ *for each* $f \in F$*) and* $n \in \mathbb{N}$ *satisfy the following conditions:*

$(*)$ $\mathbf{D}$ *is locally finite and without 2-loops.*

$(**)$ $s^{\mathbf{D}}(v_1, v_2) = n$ *and* $s^{\mathbf{D}(F,U)}(v_1, v_2) \leq n$ *for each* $v_1, v_2 \in V^{\mathbf{D}}$, $v_1 \neq v_2$.

Then there is a family R *of pairwise hyperedge-disjoint simple hypercycles such that* $\mathbf{D}(F,U) = \mathbf{D}(R)$.

Proof. We show that F can be divided onto pairwise 2-edge-disjoint finite sets in such a way that each of these sets forms a simple hypercycle.

Let $\mathbf{H} = \mathbf{D}(F,U)$ and take a two-element set $V \subseteq V^{\mathbf{D}}$. Then it easily follows from the definition of $\mathbf{H}$ that

$$|E_s^{\mathbf{H}}(V)| = \big|\big(E_s^{\mathbf{D}}(V) \setminus F\big) \cup \big\{f \in F\colon \big(I_1^{\mathbf{D}}(f) \setminus u(f)\big) \cup \{I_2^{\mathbf{D}}(f)\} = V\big\}\big|,$$

so we have

$$|E_s^{\mathbf{H}}(V)| = |E_s^{\mathbf{D}}(V)| - |E_s^{\mathbf{D}}(V) \cap F| + \big|\big\{f \in F\colon \big(I_1^{\mathbf{D}}(f) \setminus u(f)\big) \cup \{I_2^{\mathbf{D}}(f)\} = V\big\}\big|,$$

because $\mathbf{D}$ is of finite 2-type.

Hence, since by $(**)$ $s^{\mathbf{H}}(V) \leq n = s^{\mathbf{D}}(V)$, we deduce the equality

$$|E_s^{\mathbf{D}}(V)| - |E_s^{\mathbf{D}}(V) \cap F| + \big|\big\{f \in F\colon \big(I_1^{\mathbf{D}}(f) \setminus u(f)\big) \cup I_2^{\mathbf{D}}(f) = V\big\}\big| \leq |E_s^{\mathbf{D}}(V)|.$$

Thus for each two-element set $V \subseteq V^{\mathbf{D}}$,

$$\big|\big\{f \in F\colon \big(I_1^{\mathbf{D}}(f) \setminus u(f)\big) \cup \{I_2^{\mathbf{D}}(f)\} = V\big\}\big| \leq |E_s^{\mathbf{D}}(V) \cap F|. \tag{1}$$

Let $\mathbf{K}$ be the weak subdihypergraph of $\mathbf{D}$ with $V^{\mathbf{K}} = V^{\mathbf{D}}$ and $E^{\mathbf{K}} = F$. Of course, $\mathbf{K}$ has no edges and 2-loops, and also $\mathbf{K}$ is of finite 2-type n and locally finite (it easily follows from $(*)$ and $(**)$). Moreover, take the labeled dihypergraph $\langle \mathbf{K}, U\rangle$. Then by (1),

$$s^{\mathbf{K}}(v_1, v_2) \geq k^{\langle \mathbf{K},U\rangle}(v_1, v_2) \quad \text{for each } v_1, v_2 \in V^{\mathbf{K}}, v_1 \neq v_2. \tag{2}$$

Now take $v_1, v_2 \in V^{\mathbf{K}}$, $v_1 \neq v_2$, and let $\langle \mathbf{M}, W\rangle = \langle v_1, v_2\rangle_{\langle \mathbf{K},U\rangle}$. Then $s^{\mathbf{M}}(w_1, w_2) = s^{\mathbf{K}}(w_1, w_2)$ (see (1) in the previous proof) and, of course, $k^{\langle \mathbf{M},W\rangle}(w_1, w_2) \leq k^{\langle \mathbf{K},U\rangle}(w_1, w_2)$. Hence and by (2),

$$k^{\langle \mathbf{M},W\rangle}(w_1, w_2) \leq s^{\mathbf{M}}(w_1, w_2) \quad \text{for } \ w_1, w_2 \in V^{\mathbf{M}},\ w_1 \neq w_2.$$

On the other hand, $\mathbf{M}$ is a finite dihypergraph without edges of finite 2-type n, so by Lemma 3.8

$$\sum_{w_1,w_2 \in V^{\mathbf{M}}, w_1 \neq w_2} s^{\mathbf{M}}(w_1, w_2) = \sum_{w_1,w_2 \in V^{\mathbf{M}}, w_1 \neq w_2} k^{\langle \mathbf{M},W\rangle}(w_1, w_2).$$

