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# Quantum Half-Planes via Deformation Quantization[1]

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**Abstract.** We give an idea of constructing irreducible unitary representations of Lie groups by using Fedosov deformation quantization in the concrete case of the group $\mathrm{Aff}(\mathbb{R})$ of affine transformations of the real line. By an exact computation of the star-product and the operator $\hat{\ell}_Z$, we show that the resulting representations exhausted all the irreducible representations of this groups.

## 1. Introduction

Quantization normally means a procedure associating to each classical mechanical system some quantum systems, namely in the Heisenberg model or Schrödinger one. More precisely, the usual formulation of a quantization procedure is a correspondence associating to each symplectic manifold $(M, \omega)$ a Hilbert space $\mathcal{H}$ of so called quantum states and to each classical observable (i.e. each complex-valued function) $f$ a quantum observable (i.e. a normal operator) $Q(f)$, in such a way that the following relations hold

$$Q(1) = \mathrm{Id}_{\mathcal{H}} \tag{1}$$

$$[Q(f), Q(g)] = \frac{\hbar}{i} Q(\{f, g\}) \tag{2}$$

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To attack this general problem there are some approaches, such as Feynman path integral quantization, pseudo-differential operator quantization, Weyl quantization, geometric quantization, etc. .... Following the geometric quantization procedure, at first one restricts oneself to consider the set of observables to be quantized and secondly interpret the geometric quantization procedure operators, see e.g. [4],

$$Q(f) := f + \frac{\hbar}{i}\nabla_{\xi_f}$$

as operators up to the second order approximation in powers of $\hbar$, satisfying the relation (2). From this point of view the so called *Fedosov deformation quantization* can be viewed as higher order approximates of operators satisfying the relation (2). The last interpretation is the main idea behind deformation quantization. This deformation quantization essentially differs from the geometric quantization initiated by A. Kirillov, B. Kostant and J.-P. Souriau, see [1], [10].

Many mathematicians attempted to construct quantum objects related with classical ones: The first object created was the so called Podles quantum spheres. Interpreting the classical upper half-plane as the principal affine space of the special linear group $\mathrm{SL}_2(\mathbb{R})$, one introduces the quantum upper half-plane as some C*-algebra generated by some generators and relations. We are concerned with this upper half-plane from another point of view.

It is well-known that co-adjoint orbits are homogeneous symplectic manifolds with respect to the natural Kirillov form on orbits. A natural question is to associate to these orbits some quantum systems, which could be called quantum co-adjoint orbits. In the most general context, some quantum co-adjoint orbits appeared in [1]–[2]. Still it is difficult to calculate exactly the $\star$-product and the corresponding representations in concrete cases. In this paper we give such a construction for the group $\mathrm{Aff}(\mathbb{R})$ of affine transformations of the real line. The main difficulty is the fact that in the concrete case, we should find out explicit formulae. This group has only two nontrivial 2-dimensional orbits which are the upper and lower half-planes. We shall use the same notion of star-product, introduced by M. Flato and A. Lichnerowicz, see [1]. Our main result is the fact that by an exact computation we can find out explicit star-product formula and then by using the Fedosov deformation quantization, the full list of irreducible unitary representations of this group. These results show the effectiveness of the Fedosov quantization, which is not known up to now.

We introduce some notations in §2, in particular, the canonical coordinates are found in Proposition 2.1. The operators $\hat{\ell}_Z$ which define the representation of the Lie algebra $\mathrm{aff}(\mathbb{R})$ are found in §3. By exponentiating we obtain the corresponding unitary representation of the Lie group $\mathrm{Aff}_0(\mathbb{R})$ in Theorem 4.2 of §4.

## 2. Canonical coordinates on the upper half-planes

Recall that the Lie algebra $\mathfrak{g} = \mathrm{aff}(\mathbb{R})$ of affine transformations of the real straight line is described as follows, see for example [4]: The Lie group $\mathrm{Aff}(\mathbb{R})$ of affine transformations of type

$$x \in \mathbb{R} \mapsto ax + b, \text{ for some parameters } a, b \in \mathbb{R}, a \neq 0.$$

It is well-known that this group $\mathrm{Aff}(\mathbb{R})$ is a two-dimensional Lie group which is isomorphic to the group of matrices

$$\mathrm{Aff}(\mathbb{R}) \cong \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a \neq 0 \right\}.$$

We consider its connected component

$$\mathrm{G} = \mathrm{Aff}_0(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \mid a, b \in \mathbb{R}, a > 0 \right\}$$

of identity element. Its Lie algebra

$$\mathfrak{g} = \mathrm{aff}(\mathbb{R}) \cong \left\{ \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \mid \alpha, \beta \in \mathbb{R} \right\}$$

admits a basis of two generators $X, Y$ with the only nonzero Lie bracket $[X, Y] = Y$, i.e.

$$\mathfrak{g} = \mathrm{aff}(\mathbb{R}) \cong \left\{ \alpha X + \beta Y \mid [X, Y] = Y,\ \alpha, \beta \in \mathbb{R} \right\}.$$

The co-adjoint action of G on $\mathfrak{g}^*$ is given (see e.g. [2], [8]) by

$$\langle K(g)F, Z \rangle = \langle F, \mathrm{Ad}(g^{-1})Z \rangle,\ \forall F \in \mathfrak{g}^*, g \in \mathrm{G} \text{ and } Z \in \mathfrak{g}.$$

Denote the co-adjoint orbit of G in $\mathfrak{g}^*$, passing through $F$ by

$$\Omega = K(\mathrm{G})F := \{K(g)F \mid g \in \mathrm{G}\}.$$

Because the group $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$ is exponential (see [4]), for $F \in \mathfrak{g}^* = \mathrm{aff}(\mathbb{R})^*$, we have

$$\Omega = \{K(\exp(U)F \mid U \in \mathrm{aff}(\mathbb{R})\}.$$

It is easy to deduce that

$$\langle K(\exp U)F, Z \rangle = \langle F, \exp(-\operatorname{ad}_U)Z \rangle.$$

This gives

$$K(\exp U)F = \langle F, \exp(-\operatorname{ad}_U)X \rangle X^* + \langle F, \exp(-\operatorname{ad}_U)Y \rangle Y^*.$$

For a general element $U = \alpha X + \beta Y \in \mathfrak{g}$, we have

$$\exp(-\operatorname{ad}_U) = \sum_{n=0}^{\infty} \frac{1}{n!} \begin{pmatrix} 0 & 0 \\ \beta & -\alpha \end{pmatrix}^n = \begin{pmatrix} 1 & 0 \\ L & e^{-\alpha} \end{pmatrix},$$

where $L = \alpha + \beta + \frac{\alpha}{\beta}(1 - e^{\beta})$. This means that

$$K(\exp U)F = (\lambda + \mu L)X^* + (\mu e^{\alpha})Y^*.$$

From this formula one deduces from [4] the following description of all co-adjoint orbits of G in $\mathfrak{g}^*$:

- If $\mu = 0$, each point $(x = \lambda, y = 0)$ on the horizontal axis corresponds to a 0-dimensional co-adjoint orbit
$$\Omega_\lambda = \{\lambda X^*\}, \quad \lambda \in \mathbb{R}.$$
- For $\mu \neq 0$, there are two 2-dimensional co-adjoint orbits: the upper half-plane $\{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu > 0\}$ corresponds to the co-adjoint orbit
$$\Omega_+ := \{F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^* \mid \mu > 0\}, \tag{3}$$
and the lower half-plane $\{(\lambda, \mu) \mid \lambda, \mu \in \mathbb{R}, \mu < 0\}$ corresponds to the co-adjoint orbit
$$\Omega_- := \{F = (\lambda + \mu L)X^* + (\mu e^{-\alpha})Y^* \mid \mu < 0\}. \tag{4}$$