These two facts imply $k^{\langle \mathbf{M},W\rangle}(w_1, w_2) = s^{\mathbf{M}}(w_1, w_2)$ for each $w_1, w_2 \in V^{\mathbf{M}}$, $w_1 \neq w_2$, in particular $s^{\mathbf{M}}(v_1, v_2) = k^{\langle \mathbf{M},W\rangle}(v_1, v_2)$. Hence and by (2) we obtain

$$k^{\langle \mathbf{K},U\rangle}(v_1, v_2) \leq s^{\mathbf{K}}(v_1, v_2) = s^{\mathbf{M}}(v_1, v_2) = k^{\langle \mathbf{M},W\rangle}(v_1, v_2) \leq k^{\langle \mathbf{K},U\rangle}(v_1, v_2).$$

Thus, since v_1 and v_2 were arbitrary

$$s^{\mathbf{K}}(v_1, v_2) = k^{\langle \mathbf{K}, U\rangle}(v_1, v_2) \quad \text{for each } v_1, v_2 \in V^{\mathbf{K}}, v_1 \neq v_2. \tag{3}$$

Since $E^{\mathbf{K}} = F \neq \emptyset$, we can take the non-empty family $\{\langle \mathbf{K}_i, U_i\rangle : i \in I\}$ of all the labeled-connected components of $\mathbf{K}$. First, each such component of $\mathbf{K}$ is also a dihypergraph of finite 2-type n and locally finite. Secondly, by Lemma 3.7 for any $w_1, w_2 \in V^{\mathbf{K}_i}$ with $w_1 \neq w_2$ and $i \in I$,

$$s^{\mathbf{K}_i}(w_1, w_2) = s^{\mathbf{K}}(w_1, w_2) \quad \text{and} \quad k^{\langle \mathbf{K}_i, U_i\rangle}(w_1, w_2) = k^{\langle \mathbf{K}, U\rangle}(w_1, w_2)$$

or

$$s^{\mathbf{K}_i}(w_1, w_2) = 0 = k^{\langle \mathbf{K}_i, U_i\rangle}(w_1, w_2).$$

Thus (3) holds also for each $\langle \mathbf{K}_i, U_i\rangle$, i.e. $s^{\mathbf{K}_i}(w_1, w_2) = k^{\langle \mathbf{K}_i, U_i\rangle}(w_1, w_2)$. This fact and Lemma 3.11 imply that $\mathbf{K}_i$ is finite for $i \in I$. Thus now we can use Lemma 3.9 (recall that each labeled-connected component has at least one regular 2-edge) to obtain that for each $i \in I$, there is a simple hypercycle $r_i = \big(\langle e_1^i, v_1^i\rangle, \dots, \langle e_{l_i}^i, v_{l_i}^i\rangle\big)$ of $\langle \mathbf{K}_i, U_i\rangle$ (thus also of $\langle \mathbf{K}, U\rangle$) containing all labeled 2-edges of $\langle \mathbf{K}_i, U_i\rangle$.

It is trivial to see that $R = \{r_i : i \in I\}$ is a family of pairwise 2-edge-disjoint hypercycles of $\langle \mathbf{K}, U\rangle$ which contain all labeled 2-edges of $\langle \mathbf{K}, U\rangle$. Hence, $\mathbf{D}(F, U) = \mathbf{D}(R)$, since the set of all labeled 2-edges of $\langle \mathbf{K}, U\rangle$ is equal to $\big\{\langle f, u(f)\rangle : f \in F\big\}$, by the definition of $\langle \mathbf{K}, U\rangle$. □

Lemma 3.13. *Let $\mathbf{D}$ be a locally finite and total dihypergraph of finite type $\langle n_1, n_2\rangle$ and without 2-loops. Let $\mathbf{H}$ be any dihypergraph of type $\langle n_1, n_2\rangle$ such that $\mathbf{D}^* \simeq \mathbf{H}^*$. Then $\mathbf{H}$ also has no 2-loops.*