We shall work from now on for the fixed co-adjoint orbit $\Omega_+$. The case of the co-adjoint orbit $\Omega_-$ can be treated similarly. First we study the geometry of this orbit and introduce some canonical coordinates in it. It is well-known from the orbit method [8] that the Lie algebra $\mathfrak{g} = \mathrm{aff}(\mathbb{R})$ can be realized as the complete right-invariant Hamiltonian vector fields on co-adjoint orbits $\Omega \cong \mathrm{G}_F \setminus \mathrm{G}$ with flat (co-adjoint) action of the Lie group $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$. On the orbit $\Omega_+$ we choose a fix point $F = Y^*$. It is well-known from the orbit method that we can choose an arbitrary point $F$ on $\Omega$. It is easy to see that the stabilizer of this (and therefore of any) point is trivial, $\mathrm{G}_F = \{e\}$. We identify therefore G with $\mathrm{G}_{Y^*} \setminus \mathrm{G}$. There is a natural diffeomorphism $\mathrm{Id}_\mathbb{R} \times \exp(.)$ from the standard symplectic space $\mathbb{R}^2$ with symplectic 2-form $dp \wedge dq$ in the canonical Darboux $(p, q)$-coordinates, onto the upper half-plane $\mathbb{H}_+ \cong \mathbb{R} \rtimes \mathbb{R}_+$ with coordinates $(p, e^q)$, which is, from the above coordinate description, also diffeomorphic to the co-adjoint orbit $\Omega_+$. We can therefore use $(p, q)$ as the standard canonical Darboux coordinates in $\Omega_{Y^*}$. There are also non-canonical Darboux coordinates $(x, y) = (p, e^q)$ on $\Omega_{Y^*}$. We show now that in these coordinates $(x, y)$, the Kirillov form looks like $\omega_{Y^*}(x, y) = \frac{1}{y} dx \wedge dy$, but in the canonical Darboux coordinates $(p, q)$, the Kirillov form is just the standard symplectic form $dp \wedge dq$. This means that there are symplectomorphisms between the standard symplectic space $(\mathbb{R}^2, dp \wedge dq)$, the upper half-plane $(\mathbb{H}_+, \frac{1}{y} dx \wedge dy)$ and the co-adjoint orbit $(\Omega_{Y^*}, \omega_{Y^*})$. Each element $Z \in \mathfrak{g}$ can be considered as a linear functional $\tilde{Z}$ on co-adjoint orbits, as subsets of $\mathfrak{g}^*$, $\tilde{Z}(F) := \langle F, Z\rangle$. It is well-known that this linear function is just the Hamiltonian function associated with the Hamiltonian vector field $\xi_Z$, which represents $Z \in \mathfrak{g}$ following the formula
$$(\xi_Z f)(x) := \frac{d}{dt} f(x \exp(tZ))|_{t=0}, \forall f \in \mathrm{C}^\infty(\Omega_+).$$

The Kirillov form $\omega_F$ is defined by the formula
$$\omega_F(\xi_Z, \xi_T) = \langle F, [Z, T]\rangle, \forall Z, T \in \mathfrak{g} = \mathrm{aff}(\mathbb{R}). \tag{5}$$

This form defines the symplectic structure and the Poisson brackets on the co-adjoint orbit $\Omega_+$. For the derivative along the direction $\xi_Z$ and the Poisson bracket we have relation $\xi_Z(f) = \{\tilde{Z}, f\}, \forall f \in \mathrm{C}^\infty(\Omega_+)$. It is well-known in differential geometry that the correspondence $Z \mapsto \xi_Z, Z \in \mathfrak{g}$ defines a representation of our Lie algebra by vector fields on co-adjoint orbits. If the action of G on $\Omega_+$ is flat [4], we have the second Lie algebra homomorphism from strictly Hamiltonian right-invariant vector fields into the Lie algebra of smooth functions on the orbit with respect to the associated Poisson brackets.

Denote by $\psi$ the indicated symplectomorphism from $\mathbb{R}^2$ onto $\Omega_+$

$$(p,q) \in \mathbb{R}^2 \mapsto \psi(p,q) := (p, e^q) \in \Omega_+$$

**Proposition 2.1.** 1. *The Hamiltonian function $f_Z = \tilde{Z}$ in canonical coordinates $(p,q)$ of the orbit $\Omega_+$ is of the form*

$$\tilde{Z} \circ \psi(p,q) = \alpha p + \beta e^q, \text{ if } Z = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}.$$

2. *In the canonical coordinates $(p,q)$ of the orbit $\Omega_+$, the Kirillov form $\omega_{Y^*}$ is just the standard form $\omega = dp \wedge dq$.*

*Proof.* 1. Each element $F \in (\mathrm{aff}(\mathbb{R}))^*$ is of the form $F = xX^* + yY^*$. This means that the value of the function $f_Z = \tilde{Z}$ on the element $Z = \alpha X + \beta Y$ is

$$\tilde{Z}(F) = \langle F, Z \rangle = \langle xX^* + yY^*, \alpha X + \beta Y \rangle = \alpha x + \beta y.$$

It follows therefore that

$$\tilde{Z} \circ \psi(p,q) = \alpha p + \beta e^q, \text{ if } Z = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix}. \tag{6}$$

2. In canonical Darboux coordinates $(p,q)$, $F = pX^* + e^qY^* \in \Omega_+$, and for $Z = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & 0 \end{pmatrix}$, $T = \begin{pmatrix} \alpha_2 & \beta_2 \\ 0 & 0 \end{pmatrix}$, we have

$$\langle F, [Z,T] \rangle = \langle pX^* + e^qY^*, (\alpha_1\beta_2 - \alpha_2\beta_1)Y \rangle = (\alpha_1\beta_2 - \alpha_2\beta_1)e^q,$$

i.e.

$$\omega_F(\xi_Z, \xi_T) = (\alpha_1\beta_2 - \alpha_2\beta_1)e^q. \tag{7}$$

Let us consider two vector fields

$$\xi_Z = \alpha_1 \frac{\partial}{\partial q} - \beta_1 e^q \frac{\partial}{\partial p} \quad \text{and} \quad \xi_T = \alpha_2 \frac{\partial}{\partial q} - \beta_2 e^q \frac{\partial}{\partial p}.$$

We have

$$\xi_Z \otimes \xi_T = \alpha_1\alpha_2 \frac{\partial}{\partial q} \otimes \frac{\partial}{\partial q} + (\alpha_1\beta_2 - \alpha_2\beta_1)e^q \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial q} + \beta_1\beta_2 e^{2q} \frac{\partial}{\partial p} \otimes \frac{\partial}{\partial p}. \tag{8}$$

From (7) and (8) we conclude that in the canonical coordinates the Kirillov form is just the standard symplectic form $\omega = dp \wedge dq$.

### 3. Computation of generators $\hat{\ell}_Z$

Let us denote by $\Lambda$ the 2-tensor associated with the Kirillov standard form $\omega = dp \wedge dq$ in canonical Darboux coordinates. We use also the multi-index notation. Let us consider the well-known Moyal $\star$-product of two smooth functions $u, v \in \mathrm{C}^\infty(\mathbb{R}^2)$, defined by

$$u \star v = u.v + \sum_{r \geq 1} \frac{1}{r!}\Big(\frac{1}{2i}\Big)^r P^r(u,v),$$

where

$$P^r(u,v) := \Lambda^{i_1 j_1}\Lambda^{i_2 j_2}\dots\Lambda^{i_r j_r}\partial_{i_1 i_2 \dots i_r} u \partial_{j_1 j_2 \dots j_r} v,$$

with

$$\partial_{i_1 i_2 \dots i_r} := \frac{\partial^r}{\partial x^{i_1}\dots\partial x^{i_r}}, x := (p,q) = (p_1,\dots,p_n,q^1,\dots,q^n)$$

as multi-index notation. It is well-known that this series converges in the Schwartz distribution spaces $\mathcal{S}(\mathbb{R}^n)$. We apply this to the special case $n = 1$. In our case we have only $x = (x^1, x^2) = (p, q)$.

**Proposition 3.1.** *In the above mentioned canonical Darboux coordinates $(p,q)$ on the orbit $\Omega_+$, the Moyal $\star$-product satisfies the relation*

$$i\tilde{Z} \star i\tilde{T} - i\tilde{T} \star i\tilde{Z} = i\widetilde{[Z,T]},\ \forall Z, T \in \mathrm{aff}(\mathbb{R}).$$

*Proof.* Consider the elements $Z = \alpha_1 X + \beta_1 Y$ and $T = \alpha_2 X + \beta_2 Y$. Then as noted above the corresponding Hamiltonian functions are $\tilde{Z} = \alpha_1 p + \beta_1 e^q$ and $\tilde{T} = \alpha_2 p + \beta_2 e^q$. It is easy then to see that $P^0(\tilde{Z},\tilde{T}) = \tilde{Z}.\tilde{T}$,

$$P^1(\tilde{Z},\tilde{T}) = \{\tilde{Z},\tilde{T}\} = \partial_p\tilde{Z}\partial_q\tilde{T} - \partial_q\tilde{Z}\partial_p\tilde{T} = (\alpha_1\beta_2 - \alpha_2\beta_1)e^q,$$

$$P^2(\tilde{Z},\tilde{T}) = \Lambda^{12}\Lambda^{12}\partial_{pp}\tilde{Z}\partial_{qq}\tilde{T} + \Lambda^{12}\Lambda^{21}\partial_{pq}\tilde{Z}\partial_{qp}\tilde{T} + \Lambda^{21}\Lambda^{12}\partial_{qp}\tilde{Z}\partial_{pq}\tilde{T} + \Lambda^{21}\Lambda^{21}\partial_{qq}\tilde{Z}\partial_{pp}\tilde{T} = 0.$$

By analogy we have $P^k(\tilde{Z},\tilde{T}) = 0, \forall k \geq 2$. Thus,

$$i\tilde{Z} \star i\tilde{T} - i\tilde{T} \star i\tilde{Z} = \frac{1}{2i}[P^1(i\tilde{Z},i\tilde{T}) - P^1(i\tilde{T},i\tilde{Z})] = i(\alpha_1\beta_2 - \alpha_2\beta_1)e^q,$$

on one hand.