Proof. Let $\mathbf{D}_1$ be the weak subdihypergraph of $\mathbf{D}$ consisting of all vertices and all 2-edges. Of course, $\mathbf{D}_1$ contains only regular 2-edges, moreover, $s^{\mathbf{D}_1}(v_1, v_2) = s^{\mathbf{D}}(v_1, v_2) = n_2$ for $v_1, v_2 \in V^{\mathbf{D}}$, $v_1 \neq v_2$. Analogously, let $\mathbf{H}_1$ be the weak subdihypergraph of $\mathbf{H}$ consisting of all vertices and all regular 2-edges. Then $s^{\mathbf{H}_1}(w_1, w_2) \leq s^{\mathbf{H}}(w_1, w_2) \leq n_2$ for $w_1, w_2 \in V^{\mathbf{H}}$, $w_1 \neq w_2$. Observe that $\mathbf{D}_1^*$ and $\mathbf{H}_1^*$ are subhypergraphs of $\mathbf{D}^*$ and $\mathbf{H}^*$ respectively, which contain all (undirected) hyperedges with exactly three endpoints. It follows from the fact that here we consider only dihypergraphs with 1- and 2-edges, so each undirected hyperedge with three endpoints must be the image of a regular 2-edge. The inverse result is obvious. Hence we infer $\mathbf{D}_1^* \simeq \mathbf{H}_1^*$. Now, since we know that $\mathbf{D}_1$ and $\mathbf{H}_1$ have no 2-loops and edges, we can apply Lemma 3.1, and next Lemma 3.12. Thus $\mathbf{H}_1 \simeq \mathbf{D}_1(R)$, where R is a family of pairwise hyperedge-disjoint simple hypercycles. This fact and Proposition 3.5 imply $s^{\mathbf{H}_1}(v_1, v_2) = s^{\mathbf{D}_1}(v_1, v_2) = n_2$ for $v_1, v_2 \in V^{\mathbf{D}}$, $v_1 \neq v_2$. Hence we deduce that $\mathbf{H}$ cannot have 2-loops, since $\mathbf{H}$ is of type $\langle n_1, n_2\rangle$. □

Lemma 3.14. *Let a dihypergraph $\mathbf{D}$ and $F \subseteq E^{\mathbf{D}}_{reg}(1)$ and $n \in \mathbb{N}$ satisfy the following conditions:*

($*$) *$\mathbf{D}$ is locally finite.*

($**$) *$s^{\mathbf{D}}(v) = n$ and $s^{\mathbf{D}(F)}(v) \leq n$ for each $v \in V^D$.*

Then there is a family R of pairwise disjoint simple cycles such that $F = E^R$.

Proof. In [14] we prove the analogous result for digraphs. Now we must only apply this result to the digraph $\mathbf{G}$ obtained from $\mathbf{D}$ by omitting all 2-edges, since $\mathbf{G}$ contains F and each cycle of $\mathbf{G}$ is also a cycle of $\mathbf{D}$. Note that the proof of Lemma 3.12 is a generalization of ideas used in [14]. □

Theorem 3.15. *Let dihypergrahs* $\mathbf{D}$ *and* $\mathbf{H}$ *satisfy the following conditions:*

($*$) $\mathbf{D}^* \simeq \mathbf{H}^*$.

($**$) $\mathbf{D}$ *is a total dihypergraph of finite type* $\langle n_1, n_2\rangle$ *and locally finite and without 2-loops.*

($***$) $\mathbf{H}$ *is a dihypergraph of finite type* $\langle n_1, n_2\rangle$.

Then $\mathbf{S}_s(\mathbf{H}) \simeq \mathbf{S}_s(\mathbf{D})$ *and moreover,* $\mathbf{H}$ *is total and locally finite and without 2-loops.*

Proof. By Lemma 3.13, $\mathbf{H}$ has no 2-loops, so by Lemma 3.1 there are $F_1 \subseteq E^{\mathbf{D}}_{reg}(1)$ and $F_2 \subseteq E^{\mathbf{D}}_{reg}(2)$ and $U = \{u(f)\colon f \in F_2\} \subseteq V^{\mathbf{D}}$ such that $u(f) \in I^{\mathbf{D}}_1(f)$ for $f \in F_2$ and $\mathbf{H} \simeq \mathbf{D}\big(F_1; \langle F_2, U\rangle\big)$.