On the other hand, because $[Z,T] = ZT - TZ = (\alpha_1\beta_2 - \alpha_2\beta_1)Y$, we have

$$i\widetilde{[Z,T]} = i(\alpha_1\beta_2 - \alpha_2\beta_1)e^q = i\tilde{Z} \star i\tilde{T} - i\tilde{T} \star i\tilde{Z}.$$

The proposition is hence proved.

Consequently, to each adapted chart $\psi$ in the sense of [2], we associate a G-covariant $\star$-product.

**Proposition 3.2.** (See [6].) *Let $\star$ be a formal differentiable $\star$-product on $\mathrm{C}^\infty(M,\mathbb{R})$, which is covariant under* G. *Then there exists a representation $\tau$ of* G *in* Aut $N[[\nu]]$ *such that*

$$\tau(g)(u \star v) = \tau(g)u \star \tau(g)v.$$

Let us denote by $\mathcal{F}_p u$ the partial Fourier transform of the function $u$ from the variable $p$ to the variable $\eta$, i.e.

$$\mathcal{F}_p(u)(\eta, q) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ip\eta} u(p,q) dp.$$

Let us denote by $\mathcal{F}_p^{-1}(u)$ the inverse Fourier transform.

**Lemma 3.3.** *We have*

1. $\partial_p \mathcal{F}_p^{-1}(u) = i\mathcal{F}_p^{-1}(\eta.u)$,
2. $\mathcal{F}_p(p.v) = i\partial_\eta \mathcal{F}_p(v)$,
3. $P^k(\tilde{Z}, \mathcal{F}_p^{-1}(u)) = (-1)^k \beta e^q \frac{\partial^k \mathcal{F}_p^{-1}(u)}{\partial^k p}$, *with* $k \geq 2$.

*Proof.* The first two formulas are well-known from theory of Fourier transforms. We reproduce them to locate notation.

1. $\partial_p \mathcal{F}_p^{-1}(u) = \partial_p(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{ip\eta} u(\eta, q) d\eta) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} i\eta e^{ip\eta} u(\eta, q) d\eta = i\mathcal{F}_p^{-1}(\eta.u)$.
2. $i\partial_\eta \mathcal{F}_p(v) = i\partial_\eta(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ip\eta} v(p,q) dp = i\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} -ipe^{-ip\eta} v(p,q) dp =$
$= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ip\eta} pv(p,q) dp = \mathcal{F}_p(p.v)$.

3. Remark that $\Lambda = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in the standard symplectic Darboux coordinates $(p,q)$ on the orbit $\Omega_+$ and we have had $\tilde{Z} = \alpha p + \beta e^q$, then

$$P^2(\tilde{Z}, \mathcal{F}_p^{-1}(u)) = \Lambda^{12}\Lambda^{12}\partial_{pp}\tilde{Z}\partial_{qq}\mathcal{F}_p^{-1}(u)) + \Lambda^{12}\Lambda^{21}\partial_{pq}\tilde{Z}\partial_{qp}\mathcal{F}_p^{-1}(u)) +$$

$$\Lambda^{21}\Lambda^{12}\partial_{qp}\tilde{Z}\partial_{pq}\mathcal{F}_p^{-1}(u)) + \Lambda^{21}\Lambda^{21}\partial_{qq}\tilde{Z}\partial_{pp}\mathcal{F}_p^{-1}(u)) = (-1)^2\beta e^q \partial^2_{pp}\mathcal{F}_p^{-1}(u)).$$

By analogy we obtain

$$P^k(\tilde{Z}, \mathcal{F}_p^{-1}(u)) = (-1)^k \beta e^q \partial^k_{p\ldots p}\mathcal{F}_p^{-1}(u)), \forall k \geq 3.$$

The lemma is therefore proved.

For each $Z \in \mathrm{aff}(\mathbb{R})$, the corresponding Hamiltonian function is $\tilde{Z} = \alpha p + \beta e^q$ and we can consider the operator $\ell_Z$ acting on dense subspace $L^2(\mathbb{R}^2, \frac{dpdq}{2\pi})^\infty$ of smooth functions by left $\star$-multiplication by $i\tilde{Z}$, i.e. $\ell_Z(u) = i\tilde{Z} \star u$. It is then extended to the whole space $L^2(\mathbb{R}^2, \frac{dpdq}{2\pi})$. It is easy to see that, because of the relation in Proposition 3.1, the correspondence $Z \in \mathrm{aff}(\mathbb{R}) \mapsto \ell_Z = i\tilde{Z} \star .$ is a representation of the Lie algebra $\mathrm{aff}(\mathbb{R})$ on the space $N[[\frac{i}{2}]]$ of formal power series in the parameter $\nu = \frac{i}{2}$ with coefficients in $N = \mathrm{C}^\infty(M, \mathbb{R})$, see e.g. [6] for more detail.

We study now the convergence of the formal power series. In order to do this, we look at the $\star$-product of $i\tilde{Z}$ as the $\star$-product of symbols and define the differential operators corresponding to $i\tilde{Z}$. It is easy to see that the resulting correspondence is a representation of $\mathfrak{g}$ by pseudo-differential operators.

**Proposition 3.4.** *For each $Z \in \mathrm{aff}(\mathbb{R})$ and for each compactly supported* $C^\infty$*-function* $u \in C_c^\infty(\mathbb{R}^2)$, *we have*

$$\hat{\ell}_Z(u) := \mathcal{F}_p \circ \ell_Z \circ \mathcal{F}_p^{-1}(u) = \alpha(\frac{1}{2}\partial_q - \partial_\eta)u + i\beta e^{q-\frac{\eta}{2}}u.$$

*Proof.* For each $Z \in \mathfrak{g} = \mathrm{aff}(\mathbb{R})$, we have

$$\hat{\ell}_Z(u) := \mathcal{F}_p \circ \ell_Z \circ \mathcal{F}_p^{-1}(u) = \mathcal{F}_p(i\tilde{Z} \star \mathcal{F}_p^{-1}(u)) = i\mathcal{F}_p(\sum_{r\geq 0}\left(\frac{1}{2i}\right)^r P^r(\tilde{Z}, \mathcal{F}_p^{-1}(u)).$$

Remark that

$$P^1(\tilde{Z}, \mathcal{F}_p^{-1}(u)) = \{\tilde{Z}, \mathcal{F}_p^{-1}(u)\} = \alpha\partial_q\mathcal{F}_p^{-1}(u) - \beta e^q\partial_p\mathcal{F}_p^{-1}(u)$$

and applying Lemma 3.3, we obtain:

$$i\mathcal{F}_p\Big(\sum_{r\geq 0}\frac{1}{r!}\Big(\frac{1}{2i}\Big)^r P^r(\tilde{Z}, \mathcal{F}_p^{-1}(u)\Big) =$$

$$\begin{aligned}
&= i\mathcal{F}_p[(\alpha p + \beta e^q)\mathcal{F}_p^{-1}(u) + \tfrac{1}{2i}\alpha\partial_q\mathcal{F}_p^{-1}(u) - \tfrac{1}{2i}\beta e^q\partial_p\mathcal{F}_p^{-1}(u) + \\
&\quad + \tfrac{1}{2!}\left(\tfrac{-1}{2i}\right)^2\beta e^q\partial_p^2\mathcal{F}_p^{-1}(u) + \ldots + \tfrac{1}{n!}\left(\tfrac{-1}{2i}\right)^n\beta e^q\partial_p^n\mathcal{F}_p^{-1}(u) + \ldots] \\
&= i[\alpha i\partial_\eta u + \beta e^q u + \tfrac{1}{2i}\alpha\partial_q u - \tfrac{1}{2i}\beta e^q\mathcal{F}_p(i\mathcal{F}_p^{-1}(\eta.u)) + \\
&\quad + \tfrac{1}{2!}\left(-\tfrac{1}{2i}\right)^2\beta e^q\mathcal{F}_p(i^2\mathcal{F}_p^{-1}(\eta^2.u)) + \ldots + \tfrac{1}{n!}\left(-\tfrac{1}{2i}\right)^n\beta e^q\mathcal{F}_p(i^n\mathcal{F}_p^{-1}(\eta^n.u)) + \ldots] \\
&= i[i\alpha\partial_\eta u + \tfrac{1}{2i}\alpha\partial_q u + \beta e^q u - \beta e^q\tfrac{\eta}{2}u + \\
&\quad + \tfrac{1}{2!}\beta e^q\left(\tfrac{\eta}{2}\right)^2 u + \ldots + \tfrac{1}{n!}(-1)^n\beta e^q\left(\tfrac{\eta}{2}\right)^n u + \ldots] \\
&= \alpha(\tfrac{1}{2}\partial_q - \partial_\eta)u + i\beta e^q[1 - \tfrac{\eta}{2} + \tfrac{1}{2!}\left(\tfrac{\eta}{2}\right)^2 + \ldots + (-1)^n\tfrac{1}{n!}\left(\tfrac{\eta}{2}\right)^n + \ldots] \\
&= \alpha(\tfrac{1}{2}\partial_q - \partial_\eta)u + i\beta e^{q-\frac{\eta}{2}}u.
\end{aligned}$$