It is easy to show $s^{\mathbf{D}(F_2,U)}(v,w) = s^{\mathbf{D}(F_1;\langle F_2,U\rangle)}(v,w)$ and $s^{\mathbf{D}(F_1)}(v) = s^{\mathbf{D}(F_1;\langle F_2,U\rangle)}(v)$ for each $v, w \in V^{\mathbf{D}}$, $v \neq w$. Hence and by ($***$) $s^{\mathbf{D}(F_1)}(v) \leq n_1$ and $s^{\mathbf{D}(F_2,U)}(v,w) \leq n_2$ for $v, w \in V^{\mathbf{D}}$, $v \neq w$. These two facts, ($**$) and Lemmas 3.12, 3.14 imply that there is a family R of pairwise 2-edge-disjoint simple hypercycles of $\mathbf{D}$ such that $\mathbf{D}(F_2, U) = \mathbf{D}(R)$, and there is a family S of pairwise disjoint simple cycles of $\mathbf{D}$ (and thus also $\mathbf{D}(F_2, U)$) such that $F_1 = E^S$. Hence, $\mathbf{D}(F_1; \langle F_2, U\rangle) = \mathbf{D}(F_2, U)(F_1) = \mathbf{D}(F_2, U)(S) = \mathbf{D}(R)(S) = \mathbf{D}(S; R)$. Thus we obtain $\mathbf{H} \simeq \mathbf{D}(S; R)$. Hence and by Theorem 3.3 we deduce $\mathbf{S}_s(\mathbf{H}) \simeq \mathbf{S}_s(\mathbf{D})$, because obviously isomorphic dihypergraphs have isomorphic strong subdihypergraph lattices. Moreover, by Corollaries 3.4 and 3.6, $\mathbf{H}$ is total and locally finite. □

Remark. ($*$) of Theorem 3.15 can be replaced by $\mathbf{S}_w(\mathbf{D}) \simeq \mathbf{S}_w(\mathbf{H})$ (see Theorems 2.2, 2.4 and the notes after Proposition 2.1).

Theorem 3.16. *Let* $\mathbf{A} = \big\langle A, (k^{\mathbf{A}}_i)^{i=m}_{i=1}, (f^{\mathbf{A}}_i)^{i=n}_{i=1}\big\rangle$ *and* $\mathbf{B} = \big\langle B, (k^{\mathbf{B}}_1)^{i=m}_{i=1}, (f^{\mathbf{B}}_1)^{i=n}_{i=1}\big\rangle$ *be partial algebras with* m *unary and* n *binary operations such that:*

($*$) $\mathbf{S}_w(\mathbf{B}) \simeq \mathbf{S}_w(\mathbf{A})$,

($**$) $\mathbf{A}$ *is a total and locally finite algebra,*

($***$) *for each* $a, b \in A$ *and* $1 \leq i \leq n$, *if* $a \neq b$, *then* $f^{\mathbf{A}}_i(a,b) \neq a$ *and* $f^{\mathbf{A}}_i(a,b) \neq b$.

Then

(a) $\mathbf{S}_s(\mathbf{B}) \simeq \mathbf{S}_s(\mathbf{A})$.

(b) $\mathbf{B}$ *is total and locally finite, and* $\mathbf{B}$ *also satisfies* ($***$), *i.e. for each* $b_1, b_2 \in B$ *and* $1 \leq i \leq n$, $b_1 \neq b_2$ *implies* $f^{\mathbf{B}}_i(b_1,b_2) \neq b_1$ *and* $f^{\mathbf{B}}_i(b_1,b_2) \neq b_2$.

Proof. Take $\mathbf{D}(\mathbf{A})$ and $\mathbf{D}(\mathbf{B})$. Then first, they are of (dihypergraph) type $\langle m+n, 2\cdot n\rangle$ and, of course, this type is finite. Secondly, by Theorem 2.4, $\mathbf{D}^*(\mathbf{A}) \simeq \mathbf{D}^*(\mathbf{B})$. Thirdly, since $\mathbf{A}$ is total and locally finite, we have by Propositions 2.1(b) and 2.3 that $\mathbf{D}(\mathbf{A})$ is total and locally finite. Fourthly, $\mathbf{A}$ satisfies ($***$), so by Proposition 2.5, $\mathbf{D}(\mathbf{A})$ has no 2-loops. These facts, Theorems 2.2 and 3.15 imply $\mathbf{S}_s(\mathbf{A}) \simeq \mathbf{S}_s\big(\mathbf{D}(\mathbf{A})\big) \simeq \mathbf{S}_s\big(\mathbf{D}(\mathbf{B})\big) \simeq \mathbf{S}_s(\mathbf{B})$. Moreover, by Theorem 3.15, $\mathbf{D}(\mathbf{B})$ is total and locally finite and without 2-loops, so by Propositions 2.1(b), 2.3 and 2.5 we obtain (b). □

Obviously the result on unary algebras proved in [14] is a particular case of Theorem 3.16. Thus we obtain, in particular, the necessity of the following three conditions of Theorem 3.16: **A** is total and locally finite and has only finitely many operations; and $(**)$ and $(***)$ of Theorem 3.15. In [14] we showed that these conditions are necessary for unary algebras and directed graphs.

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