The proposition is therefore proved.

**Remark 3.5.** Set $s = q - \frac{\eta}{2}$, $t = q + \frac{\eta}{2}$, we have

$$\hat{\ell}_Z(u) = \alpha\frac{\partial u}{\partial s} + i\beta e^s u, \quad \text{i.e.} \quad \hat{\ell}_Z = \alpha\frac{\partial}{\partial s} + i\beta e^s, \tag{9}$$

which provides a representation of the Lie algebra $\mathrm{aff}(\mathbb{R})$.

## 4. The associate irreducible unitary representations

Our aim in this section is to exponentiate the obtained representation $\hat{\ell}_Z$ of the Lie algebra $\mathrm{aff}(\mathbb{R})$ to the corresponding representation of the Lie group $\mathrm{Aff}_0(\mathbb{R})$. We shall prove that the result is exactly the irreducible unitary representation $T_{\Omega_+}$ obtained from the orbit method or Mackey small subgroup method applied to the group $\mathrm{Aff}(\mathbb{R})$. Let us recall first the well-known list of all the irreducible unitary representations of the group of affine transformation of the real line.

**Theorem 4.1.** [7] *Every irreducible unitary representation of the group* $\mathrm{Aff}(\mathbb{R})$ *of all the affine transformations of the real line, up to unitary equivalence, is equivalent to one of the pairwise nonequivalent representations:*

- *the infinite-dimensional representation* $S$*, realized in the space* $L^2(\mathbb{R}^*, \frac{dy}{|y|})$*, where* $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ *and defined by the formula*

$$(S(g)f)(y) := e^{iby} f(ay), \text{ where } g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix},$$

- *the representation* $U^\varepsilon_\lambda$*, where* $\varepsilon = 0, 1$*,* $\lambda \in \mathbb{R}$*, realized in the 1-dimensional Hilbert space* $\mathbb{C}^1$ *and given by the formula*

$$U^\varepsilon_\lambda(g) = |a|^{i\lambda}(\operatorname{sgn} a)^\varepsilon.$$

Let us consider now the connected component $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$. The irreducible unitary representations can be obtained easily from the orbit method machinery.

**Theorem 4.2.** *The representation* $\exp(\hat{\ell}_Z)$ *of the group* $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$ *is exactly the irreducible unitary representation* $T_{\Omega_+}$ *of* $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$ *associated following the orbit method construction, to the orbit* $\Omega_+$*, which is the upper half-plane* $\mathbb{H} \cong \mathbb{R} \rtimes \mathbb{R}^*$*, i.e.*

$$(\exp(\hat{\ell}_Z)f)(y) = (T_{\Omega_+}(g)f)(y) = e^{iby}f(ay), \forall f \in L^2(\mathbb{R}^*, \frac{dy}{|y|}),$$

*where* $g = \exp Z = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

*Proof.* We choose an admissible Lie sub-algebra $\mathfrak{h} = \langle X \rangle$. Let us denote by $H$ the corresponding analytic subgroup of G with Lie algebra $\mathfrak{h}$. The corresponding representation is $\mathrm{Ind}_H^{\mathrm{G}} \chi_F = \mathrm{Ind}_H^{\mathrm{G}} \chi_{Y^*}$. The homogeneous space $H \setminus \mathrm{G}$ is homeomorphic to $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ with the quasi-invariant measure $\frac{dy}{|y|}$. The corresponding representation $T_{\Omega_+}$ is given exactly by the same formula as the representation $S$ in Theorem 4.1. More precisely, for the element $Z = \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} = \mathrm{aff}(\mathbb{R})$,

$$\exp Z = \exp \begin{pmatrix} \alpha & \beta \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} e^\alpha & \frac{\beta}{\alpha}(e^\alpha - 1) \\ 0 & 0 \end{pmatrix} & \text{if } \alpha \neq 0 \\ \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} & \text{if } \alpha = 0 \end{cases}$$

It is reasonable to simplify the notation, to consider the second case. Remark that because $y = e^q$ is the natural but non-canonical coordinate in $\mathbb{R}^* \cong H \setminus \mathrm{G}$ we can write the induced representation obtained from the orbit method construction as

$$T_{\Omega_+}(\exp Z)f(e^s) = \exp(i\frac{\beta}{\alpha}(e^\alpha - 1)e^s)f(e^{\alpha+s}). \tag{10}$$

Therefore for the one-parameter subgroup $\exp(tZ), t \in \mathbb{R}$, the action is given by the formula

$$T_{\Omega_+}(\exp tZ)f(e^s) = \exp(i\frac{\beta}{\alpha}(e^{t\alpha} - 1)e^s)f(e^{t\alpha+s}).$$

By a direct computation, we obtain

$$\begin{aligned} &\frac{\partial}{\partial t}T_{\Omega_+}(\exp tZ)f(e^s) = \\ &= i\tfrac{\beta}{\alpha}e^s\alpha e^{t\alpha}\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)f(e^{ts+s}) + \exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)\alpha e^{t\alpha+s}.\partial_s f \\ &= e^{t\alpha+s}\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)[i\beta f(e^{t\alpha+s}) + \alpha.\partial_s f], \end{aligned} \tag{11}$$

on one hand. On the other hand, we have

$$\begin{aligned} &\hat{\ell}_Z T_{\Omega_+}(\exp tz)f(e^s) = \\ = \;& (i\beta e^s + \alpha.\partial_s)[\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)f(e^{t\alpha+s})] = i\beta e^s\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)f(e^{t\alpha+s}) + \\ & +\alpha[i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)f(e^{t\alpha+s}) + \exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)e^{t\alpha+s}.\partial_s f] \\ = \;& e^{t\alpha+s}\exp(i\tfrac{\beta}{\alpha}(e^{t\alpha}-1)e^s)[i\beta f(e^{t\alpha+s}) + \alpha.\partial_s f]. \end{aligned} \tag{12}$$

From (11) and (12) follows that

$$\frac{\partial}{\partial t}T_{\Omega_+}(\exp(tZ))f(y) = \hat{\ell}_Z T_{\Omega_+}(\exp(tZ))f(y).$$

Obviously, $T_{\Omega_+}(\exp(tZ))f(y)|_{t=0} = f(y)$. This means that $T_{\Omega_+}(\exp(tZ))f(y)$ is the unique solution of the Cauchy problem

$$\begin{cases} \frac{\partial}{\partial t}U(t,y) & = \hat{\ell}_Z U(t,y) \\ U(0,y) & = \mathrm{Id} \end{cases}$$

This gives $\exp(\hat{\ell}_Z)f(y) \equiv T_{\Omega_+}(\exp Z)f(y)$. The proof of the theorem is therefore achieved.

By analogy, we have also

**Theorem 4.3.** *The representation* $\exp(\hat{\ell}_Z)$ *of the group* $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$ *is exactly the irreducible unitary representation* $T_{\Omega_-}$ *of* $\mathrm{G} = \mathrm{Aff}_0(\mathbb{R})$ *associated following the orbit method construction, to the orbit* $\Omega_-$*, which is the lower half-plane* $\mathbb{H} \cong \mathbb{R} \rtimes \mathbb{R}^*$*, i.e*

$$(\exp(\hat{\ell}_Z)f)(y) = (T_{\Omega_-}(g)f)(y) = e^{iby}f(ay), \forall f \in L^2(\mathbb{R}^*, \frac{dy}{|y|}),$$

*where* $g = \exp Z = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$.

**Remark 4.4.**

1. We have demonstrated how all the irreducible unitary representations of the connected group of affine transformations could be obtained from deformation quantization. It is reasonable to refer to the algebras of functions on co-adjoint orbits with this $\star$-product as quantum ones.

2. In a forthcoming work, we shall do the same calculation for the group of affine transformations of the complex straight line $\mathbb{C}$. This achieves the description of *quantum* $\overline{\mathrm{MD}}$ *co-adjoint orbits*, see [4] for definition of $\overline{\mathrm{MD}}$ Lie algebras.

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